DISCRIMINANTS OF POLYNOMIALS IN MANY VARIABLES

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Introduction. Let $f(x_1, \ldots, x_n) = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$ be a homogeneous polynomial of degree d. Its discriminant $\Delta(f)$ is an irreducible polynomial with integer coefficients in the coefficients $a_{i_1,\ldots,i_n}$; its being zero means that the projective hypersurface given by f = 0 has a singular point. The polynomial $\Delta(f)$ for n > 2 was first considered in 1841 by G. Boole, who found that its degree was $n(d-1)^{n-1}$. Apart from that, little is known about the discriminants $\Delta(f)$. In the present work we study from a geometric viewpoint the set of monomials entering $\Delta(f)$. Namely, we consider the space $\mathbf{R}^N$, where $N = \binom{n+d-1}{d}$, points of which are the collections $\eta = (\eta(i_1,\ldots,i_n))$ of real numbers, where $(i_1,\ldots,i_n)$ run through all sequences of nonnegative integers with sum equal to d. To each monomial $c_\eta \prod a_{i_1,\ldots,i_n}^{\eta(i_1,\ldots,i_n)}$, in $\Delta(f)$ corresponds a point in the space $\mathbf{R}^N$. Let us define the polyhedron $M \subset \mathbf{R}^N$ as the convex hull of all such points η (for which $c_\eta \neq 0$). We shall give a complete description of the vertices of the polyhedron M and of the coefficients c_η. It turns out that the vertices of M correspond to certain triangulations of the (n − 1)-dimensional simplex (the Newton polyhedron of a general polynomial of degree d). In particular, for n = 2 (i.e., for the classical discriminant of a polynomial of one variable) the polyhdron M is a "skewed" cube.

In fact, we solve in this paper a more general problem that contains in itself the problems concerning hyperdeterminants of cubic and multidimensional matrices and concerning the resultant of a system of polynomials in many variables. As regards the setting of the more general problem, see [2, 3] and Sec. 1. The present short paper contains no proofs. The proofs of the results stated here as well as in the notes [2, 3], will be published later.

1. Regular A-Determinants. Let A be a finite set of Laurent monomials in $x_1,\ldots,x_n$, which we identify with points of the lattice $\mathbf{Z}^n$. Let us consider the space $\mathbf{C}^A$ of Laurent polynomials that are linear combinations of monomials in A. The coefficients $(a_\omega)_{\omega\in A}$ of the polynomials are coordinates in $\mathbf{C}^A$. In [2, 3] we introduce three rational functions on $\mathbf{C}^A$, defined up to sign: the A-discriminant Δ_A, the principal A-determinant E_A and the regular A-determinant D_A.

The A-discriminant Δ_A is defined to be [2] a nonzero polynomial in $(a_\omega)_{\omega\in A}$ with the integer coefficients, irreducible in the ring $\mathbf{Z}[(a_\omega)]$ of such polynomials and having the following property: if $f \in \mathbf{C}^A$ is a nonzero Laurent polynomial for which there exist nonzero numbers $x_1^{(0)}, \ldots, x_n^{(0)} \in \mathbf{C}^*$ such that $f(x_1^{(0)}, \ldots, x_n^{(0)}) = (\partial f/\partial x_i)(x_1^{(0)}, \ldots, x_n^{(0)}) = 0$ for all i, then $\Delta_A(f) = 0$. In particular, it is not hard to see that if A consists of all homogeneous monomials in $x_1,\ldots,x_n$ of degree d (not containing negative powers), then $\Delta_A(f)$ is the discriminant of the form f of Introduction.

The polynomial E_A was studied in [3]. In this paper we consider the function D_A. Without giving its definition here (see [2]), we observe that in a number of important particular cases D_A coincides with Δ_A (namely, in the case of smoothness of the toroidal manifold connected with A). In Proposition 1 we have collected all the important properties of D_A, which are sufficient in order to understand the rest of the paper.

Let $Q \subset \mathbf{R}^n$ be the convex hull of the set $A \subset Z^n$. We say that the polyhedron Q is simple (or that it has simplicial corners) if every face $\Gamma \subset Q$ of codimension i is contained in exactly i faces of codimension 1 (here i = 0, 1,...,n).

Proposition 1. a) If the polyhedron Q is simple, then the rational function D_A is a polynomial in $(a_\omega)_{\omega\in A}$ with integer coefficients, which are all relative prime.

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b) D_A coincides with Δ_A in each of the following cases:

1) A consists of all monomials in $x_1,\ldots,x_n$ of degree d that do not contain negative powers [in the case D_A is $\Delta(f)$ of Introduction, while Q is a $(n-1)$-dimensional simplex].

2) A consists of polylinear monomials $x^{(1)}_{i_1}\ldots x^{(d)}_{i_d}$ in d groups of variables: $x^{(1)}_{i_1}$, $i_1\in[0, l_1]$, $\ldots$, $x^{(d)}_{i_d}$, $i_d\in[0, l_d]$ (in this case D_A is the hyperdeterminant of a d-dimensional matrix [4], while Q is a product of d simplices of dimensions $l_1,\ldots,l_d$).

If one so wishes, one can consider instead of D_A the discriminant Δ_A in one of the cases of part b).

In the general case the decomposition of D_A into irreducible factors, is described in [2]. Let us note that the condition of simplicity of the polyhedron, which ensures that D_A is a polynomial, was missed in [2] (in the general case D_A is not a polynomial).

We shall assume in the following that the polyhedron Q is simple, and we shall write the polynomial D_A in the form

$$D_A(f)=\sum_\eta c_\eta\prod_{\omega\in A}a_\omega^{\eta(\omega)}, \tag{1}$$

where $\eta\in\mathbf{Z}_+^A$ run through sum functions $A\to\mathbf{Z}_+$. Let us denote by $(A)\subset\mathbf{R}^A$ the convex hull of the set of $\eta\in\mathbf{Z}_+^A$, for which $c_\eta\neq 0$ (i.e., the Newton polyhedron of the polynomial D_A).

2. Description of the Polyhedron M(A). We shall consider triangulations of the polyhedron Q with vertices in A, that is, collections T of simplices in Q, such that all their vertices are in A, intersection of any two simplices is either empty or is a face common to them both, a face of any simplex in T is also included in T, while the union of all simplices in T coincides with Q. Moreover, we shall assume that the condition of regularity of triangulation [3, 5] is satisfied. We shall call a simplex σ (of arbitrary dimension $j\geqslant 0$) massive if it is contained in a j-dimensional face of the polyhedron Q. We shall denote this face by $\Gamma(\sigma)$.

If a lattice L is defined in a real affine space W, then we define a volume form on W, setting the volume of a standard simplex with vertices in L equal to 1. For every face $\Gamma\subset Q$ (in particular, for $\Gamma=Q$) we introduce a volume form Vol_Γ on Γ, starting with the affinely generated lattice over $\mathbf{Z}$ by the set $A\cap\Gamma$.

For each triangulation T of the polyhedron Q with vertices in A we define the integer valued function $\eta_T\colon A\to\mathbf{Z}$, by setting

$$\eta_T(\omega)=\sum_\sigma(-1)^{\dim Q-\dim\sigma}\mathrm{Vol}_{\Gamma(\sigma)}(\sigma),$$

where σ runs through all massive simplices of the triangulation T, having ω as a vertex. We call two triangulations T and T' D-equivalent if $\eta_T=\eta_{T'}$.

THEOREM 1. Let the set A be such that the polyhedron Q is simple. Then:

a) if T is a regular triangulation of Q with vertices in A, then for each $\omega\in A$ we have $\eta_T(\omega)\geqslant 0$;

b) the vertices of the polyhedron M(A) are exactly the functions η_T for all regular triangulations T (so that the vertices are in bijective correspondence with classes of D-equivalence of regular triangulations of Q with vertices in A);

c) for a regular triangulation T the coefficient c_{η_T} in Eq. (1) is

$$\delta_A(T)\prod_\sigma\mathrm{Vol}_{\Gamma(\sigma)}(\sigma)^{\mathrm{Vol}_{\Gamma(\sigma)}(\sigma)\cdot\nu(\sigma)},$$

where σ runs through all massive simplices of the triangulation T, $\nu(\sigma)=(-1)^{\dim Q-\dim\sigma}$, while $\delta_A(T)=\pm 1$.

Remarks. 1) The sign of $\delta_A(T)$ equals $\prod_\Gamma\varepsilon_{A\cap\Gamma}(T)$, where Γ runs through all the faces of Q; the signs of $\varepsilon_{A\cap\Gamma}$ are defined in [3].

2) For D-equivalent triangulations T and Σ we have $c_{\eta_T}=c_{\eta_\Sigma}$.

3. Combinatorial Description of D-Equivalence. We shall denote by $\langle I\rangle \subset Q$ the convex hull of a set $I \subset Q$. A subset $Z \subset A$ is called a cycle if all its proper subsets are sets of vertices of some simplex, while it itself does not have that property (compare with [3, 6]). In this case the polyhedron $\langle Z\rangle$ has exactly two triangulations with vertices in Z, which we denote by $T^{*}(Z)$; each simplex of the form $\langle Z - \{\omega\}\rangle$ is contained in exactly one of these (see [6]). We shall say that a triangulation T is supported by a cycle Z, if T induces on $\langle Z\rangle$ one of the triangulations $T^{*}(Z)$ and if the following condition holds:

> let $\langle I\rangle$ and $\langle I'\rangle$ be two simplices in the same triangulation $\langle Z\rangle$; then for any subset $J \subset A - Z$, for which $I \cup J$ is the set of vertices of a simplex in the triangulation T, $I' \cup J$ is also the set of vertices of a simplex in T.

Let a triangulation T be supported by a cycle Z and let it induce on $\langle Z\rangle$, say, a triangulation $T^{*}(Z)$. Let us denote by $s_Z(T)$ the new triangulation of the polyhedron Q obtained by deleting all the simplices of the form $\langle I \cup J\rangle$ with $\langle I\rangle \in T^{+}(Z)$ and adding instead simplices of the form $\langle I' \cup J\rangle$ with $\langle I'\rangle \in T^{-}(Z)$ and the same J (see [3]). We say that $T' = s_Z(T)$ is obtained by a rearrangement from T by a rearrangement along the cycle Z. We call a subset $J \subset A - A$ subseparating (respectively, separating) for T and T' if there exists $\omega \in Z$, such that $J \cup Z - \{\omega\}$ is the set of vertices of a simplex (respectively, maximal simplex) in T.

Proposition 2. Let T, Σ be two regular non D-equivalent triangulations of Q with vertices in A. The vertices η_T, η_Σ of the polyhedron M(A) are connected by an edge if and only if there exist regular triangulations T', Σ', which are D-equivalent to T, Σ, respectively, which are obtained one from another by a rearrangement.

Proposition 3. If regular triangulations T and T' are D-equivalent, then T' can be obtained from T by a chain of rearrangements in such a way, that at each step we have a D-equivalent regular triangulation.

For each subset $I \subset A$ we denote by Aff(I) the affine subspace over Q generated by I. Aff(I) is equipped with a lattice $\mathrm{Aff}_{\mathbf{Z}}(I)$, generated over $\mathbf{Z}$ by the set I. Let $U_1, U_2 \subset$ Aff(I) be affine subspaces, $\dim U_1 + \dim U_2 = \dim \mathrm{Aff}(I) - 1$. Let us denote by $\rho(\mathrm{Aff}(I), U_1, U_2)$ the volume [relative to the lattice $\mathrm{Aff}_{\mathbf{Z}}(I)$)] of a simplex $\sigma \subset \mathrm{Aff}(I)$, such that $\sigma \cap U_i$ is an elementary simplex with vertices in $\mathrm{Aff}_{\mathbf{Z}}(I) \cap U_i$ [clearly, $\rho(\mathrm{Aff}(I), U_1, U_2$ is independent of the choice of σ].

By Proposition 3, it suffices to describe the rearrangements that lead to D-equivalent triangulations. Let us state a useful sufficient condition of D-equivalence.

Proposition 4. Assume that the polyhedron Q is simple and that for each its face Γ the set $A \cap \Gamma$ affinely generates over $\mathbf{Z}$ the lattice $\mathrm{Aff}(A \cap \Gamma) \cap \mathbf{Z}^n$. Let T, T' be regular triangulations of Q with vertices in A, and $T' = s_Z(T)$ for some cycle $Z \subset A$. Triangulations T and T' are D-equivalent as long as every subseparating subset $J \subset A - Z$ satisfies either of the two following conditions:

a) there exists $\omega \in J$ such that $\rho(\mathrm{Aff}(J), \omega, \mathrm{Aff}(J - \{\omega\})) = 1$;

b) there exists $\omega \notin J$ such that every separating subset containing J necessarily contains ω, while all the simplices $\langle Z - \{z\} \cup J \cup \omega\rangle$, $z \in Z$, are massive.

We observe that the assumptions we have made are satisfied in the case of Proposition 1b.

4. Example. Let $A = \{0, 1, \ldots, d\} \subset \mathbf{Z}$, that is, the space $\mathbf{C}^A$ consists of polynomials $f(x) = a_0 + \ldots + a_d x^d$ of one variable. Then $D_A(f) = \Delta_A(f)$ is the classical discriminant of a polynomial in one variable [7]. There are 2^{d-1} regular triangulations of the interval [0, d] = Q. They are numbered by the sequences $0 = r_0 < r_1 < \ldots < r_{k-1} < r_k = d$; the corresponding triangulation consists of intervals $[r_i, r_{i+1}]$, $i = 0, 1, \ldots, k - 1$. A monomial in Δ_A, that corresponds by Theorem 1 to this triangulation, is

$$\left(\prod_{i=1}^{k} (-1)^{(r_i - r_{i-1})(r_i - r_{i-1} - 1)/2} (r_i - r_{i-1})^{(r_i - r_{i-1})}\right) a_{r_0}^{r_1 - 1} a_{r_1}^{r_2 - r_0} a_{r_2}^{r_3 - r_1} \ldots a_{r_{k-2}}^{r_{k-1} - r_{k-3}} a_{r_{k-1}}^{r_k - r_{k-2}} a_{r_k}^{r_k - r_{k-1} - 1}.$$

None of these triangulations are D-equivalent. It is not hard to see that the polyhedron M(A), which is the convex hull of these monomials is combinatorially equivalent to an (d − 1)-dimensional cube (through its opposite faces are not, in general, parallel).

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INTEGRAL GEOMETRY AND MANIFOLDS OF MINIMAL DEGREE IN $\mathbf{CP}^n$

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1. INTRODUCTION

1. An n-parameter family of submanifolds $B_\xi \subset B$, $\dim B = n$, is said to be admissible if the value of any smooth function f at each point x can be reconstructed, knowing only the integrals of f over the submanifolds of the family passing through an infinitesimal neighborhood of the point x. (A rigorous definition will be given in Sec. 2.)

The classical example is the family of all hyperplanes in $\mathbf{R}^{2n+1}$ or $\mathbf{C}^n$. Its admissibility follows from the locality of the inversion formula for the Radon transformation (cf. Sec. 2).

The goal of this paper is to construct a large class of admissible families of hypersurfaces. In Sec. 7 we prove that in this way one gets all admissible families of curves on algebraic surfaces up to birational isomorphism. Explicit local inversion formulas are obtained.

2. We recall that if X is a submanifold in $\mathbf{CP}^n$, which does not lie in a hyperplane (nondegenerate submanifold), then

$$\deg X \geqslant \operatorname{codim} X + 1, \tag{1}$$

where $\deg X$ is the number of points of intersection of X with a generic plane of complementary dimension. Indeed neither the degree nor the codimension changes under passage to a hyperplane section so that arguing by induction one can assume that $\dim X = 0$. In this case X is a collection of points not lying in any hyperplane.

In 1885 geometer Federigo Enriques discovered that all nondegenerate irreducible submanifolds for which equality holds in (1) can be simply and beautifully described ([12], cf. also Sec. 3).

Example 1.1. a) Let X_d be an irreducible nondegenerate curve of degree d in $\mathbf{CP}^d$. Then it is projectively equivalent to the Veronese curve $(x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d)$ (it is also called a rational normal curve [11, p. 196]).

b) Del Pezzo proved [11, p. 561] that any irreducible nondegenerate surface of degree $n - 1$ in $\mathbf{CP}^n$ is either a Veronese surface

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2) \tag{2}$$

in $\mathbf{CP}^5$, or a surface S_k constructed as follows:

We take two Veronese curves lying in crossing planes of dimensions k and $n - k - 1$ in $\mathbf{CP}^n$ and we establish an isomorphism between them. The surface S_k consists of lines joining

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