A NONCOMMUTATIVE TAYLOR FORMULA AND FUNCTIONS OF TRIANGLE OPERATORS

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1. The functional calculus of operators is defined (cf., [1]) by a class of operators $\mathfrak{B}$ in a Banach(or Hilbert) space $\mathfrak{A}$ and some algebra F of scalar functions of the real or complex variable with the help of a homomorphism $f \mapsto f(A)$ of the algebra F into the algebra $L(\mathfrak{B})$ of bounded linear mappings in $\mathfrak{B}(A \in \mathfrak{A})$.

A transformer of order k will be called a k-linear continuous mapping S_k: $B_1 \times \ldots \times B_k \mapsto S_k(B_1, \ldots, B_k)$ of the space $L(\mathfrak{B})$. Further on we shall consider functional transformers generated by an operator $A \in \mathfrak{A}$ and scalar functions $f(\lambda_0, \lambda_1, \ldots, \lambda_k)$ (A-transformers). It is convenient to describe the action of an A-transformer using a method proposed by R. Feynman and in detail worked by V. P. Maslov (cf., [2]), and which has the form

$$S_k(f, A): \overset{1}{B} \times \overset{3}{B} \times \ldots \times \overset{2k-1}{B} \mapsto f(\overset{0}{A}, \overset{2}{A}, \ldots, \overset{2k}{A}) \overset{1}{B} \cdot \overset{3}{B} \ldots \overset{2k-1}{B},$$

where the upper index indicates the order of factors. The way of computation ("untangling" of indices) of this expression can be explained by the following fundamental examples:

a) If $A = \int \lambda E(d\lambda)$ is a self-adjoint operator in a Hilbert space $\mathfrak{B}$ and f is a sufficiently smooth function, bounded on the spectrum of A, then

$$S_k(f, A)(B, \ldots, B) = \int \ldots \int f(\lambda_0, \lambda_1, \ldots, \lambda_k) E(d\lambda_0) BE(d\lambda_1) B \ldots BE(d\lambda_k), \tag{1}$$

where the integral on the right is the multiple operator Stiltjes integral (m.o.i). M.o.i. have been introduced by S. G. Krein and the author in connection with the formulas of differentiation with respect to a parameter of functions of self-adjoint operators [3-6]. The theory and applications of m.o.i. have been subsequently developed by M. Sh. Birman and M. Z. Solomyak [7-10].

b) If A is the generating operator of a strongly continuous group $e^{tA} = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} (\lambda - A)^{-1} d\lambda$, where γ is a suitable contour separating domain D containing the spectrum of operator A and $f(\lambda_0, \lambda_1, \ldots, \lambda_k)$ is analytic in D^{k+1}, vanishing at infinity, so that the corresponding integrals converge, then

$$S_k(f, A)(B, \ldots, B) = \left(\frac{1}{2\pi i}\right)^{k+1} \int_{\gamma^{k+1}} f(\lambda_0, \lambda_1, \ldots, \lambda_k)(\lambda_0 - A)^{-1} B (\lambda_1 - A)^{-1} B \ldots B (\lambda_k - A)^{-1} d\lambda_0 \ldots d\lambda_k. \tag{2}$$

We shall consider formal power series with respect to operator B, with coefficients which are A-transformers (A-series)

$$F(A, B) = \sum_{k=0}^{\infty} S_k(f_k, A)(B, \ldots, B).$$

We shall introduce the disjoint differences of the function $f(\lambda)$:

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$$\Delta^{(k)}f(\lambda_0, \lambda_1, \ldots, \lambda_k) = \int_0^1\int_0^{t_1}\cdots\int_0^{t_{k-1}} f^{(k)}\Big(\sum_{j=0}^{k}\lambda_j(t_j - t_{j+1})\Big)\,dt_1\ldots dt_k, \qquad (t_0 = 1,\ t_{k+1} = 0)$$

and their corresponding A-transformers

$$f^{(k)}(A)(B,\ldots,B) = S_k(\Delta^{(k)}f, A)(B,\ldots,B) = \int_0^1\int_0^{t_1}\cdots\int_0^{t_{k-1}} f^{(k)}\Big(\sum_{j=0}^{k}\overset{2j}{A}(t_j - t_{j+1})\Big)\,dt_1\ldots dt_k\cdot\overset{1}{B}\cdot\overset{3}{B}\ldots\overset{2k-1}{B}. \tag{3}$$

The formal series

$$T(f)(A, B) = f(A) + \sum_{k=1}^{\infty} f^{(k)}(A)(B,\ldots,B) \tag{4}$$

will be caled the Taylor series for the operator f(A + B) (for functions of self-adjoint operators in terms of the m.o.i. (1) the derivatives $f^{(k)}(A)$ have been calculated and used in [3-6] to construct the Taylor series; a further development has been carried out in the above quoted book by V. P. Maslov [2], see also [11]). From the properties of the disjoint differences it is easy to derive that the mapping $f \mapsto T(f)(A, B)$ is a homomorphism of the algebra of smooth functions into the algebra of formal series. In the case when the corresponding series is convergent by definition one can put

$$f(A + B) = T(f)(A, B) \tag{5}$$

and thus obtain an extension of the operator functional calculus onto a wider class of operators $\widetilde{\mathfrak{A}} = \{A + B,\ A \in \mathfrak{A},\ B \in L(\mathfrak{B})\}$. The convergence of series can be studied with the help of estimations obtained in [2-10]. Below we shall consider only the cases in which the corresponding expansions are finite.

2. At first we shall consider a scalar triangular matrix

$$A = \|a_{jk}\|, \quad a_{jk} = 0\ (j > k), \quad a_{jj} \in [\alpha, \beta] \quad (j, k = 1, \ldots, n). \tag{6}$$

<u>THEOREM 1.</u> For a function $f \in C_{n-1}[\alpha, \beta]$ the matrix $f(A) = \|f_{jk}\|$ has the form

$$f_{jk} = \sum_{r=1}^{k-j}\ \sum_{j<j_1<\cdots<j_{r-1}<k} \Delta^{(r)}f(a_{jj}, a_{j_1j_1}, \ldots, a_{j_{r-1}j_{r-1}}, a_{kk})\, a_{jj_1}a_{j_1j_2}\ldots a_{j_{r-1}k},$$
$$f_{jj} = f(a_{jj}). \tag{7}$$

<u>Remark.</u> For $a_{jk} = a_{k-j}$ formula (7) leads to the relation

$$f_{jk} = \frac{1}{(k-j)!}\,\frac{d^{k-j}}{dt^{k-j}} f\Big(\sum_{\sigma=0}^{n-1} a_\sigma t^\sigma\Big)\Big|_{t=0} \qquad (j \leqslant k). \tag{8}$$

In particular, for the Jordan box $(a_1 = 1;\ a_k = 0,\ k > 1)$ we obtain the classical formula of linear algebra: $f_{jk} = \frac{1}{(k-j)!} f^{(k-j)}(a_0)$.

Let now $\mathfrak{B} = \mathfrak{B}_1 \dotplus \mathfrak{B}_2 \dotplus \ldots \dotplus \mathfrak{B}_n$ be the direct sum of subspaces and let the operator $A = \|a_{jk}\|$ be defined by the matrix linear operators

$$a_{jk} : \mathfrak{B}_k \to \mathfrak{B}_j,\ \|a_{jk}\| < \infty \quad (k > j), \quad a_{jj} \in \mathfrak{A}, \quad a_{jk} = 0 \quad (j > k). \tag{9}$$

<u>THEOREM 2.</u> Formula (7) remains valid for the function f(A) of operator (9) if understood from the point of view of the above mentioned calculus ordered operators, putting $a_{jk} < a_{r\ell}$ for $j \le r$, $k \le \ell$.

Remark. We shall represent operator matrix A in the form of the sum $A = \sum_{\sigma=0}^{n-1} a_\sigma$ of "one-diagonal" matrices $a_\sigma = \| a_{jk}\delta_{kj+\sigma} \|$. Then formula (8) holds, where the derivatives with respect to a parameter operator functions are calculated according to the formulas indicated in [2-6].

We remark that formula (7) gives explicit expressions for concrete elementary functions of triangular operators, in particular, for spectral projections (cf., [4; 5]).

3. Let us consider a continuous analog of the above relations. We shall consider in the space C(G) of continuous functions f(x) in a closed domain $G \subset \mathbf{R}^m$ an integral operator

$$A\varphi(x) = a(x)\varphi(x) + \int_G b(x, y)\varphi(y)\,dy. \tag{10}$$

Let function f be analytic in a convex domain containing the values of a(x) (x ∈ G). Then from (5) the formula

$$f(A)\varphi(x) = f(a(x))\varphi(x) + \int_G F(x, y)\varphi(y)\,dy,$$

follows, where

$$F(x, y) = \Delta^{(1)} f(a(x), a(y))\, b(x, y) + \sum_{r=2}^{\infty} \int_{G^{r-1}} \Delta^{(r)} f(a(x), a(x_1), \ldots, a(x_{r-1}), a(y))\, b(x, x_1)\, b(x_1, x_2) \ldots b(x_{r-1}, y)\, dx_1 \ldots dx_{r-1}. \tag{11}$$

The convergence of this expansion follows from the analytic properties of function f at least for small kernels b. The convergence improves in the triangular case when b(x, y) = 0 (x > y). We shall notice that formula (11) is preserved also for the integral operator (10) with respect to some functions under the above described condition of the "untangling" of noncommutative expressions.

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