# DECOMPOSITION OF SIMPLE SINGULARITIES OF FUNCTIONS

O. V. Lyashko

The goal of this article is to study the decompositions of simple singularities of functions under deformations. All possible decompositions are described in terms of Dynkin diagrams of the singularities.

## §1. DEFINITIONS

Let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of an analytic function with a critical point of multiplicity $\mu$ at the origin. The germ of the analytic function $F: (\mathbf{C}^n \times \mathbf{C}, 0) \to (\mathbf{C}, 0)$ for which $F(x, 0) \equiv f(x)$ is called the deformation F of the function f with base C.

The critical point 0 of the function f under the deformation F can be decomposed with respect to the type $X = (X_1, \ldots, X_k)$, where $X_i = (X_{i,1}, \ldots, X_{i,j_i})$, if there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ there exists an $\varepsilon_0(\delta)$ such that, for all $\varepsilon$ such that $0 < |\varepsilon| < \varepsilon_0$, the function $F(\cdot, \varepsilon)$ of the variable $x \in \mathbf{C}^n$ in the sphere $|x| < \delta$ has exactly k different critical values, the i-th critical value being attained at critical points of the types $X_{i,1}, \ldots, X_{i,j_i}$.

Let g be the germ of an analytic function on the sphere $V \subset \mathbf{C}^n$. We will say that the germ g has a singularity of type X, if $dg \neq 0$ on the boundary $\partial V$ of the sphere, the function g attains exactly k critical values inside the sphere V, and the i-th critical value is attained at critical points of the types $X_{i,1}, \ldots, X_{i,j_i}$. By definition, the multiplicity of the singularity is $\mu(X) = \sum_{ij} \mu(X_{ij})$, where the $\mu(X_{ij})$ are the multiplicities of the singularities $X_{ij}$. The germ of the function $G: \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$ on the set $V \times \{0\}$ that is such that $G(x, 0) \equiv g(x)$ is called the deformation G of g. The decomposition of a function g of type X with respect to the type Y is defined analogously. A singularity of type X abuts a singularity of type Y, if X can be decomposed with respect to the type (Y, Y').

A Dynkin diagram of the critical point 0 of multiplicity $\mu$ of a function of n variables, where $n \equiv 3 \pmod 4$ is a connected graph with $\mu$ vertices numbered $1, 2, \ldots, \mu$ that correspond to the vanishing cycles of a distinguished basis (see [3]). Two vertices are connected by k simple (dotted) edges, if the intersection index of the corresponding vanishing cycles is equal to k (or minus k).

Let $E = (E_1, \ldots, E_k)$ be a partition of the set of vertices of the Dynkin diagram. We will call the partition E proper, if every $E_i$ consists of vertices numbered by consecutive integers. The part of the Dynkin diagram formed by the vertices of $E_i$ and the edges connecting them will be called the inner diagram for $E_i$.

This diagram is not necessarily connected. Denote the connected components of this diagram by $(E_{i,1}, \ldots, E_{i,j_i})$. We will say that the original Dynkin diagram can be decomposed with respect to the type E into the diagrams $E_{ij}$.

We will say that the decomposition of the Dynkin diagram with respect to the type E is consistent with the decomposition of the critical point with respect to the type X, if, for every i, the set $(E_{i,1}, \ldots, E_{i,j_i})$ is a set of Dynkin diagrams of critical points of the types $(X_{i,1}, \ldots, X_{i,j_i})$ corresponding to the i-th critical value.

Simple critical points (points of types $A_k$, $D_k$, $E_6$, $E_7$, $E_8$) are defined in [1].

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## §2. FORMULATION OF THE RESULTS

THEOREM 1. Assume that the critical point 0 of the function f can be decomposed with respect to the type X. Then there exist a distinguished basis and a proper partition E of this basis such that the decomposition of the Dynkin diagram with respect to the type E is consistent with the decomposition of the critical point with respect to the type X.

THEOREM 2. The simple critical point 0 of the function f can be decomposed with respect to the type X if and only if there exist a distinguished basis and a proper partition E of this basis such that the decomposition of the Dynkin diagram with respect to the type E is consistent with the decomposition of the critical point with respect to the type X.

THEOREM 3. A simple critical point can be decomposed with respect to the type $X = (X_1, \ldots, X_k)$ if and only if it can be decomposed with respect to the type $(X_1, \ldots, X_{k-2}, X')$, $X'$ being decomposable with respect to the type $(X_{k-1}, X_k)$.

Theorem 2 permits one to establish that some decomposition of a simple singularity is realizable. In order to do this, starting from the original diagram (from a standard diagram of $A_k$, $D_k$, $E_k$, for example) and changing the homotopic classes of the paths of the distinguished basis, it is necessary to convert to other diagrams and investigate their partitions. Theorem 3 shows that it is sufficient to study the decompositions into pairs of critical values.

A simple singular point abuts precisely those simple singular points, the canonical Dynkin diagrams of which can be imbedded in the canonical Dynkin diagram of the original singular point (V. I. Arnol'd [1], D. Siersma [9]). Using Theorem 2, one can obtain the following result, which belongs to Grothendieck (see [5]).

THEOREM 4. Assume that, under some decomposition of a simple critical point S, singular points of the types $X_1, \ldots, X_k$ correspond to some critical value. Then the canonical Dynkin diagram for S can be decomposed into Dynkin diagrams of the singularities $X_i$ after elimination of some number of vertices with all of the edges going into them. The converse is also true.

All decompositions of simple singularities into pairs of critical values are described by the following theorem.

THEOREM 5. I. Assume that a critical point of type X can be decomposed with respect to the type $(X_1, X_2)$. Then

1) $\mu(X_1) + \mu(X_2) = \mu(X)$,

2) there exist abuttings $X \to X_i$ $(i = 1, 2)$,

3) if X is of type $D_k$, then there exists an i such that $A_{k-2}$ abuts $X_i$.

II. All decompositions of simple singularities satisfying conditions 1), 2), and 3) are realizable except for the decomposition $E_6 \to (A_1, A_1)(A_2, A_2)$, which is not.

Remark. In terms of Dynkin diagrams, conditions 2) and 3) are described by Theorem 4. Condition 2) means that the canonical diagram of each $X_i$ can be imbedded in the diagram of X. Condition 3) means that the diagram of one of the $X_i$ can be imbedded in the diagram of $A_{k-2}$. For example, $D_4$ cannot be decomposed with respect to the type $(A_1A_1)(A_1A_1)$, since the diagram of $(A_1A_1)$ cannot be imbedded in the diagram of $A_2$ and, therefore, condition 3) is not fulfilled.

The number of different ways of decomposing simple singularities into pairs of critical values is contained in the following table ($\mu \leq 8$):

| $\mu$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $A_\mu$ | 1 | 2 | 5 | 9 | 17 | 29 | 51 |
| $D_\mu$ | — | — | 4 | 10 | 22 | 38 | 74 |
| $E_\mu$ | — | — | — | — | 19 | 41 | 75 |

S. Zdravkovskaya [6] has investigated degeneracies of polynomials of one variable. For $\mu \leq 5$, the number of irreducible strata has been calculated. For $\mu \geq 5$, the closure of the set of polynomials of type X need not be an irreducible variety.

## §3. PROOFS

3.1. Proof of Theorem 1. Assume that, under a deformation F, the function f can be decomposed with respect to the type X. Consider a minimal versal deformation G of f:

$$G : (\mathbf{C}^n \times \mathbf{C}^\mu,\ \ 0 \times 0) \to (\mathbf{C}, 0); \quad (x, \lambda) \mapsto G(x, \lambda).$$

Then the deformation F is equivalent to a deformation F' induced by the versal: $F'(x, h) = G(x, \lambda(h)),\ h \in \mathbf{C}$. Fix some small $h_0$. Then the function $g_0 = G(\cdot, \lambda(h_0))$ has k different critical values and the i-th critical value is attained at the points $X_{i,1}, \ldots, X_{i,j_i}$. In the plane of values of $g_0$, draw k nonintersecting circles $B_i$ with centers at the critical values of $g_0$. On the boundary of each circle, single out a point $d_i$. Choose a point $u \in \mathbf{C} \setminus \bigcup_i B_i$ such that $\operatorname{Im} z > \operatorname{Im} u$ and $\forall z \in \bigcup_i B_i$. Then $\bigcup_i B_i$ is contained in the half-plane $H = \{z \in \mathbf{C} \mid \operatorname{Im} z \geqslant \operatorname{Im} u\}$.

Consider the path $\gamma(t)$, $t \in [0, 1]$, in the base of the versal deformation of f such that $\gamma(0) = \lambda(h_0)$ and, for all $t \in (0, 1]$, the function $g_t \equiv G(\cdot, \gamma(t))$ has $\mu$ different critical values contained in $\bigcup_i B_i$. Now, in the half-plane H, draw a system of nonintersecting paths $\pi_i$ in $H \setminus \bigcup_i B_i$ connecting the point u with $d_i$. Inside each circle $B_i$, draw a system of nonintersecting paths $\pi_{ij}$ connecting the point $d_i$ with the critical values of $g_1$. Number the paths consecutively $1, 2, \ldots, \mu$, the number of $\pi_{ij} \cup \pi_i$ being larger than the number of $\pi_{lm} \cup \pi_l$, if $\arg \frac{d\pi_i}{dt}\Big|_{t=0} < \arg \frac{d\pi_l}{dt}\Big|_{t=0}$. In the case $i = l$, the number of $\pi_{ij} \cup \pi_i$ is larger than the number of $\pi_{im} \cup \pi_i$, if $\arg \frac{d\pi_{ij}}{dt}\Big|_{t=0} < \arg \frac{d\pi_{im}}{dt}\Big|_{t=0}$. Then the paths $\pi_{ij} \cup \pi_i$ give a distinguished basis of the vanishing cycles in the homologies of $g_1^{-1}(u)$. The circles $B_i$ give a partition of the critical values of the function $g_1$ and, consequently, a partition E of the Dynkin diagram constructed from the $\pi_{ij} \cup \pi_i$. This partition is proper.

Note that, if the critical points corresponding to the critical values that lie in one circle do not merge as $t \to 0$, then the intersection index of the corresponding cycles is equal to zero. On the other hand, the Dynkin diagram of a singular point is connected (see [4, 7]). From this, it follows that the partition E is consistent with X. Theorem 1 is proved.

3.2. Consider the base $C^\mu$ of the versal deformation of the function f. To the point $\lambda \in C^\mu$, there corresponds the set of $\mu$ critical values of the function $G(\cdot, \lambda)$ (taking their multiplicities into account). With the point $\lambda$, associate the polynomial p(z) of degree $\mu$ with leading coefficient one the roots of which are the critical values of $G(\cdot, \lambda)$. We will obtain the germ of the mapping $\varphi : (\mathbf{C}^\mu, 0) \to (P, 0)$, where $P \sim C^\mu$, the space of polynomials. In P, there is a natural stratification: The stratum $S(\mu_1, \ldots, \mu_s)$ is the polynomials that have s different roots with multiplicities $\mu_1, \ldots, \mu_s$ $(\mu_1 + \mu_2 + \ldots + \mu_s = \mu)$. For a simple singular point, $\varphi$ is a characteristic holomorphic mapping, and, for the stratum $S(1, \ldots, 1)$, the mapping $\varphi$: $\varphi^{-1}(S(1, 1, \ldots, 1) \to S(1, \ldots, 1)$ is a covering (see [8]).

3.3. Proof of Theorem 2. Assume that f has a simple singular point of multiplicity $\mu$ at the origin. Consider a quasi-homogeneous versal deformation of f:

$$G(x, \lambda) = f(x) + \sum_{i=1}^{\mu} \lambda_i \varphi_i(x),$$

where the $\varphi_i(x)$ are monomials that define a basis of the local ring $Q_f$ [after factorization with respect to the ideal $(\partial f/\partial x_i)$] (see [2]). Consider also the Dynkin diagram of the function f and a proper partition E of this diagram. The diagram is given by the set of paths $\pi_i(t)$ in the plane of values of $G(\cdot, \lambda_0)$, where $\lambda_0$ does not belong to the bifurcation set of the functions in $C^\mu$. Let $\lambda_1 \in C^\mu$ be such that $G(\cdot, \lambda_1)$ is of type X. Since G is a quasi-homogeneous versal deformation, there exists a curve $\lambda_1(h)$ $(h \in C)$ such that $\lambda_1(0) = 0$, $\lambda_1(1) = \lambda_1$ and $G(\cdot, \lambda_1(h))$ is of type X for any $h \neq 0$. Therefore, in order to prove Theorem 2, it is sufficient to find such a point $\lambda_1$.

In the plane of values of $G(\cdot, \lambda_0)$, construct a system of paths $\gamma_i(\tau)$ such that: 1) $\gamma_i(0) = \pi_i(1)$, 2) if the i-th and j-th vertices of the diagram lie in one subset of the partition E, then $\gamma_i(1) = \gamma_j(1)$, and 3) the paths $\gamma_i$ do not mutually intersect and do not intersect the paths $\pi_j$.

Such a system of paths $\gamma_i$ gives a path $\gamma$ in the space P, and $\gamma(\tau) \in S(1, \ldots, 1)$ for $\tau \in [0, 1)$. Now raise the path $\gamma$ in P to a path $\lambda(t)$ in $C^\mu$, such that $\lambda(0) = \lambda_0$ (note that $\varphi(\lambda_0) = \gamma(0)$). Then $\lambda(1)$ is the desired point $\lambda_1$. The proof is finished.

3.4. A braid group of $\mu$ filaments acts transitively on the set of Dynkin diagrams of the singularity of f in the following manner. Let $\pi_i$ be a distinguished basis of paths giving some Dynkin diagram of the singularity of f, let the $e_i$ be the vanishing cycles corresponding to these paths, and let the $t_j$ ($j = 1, \ldots, \mu - 1$) be the standard generators of the braid group. The braid $t_j$ transforms the system of paths $\pi_i$ into the system $\pi_i'$:

$$\pi_i' = \pi_i \ (i \neq j, j+1); \ \pi_j' = \pi_{j+1}; \quad \pi_{j+1}' = \pi_j \cup \beta_{j+1},$$

where $\beta_{j+1}$ is the simple loop corresponding to $\pi_{j+1}$. From this, making use of the Picard–Lefschetz formulas, we can describe the action of the braid on the vanishing cycles and, thus, on the Dynkin diagram:

$$t_j\colon (e_1, \ldots, e_\mu) \mapsto (e_1, \ldots, e_{j-1}, e_{j+1}, e_j + \langle e_j, e_{j+1}\rangle e_{j+1}, \ldots, e_\mu).$$

Proposition 1. 1) Let L be a subdiagram of a Dynkin diagram of f. One can transform the original diagram so that L remains a subdiagram and the vertices of L are numbered consecutively.

2) Let $(E_1, E_2)$ be a proper partition of a Dynkin diagram of f and let the numbers of the vertices of $E_1$ be smaller than the numbers of the vertices of $E_2$. Then there exists a diagram of f with proper partition $(E_1, E_2)$ such that the numbers of the vertices of $E_1$ are larger than the numbers of the vertices of $E_2$.

The proof of this proposition is not complicated. We omit it.

3.5. Proof of Theorem 3. Assume that f can be decomposed with respect to the type $\bar{X} = (X_1, \ldots, X')$. Then there exist a distinguished basis and a proper partition $\bar{E}$ of this basis consistent with $\bar{X}$. Since $X'$ can be decomposed with respect to the type $(X_{k-1}, X_k)$, there exist a distinguished basis for $X'$ and a proper partition $(E_{k-1}, E_k)$ of this basis that corresponds to this decomposition. One can choose a distinguished basis for f in such a manner that the Dynkin diagram for $X'$ is a subdiagram of the Dynkin diagram for f and the partition $E'$ into $(E_{k-1}, E_k)$ is a subpartition of the partition $\bar{E}$. Then, by Theorem 2, f can be decomposed with respect to the type X.

Conversely, if f can be decomposed with respect to the type X, then there exist a distinguished basis and a partition E of the diagram consistent with X. By changing the distinguished basis, one can make the numbers of the vertices of $E_k$ and $E_{k-1}$ be consecutive, while, at the same time, the inner diagrams of the $E_i$ do not change (item 3.4). Consider the partition $(E_1, \ldots, E_{k-2}, E_{k-1} \cup E_k)$ of the diagram of f. The decomposition of f with respect to the type $(X_1, \ldots, X_{k-2}, X')$ corresponds to this partition and $X'$ can be decomposed into $(X_{k-1}, X_k)$. The theorem is proved.

3.6. Proof of Theorem 4. Assume that after elimination of some number of vertices of the canonical Dynkin diagram of f, this diagram can be decomposed into Dynkin diagrams of the singularities $X_i$. Then, for some deformation, f can be decomposed so that singular points of the types $X_1, \ldots, X_k$ correspond to one critical value. This follows from Theorem 2 and Proposition 1. Let us now prove the converse.

Consider a root system R (of type $A_\mu$, $D_\mu$, or $E_\mu$) in the space $\mathbf{R}^\mu$ and the Weyl group W(R). From each $\mu$ root $e_1, \ldots, e_\mu$ that generates the whole Weyl group, one can construct a diagram L: Its vertices correspond to the roots of the set and two vertices are connected by straight (dotted) edges whenever the scalar product of the corresponding roots is equal to $-1$ ($+1$). These diagrams will be called the Dynkin diagrams of $A_\mu$ $D_\mu$, and $E_\mu$, respectively. If we reflect the $e_i$ in the mirror corresponding to $e_j$, we obtain a new root set that generates W(R) and its Dynkin diagram $t_{ij}(L)$.

Assume that the subdiagram $L'$ of the diagram L constructed from $e_2, \ldots, e_\mu$ is a canonical Dynkin diagram. Then one can reduce the diagram L to canonical form by reflecting $e_1$ in the mirrors of the $e_i$ ($i \geq 2$). Indeed, consider the hyperplane $H \subset \mathbf{R}^\mu$ generated by the vectors $e_2, \ldots, e_\mu$. The reflections in the mirrors of $e_2, \ldots, e_\mu$ generate the Weyl group $W(R \cap H)$ in the hyperplane H. The $e_i$ ($i \geq 2$) are a system of simple roots of the root system $H \cap R$. Let $\bar{e}_1$ be the projection of $e_1$ on H and C be the closure of the Weyl chamber corresponding to the basis $e_2, \ldots, e_\mu$. By reflection in the mirrors of the $e_i$ ($i \geq 2$), the vector $\bar{e}_1$ can be transformed into $\bar{e}_1' \in -C$. Then this sequence of reflections in the space $\mathbf{R}^\mu$ transforms $e_1$ into $e_1'$. The roots $e_1', e_2, \ldots, e_\mu$ are a basis for the system R. Therefore, this sequence of reflections transforms the diagram L into a canonical Dynkin diagram (in the process, the diagram $L'$ does not change at all).

Assume now that under deformation of a singularity of f, critical points of the types $X_1, \ldots, X_k$ correspond to one critical value. Then the diagram $E = (E_1, \ldots, E_k)$ (the $E_i$ are the connected components of E corresponding to the $X_i$) can be imbedded in some Dynkin diagram L of the singularity of f. We may assume that the $E_i$ are canonical diagrams of simple singularities. We will prove that E can be imbedded in the canonical diagram of f. Let $\alpha$ be a vertex of L not lying in E. Then one can reduce all of the connected components of $\{\alpha\} \cup E$ to canonical form by reflecting $e_\alpha$ in the mirrors of the roots of E. In this process, the diagram L will transform into the diagram L' and $\{\alpha\} \cup E$ will transform into $E' = (E_1', \ldots, E_{k'}')$. Now choose an $\alpha' \in L' \setminus E'$ and do the same operation. After several steps, we obtain the canonical diagram of f. Note that, on each step, the subdiagram E did not change and, thus, it can be imbedded in the canonical diagram of f.

## §4. DYNKIN DIAGRAMS

The proof of Theorem 5 is rather tedious, so we will only outline it. The proof is based on investigating the possible Dynkin diagrams of the singularities $A_k$ and $D_k$ (for the singularities $E_6$, $E_7$, and $E_8$, the proof is carried out by checking out each case). We will present the corresponding assertions here without proof.

A. Let $\varepsilon_1, \ldots, \varepsilon_{\mu+1}$ be a canonical basis in $R^{\mu+1}$ and $\varepsilon_i - \varepsilon_j$ $(i \neq j)$ be a root system of $A_\mu$. Consider a set of $\mu$ roots $\{e_1, \ldots, e_\mu\}$ that generate the Weyl group of $A_\mu$. Every such set defines a Dynkin diagram. We will call the set of all such diagrams the Dynkin diagrams of $A_\mu$. If, in the set, the root $e_i$ is replaced by $-e_i$, the corresponding vertex in the Dynkin diagram changes orientation: All simple edges emanating from it become dotted lines and vice versa. We will say that two diagrams coincide to within orientation of their vertices if one of them can be obtained from the other by changing the orientation of several if its vertices.

The Weyl group of $A_\mu$ coincides with the permutation group of the set $\{1, \ldots, \mu + 1\}$. The transposition (i, j) corresponds to reflection in the mirror of the root $\varepsilon_i - \varepsilon_j$. Assume that the root set $\{e_1, \ldots, e_\mu\}$ generates the whole Weyl group of $A_\mu$. Then the transpositions that correspond to reflections in the roots generate the permutation group of the set $\{1, \ldots, \mu + 1\}$. Consider an oriented graph with $\mu + 1$ vertices numbered $1, \ldots, \mu + 1$: i and j are connected by an edge going from i to j, if the root $\varepsilon_i - \varepsilon_j$ is in the set $\{e_1, \ldots, e_\mu\}$. Such a graph uniquely defines the original root set. Now neglect the numbering of the vertices of the graph. We obtain an oriented graph. Denote it by $\Gamma$. Denote the nonoriented graph corresponding to $\Gamma$ by $\bar{\Gamma}$. The graph $\Gamma$ has $\mu + 1$ vertices and $\mu$ edges and is connected, since the set of transpositions generates a permutation group. Therefore, $\Gamma$ is a tree. The graph $\Gamma$ defines a Dynkin diagram of $A_\mu$. After its vertices have been numbered, it defines a root set. We associate the Dynkin diagram of this set with the graph $\Gamma$ (obviously, the diagram does not depend on the manner in which the vertices of $\Gamma$ were numbered).

Let $\Gamma_1$ and $\Gamma_2$ be trees with oriented edges such that $\bar{\Gamma}_1 = \bar{\Gamma}_2$. Then the Dynkin diagrams corresponding to $\Gamma_1$ and $\Gamma_2$ coincide to within the orientation of their vertices. For any $\Gamma$, there exists an oriented graph $\Gamma'$ such that $\bar{\Gamma} = \bar{\Gamma}'$ and such that, for any vertex of $\Gamma'$, all edges go into it or emanate from it simultaneously. All edges of the Dynkin diagram corresponding to $\Gamma'$ are dotted. Therefore, every Dynkin diagram of $A_\mu$ coincides to within orientation with a diagram that has all edges dotted.

Any diagram of $A_\mu$ with dotted edges can be obtained in the following manner. Consider an arbitrary tree $\bar{\Gamma}$ with $\mu$ edges. Assume that it has k vertices out of which emanate $\mu_i \geq 2$ edges $(i = 1, \ldots, k)$. With each such vertex of $\bar{\Gamma}$ associate a complete graph with $\mu_i$ vertices, all edges of which are dotted. We will paste two such complete graphs together at one vertex if and only if the corresponding vertices of $\Gamma$ are connected by an edge.

A braid group acts on the trees $\bar{\Gamma}$ with numbered edges: $t_j : \bar{\Gamma} \mapsto t_j(\bar{\Gamma})$, where $t_j$ is a generator of the braid group. The graph $t_j(\bar{\Gamma})$ can be obtained from $\bar{\Gamma}$ in the following manner. If the j and j + 1 edges of $\bar{\Gamma}$ do not intersect, interchange their numbers. If the j and j + 1 edges have a vertex in common, then in $t_j(\bar{\Gamma})$, the j edge connects the two other vertices of these edges of $\bar{\Gamma}$ and the j + 1 edge coincides with the j edge of $\bar{\Gamma}$. The remaining edges in $t_j(\bar{\Gamma})$ are the same as in $\bar{\Gamma}$.

Proposition 2. Let $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ be two trees with numbered edges. Then $\bar{\Gamma}_1$ can be transformed into $\bar{\Gamma}_2$ by the action of a braid group.

Let f be a singularity of type $A_\mu$. A distinguished basis of the vanishing cycles generates a root system and the Weyl group of $A_\mu$ in the homologies of the nonsingular fiber $V \simeq R^\mu$. In the space $R^1 \times V$, choose a basis $\varepsilon_1, \ldots, \varepsilon_{\mu+1}$ such that the root system coincides with the set of vectors $\varepsilon_i - \varepsilon_j$ $(i \neq j)$. Then every distinguished basis of the vanishing cycles defines the graphs $\Gamma$ and $\bar{\Gamma}$. Their edges are numbered (in a manner corresponding to the numbering of the vanishing cycles).

Proposition 2 implies that any Dynkin diagram with vertices numbered in an arbitrary manner is a Dynkin diagram of the singularity $A_\mu$.

Let us prove Theorem 5 in the case of $A_\mu$. Assume that the diagrams of $(A_{\mu_1}, \ldots, A_{\mu_s})$ and $(A_{\mu'_1}, \ldots, A_{\mu'_k})$ can be imbedded in the canonical diagram of $A_\mu$ (that is, $\sum\mu_i + s - 1 \leqslant \mu$ and $\sum\mu'_j + k - 1 \leqslant \mu$) and that $\sum\mu_i + \sum\mu'_j = \mu$. To be definite, let $s \geq k$. Divide the set of numbers $\{1, \ldots, \mu\}$ into s + k nonintersecting subsets $M_1, \ldots, M_s, M'_1, \ldots, M'_k$ in such a manner that every subset $M_i(M'_j)$ consists of $\mu_i(\mu'_j)$ consecutive numbers and that the numbers of $M_i$ are less than the numbers of $M'_j$ for any j and i. We will construct complete graphs on the sets $M_1 \cup M'_1, \ldots M_k \cup M'_k, M_{k+1}, \ldots, M_s$.

Connect an arbitrary point of $M'_i$ with a point of $M_{i+1}$ $(1 \leqslant i \leqslant k-1)$ by an edge. Let S be the set of vertices of these edges lying in $\bigcup M'_i$. In the set $\bigcup M'_i \setminus S$, choose s − k points (in the set $\bigcup M'_i \setminus S$, there are $\sum\mu'_k - (k-1)$ elements, and $\sum\mu'_k - (k-1) \geqslant s-k$). Connect each of them with one of the vertices of $M_i$ $(k+1 \leq i \leq s)$, by an edge, different points being connected with different $M_i$. We have constructed a Dynkin diagram of the singularity $A_\mu$. A partition of the set $\{1, \ldots, \mu\}$ into $\cup M_i$ and $\cup M'_j$ defines a proper partition of the Dynkin diagram. Obviously, the partition corresponds to the decomposition of $A_\mu$ with respect to the type $(A_{\mu_1} \ldots A_{\mu_s})(A_{\mu'_1} \ldots A_{\mu'_k})$. Hence, by Theorem 2, such a decomposition is realizable.

D. Let the $\varepsilon_i$ be a canonical basis in $R^\mu$ and $\pm\varepsilon_i \pm \varepsilon_j$ $(i \neq j)$ be a root system of $D_\mu$. An element of the Weyl group of $D_\mu$ can be considered to be permutation of the $\varepsilon_i$ with a change of sign of an even number of vectors. Reflections are ordinary transpositions and transpositions that change the signs of the permuted vectors. Consider a set of $\mu$ roots that generate the whole Weyl group. Construct the graph $\Gamma$: connect, $i, j \in \{1, \ldots, \mu\}$ by an edge going from i to j, if the root $\varepsilon_i - \varepsilon_j$ is in the set. If $\varepsilon_i + \varepsilon_j$ $(-\varepsilon_i - \varepsilon_j)$ is in the set, connect i and j by a dotted edge and place the sign + (−) on it. Finally, neglect the numbering of the vertices of the graph so obtained.

Note that, when the signs of the roots of the set are changed, we can obtain any orientation of the solid lines and any signs on the dotted lines. If the graph $\Gamma$ is connected, it has one and only one cycle, since there are $\mu$ vertices and edges.

Proposition 3. A set of $\mu$ roots generates the Weyl group of $D_\mu$ if and only if the graph $\Gamma$ is connected and there is an odd number of dotted edges on the cycle.

In the same manner as in the case of $A_\mu$, the graph $\Gamma$ defines a Dynkin diagram. From $\Gamma$, construct the graph $\bar{\Gamma}$. Make all edges solid and neglect the orientation and the signs on the edges.

Proposition 4. The graph $\bar{\Gamma}$ defines a Dynkin diagram to within the orientation of its vertices.

Consider the set of all connected graphs $\Gamma$ with $\mu$ vertices and $\mu$ edges. Number the edges from 1 to $\mu$. The braid group acts on the set of such graphs: $t_j$: $\bar{\Gamma} \mapsto t_j(\bar{\Gamma})$. If the j-th and (j + 1)-th edges of $\bar{\Gamma}$ have two vertices in common, then $t_j(\bar{\Gamma}) = \bar{\Gamma}$. Otherwise, $t_j(\bar{\Gamma})$ is defined in the same manner as in A.

Let f be a singularity of type $D_\mu$. Then a distinguished basis of the vanishing cycles defines a root system of type $D_\mu$ and, hence, a graph $\bar{\Gamma}$ with numbered edges. Under the circumstances, the numbering of the edges cannot be arbitrary.

Proposition 5. Let $\bar{\Gamma}$ be the graph constructed from a distinguished basis of the vanishing cycles. Then:

1) When going around a cycle of the graph $\bar{\Gamma}$, the numbering of its edges must be monotonic.

2) Let a cycle of $\bar{\Gamma}$ consist of $l$ edges, numbered $1, \ldots, l$. Then other edges do not emanate from the vertex that the edges 1 and $l$ have in common.

Making use of Proposition 5, one can show that, if the Dynkin diagram of $X_i$ cannot be imbedded in the canonical diagram of $A_{\mu-2}$ (for i = 1, 2), then $D_\mu$ cannot be decomposed with respect to the type $(X_1, X_2)$. On the other hand, the proof of Theorem 5 for $D_\mu$ can be carried out by constructing an appropriate diagram of $D_\mu$ and a proper partition of this diagram.

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# FUNCTIONS WITH ISOMORPHIC JACOBIAN IDEALS

A. N. Shoshitaishvili

§0. Let us consider a set of germs at 0 of functions F that are holomorphic at 0, i.e., F: $C^n \to C$, $F(0) = 0$, and grad $F(0) = 0$. Let us denote it by J. Let us consider a C algebra of formal power series $C[[x_1, \ldots, x_n]]$. For any $F \in J$ let us consider an ideal $i(F) \subset C[[x_1, \ldots, x_n]]$, spanned over the partial derivatives $\partial_{x_i} F$, $i = 1, \ldots, n$.

For any integer $k \geq 1$ let us denote by $Q_k(F)$ the factor algebra $C[[x_1, \ldots, x_n]]/(i(F) \bigcup M^k)$, where M is a maximal ideal in $C[[x_1, \ldots, x_n]]$.

With the aid of a contact group [1], we shall prove in Theorem 1 that the image in k jets of the set

$$W^k(F) = \{\psi \in J: \text{ the C-algebras } Q_k(F) \text{ and } Q_k(\psi) \text{ are isomorphic}\}$$

is a manifold.

Let us consider $Q(F) = C[[x_1, \ldots, x_n]]/i(F)$. We shall say that a germ $G \in J$ is Q equivalent to a germ $F \in J$ if the C algebras Q(G) and Q(F) are isomorphic. The set of germs that are Q equivalent to a germ of F will be denoted by W(F).

Let F have finite multiplicity, i.e., $\dim_C Q(F) < \infty$. In Theorems 2 and 3 we shall consider two classes of finite-multiplicity germs of functions. The first of them is a class of functions $F \in J$ such that W(F) coincides with the orbit of action of an R group, i.e., of a group of germs at 0 of holomorphisms h at 0, i.e., h: $C^n \to C^n$, $h(0) = 0$, that acts in J according to the law $(h, F) \mapsto F \circ h$. This class of functions coincides with R-equivalent functions (i.e., which lie in the same orbit of an R group) that are quasihomogeneous. The second class consists of functions $F \in J$ for which W(F) coincides with the orbit of action of an RL group, i.e., a group which is a direct product of an R group and a group p of germs at 0 of holomorphisms at 0, i.e., p: $C \to C$, $p(0) = 0$, that act according to the law $((p, h), F) \mapsto p \circ F \circ h$.

§1. Let us consider a function $F \in J$. In $C[[x_1, \ldots, x_n]]$ let us define two subalgebras $a^k(F)$ and $A^k(F)$: $a^k(F) = \{g \in M \subset C[[x_1, \ldots, x_n]]: \ i(g) \subset (i(F) \bigcup M^k)\}$; $A^k(F)$ is a linear space spanned over $a^k(F)$ and $i(F) \cdot M$.

Let us denote $a_k(F)$, $A_k(F)$, $(i(F) \cdot M)_k$ the images of the algebras $a^k(F)$, $A^k(F)$, $i(F) \cdot M$ under a factorization mapping $C[[x_1, \ldots, x_n]] \to C[[x_1, \ldots, x_n]]/M^{k+1}$.

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