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SOME ALGEBRAIC STRUCTURES CONNECTED WITH THE YANG—BAXTER EQUATION

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One of the strongest methods of investigating the exactly solvable models of quantum and statistical physics is the quantum inverse problem method (QIPM; see the review papers [1-3]). The problem of enumerating the discrete quantum systems that can be solved by the QIPM reduces to the problem of enumerating the operator-valued functions L(u) that satisfy the relation

$$R(u-v)\,L'(u)\,L''(v)=L''(v)\,L'(u)\,R(u-v) \quad (1)$$

for a fixed solution R(u) of the so-called quantum Yang—Baxter equation

$$R_{12}(u-v)\,R_{13}(u)R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \quad (2)$$

Here we use the notation $L'=L\otimes 1$, $L''=1\otimes L$ (see [1, 3]). More detailed information concerning equations (1) and (2) and the notation used here can be found in the review papers to which we have already referred.

In the classical case, (1) is replaced by the equation

$$\{L'(u),L''(v)\}=[r(u-v),L'(u)\,L''(v)]_-, \quad (3)$$

while (2) becomes the classical Yang—Baxter equation

$$[r_{12}(u-v),\ r_{13}(u)]_-+[r_{12}(u-v),\ r_{23}(v)]_-+[r_{13}(u),\ r_{23}(v)]_-=0. \quad (4)$$

Here we use {,} to denote the Poisson bracket, and $[A, B]_- = AB - BA$ stands for the commutator of the matrices A and B. We shall also make use of the notation $[A, B]_+ = AB + BA$ for the anticommutator.

The problem of enumerating the solutions to equations (1) and (3) has received little attention. This contrasts with the intense study of both the quantum and classical Yang—Baxter equations, which has led to a number of successes. These have revealed, in particular, the deep relationship between the Yang—Baxter equation, the theory of Lie groups [4, 5], and algebraic geometry [6, 7]. However, important results were obtained in [8, 9], where solutions to (1) and (3) corresponding to lattice versions of the nonlinear Schrödinger and sine—Gordon equations were found.

The present paper is devoted to a study of equations (1) and (3) in the case when R(u) and r(u) are, respectively, the simplest solution to equation (1), found by R. Baxter [10], and its classical analog [11]. During our investigation it turned out that it is necessary to bring into the picture new algebraic structures, namely, the quadratic algebras of Poisson brackets and the quadratic generalization of the universal enveloping algebra of a Lie algebra. The theory of these mathematical objects is surprisingly reminiscent of the theory of

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Lie algebras, the difference being that it is more complicated. In our opinion, it deserves the greatest attention of mathematicians.

In the basic text we shall use the following convention on indices: Latin indices a, b, c run over the values 0, 1, 2, 3, while the Greek indices α, β, γ run over the values 1, 2, 3. In formulas (11)-(13), (22), (26), and (30)-(32), the triple of indices α, β, γ denotes a cyclic permutation of (1 2 3).

1. Classical Case

Let r(u) be the simplest solution to equation (4), obtained in [11]:

$$r(u)=\sum_{\alpha=1}^{3} w_\alpha(u)\,\sigma_\alpha\otimes\sigma_\alpha, \tag{5}$$

where σ_α are the standard Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

and the coefficients $w_\alpha(u)$ can be expressed in terms of the Jacobi elliptic functions

$$w_1(u)=\rho\frac{1}{sn\,(u,k)},\quad w_2(u)=\rho\cdot\frac{dn\,(u,k)}{sn\,(u,k)},\quad w_3(u)=\rho\frac{cn\,(u,k)}{sn\,(u,k)} \tag{6}$$

($\rho > 0$ and $k\in[0,\ 1]$ are fixed).

Notice that the coefficients w_α lie on a quadratic

$$w_\alpha^2-w_\beta^2=J_{\alpha\beta}, \tag{7}$$

which is uniformized by the parameter u. The constants $J_{\alpha\beta}$, which can be easily expressed through ρ and k using (6), satisfy the obvious equality

$$J_{12}+J_{23}+J_{31}=0. \tag{8}$$

In the sequel it will be convenient to represent $J_{\alpha\beta}$ in the form

$$J_{\alpha\beta}=J_\alpha-J_\beta, \tag{9}$$

where the constants J_α are determined modulo the transformation $J_\alpha \mapsto J_\alpha + c$.

We shall seek a solution L(u) to equation (3) in the form

$$L(u)=S_0+i\sum_{\alpha=1}^{3} w_\alpha(u)\,S_\alpha\sigma_\alpha, \tag{10}$$

where the S_a are, for the moment, unknown. Now substitute (5) and (10) into (3) and take advantage of the easily verifiable identities

$$w_\alpha(u-v)\,w_\gamma(v)-w_\beta(u-v)\,w_\gamma(u)=w_\alpha(u)\,w_\beta(v), \tag{11}$$

$$w_\alpha(u-v)\,w_\beta(u)\,w_\alpha(v)-w_\beta(u-v)\,w_\alpha(u)\,w_\beta(v)=-J_{\alpha\beta}w_\gamma(u). \tag{12}$$

Observe that, in fact, one needs to verify only (12), because (11) is equivalent to the Yang–Baxter equation (4). In the end, we obtain the following quadratic algebra of Poisson brackets for the variables S_a:

$$\{S_\alpha,S_0\}=2J_{\beta\gamma}S_\beta S_\gamma,\qquad \{S_\alpha,S_\beta\}=-2S_0S_\gamma. \tag{13}$$

Here is the place to give a number of general definitions concerning algebras of Poisson brackets.

Definition 1. A *Poisson brackets algebra* (PBA) is a set endowed with two structures: one of a commutative ring over the field **C** with multiplication f, g $\mapsto$ fg = gf, and one of a Lie algebra with binary operation (Poisson bracket) f, g $\mapsto$ {f, g} = −{g, f}, the two of them being related by the Leibniz relation {fg, h} = f{g, h} + g{f, h}.

Definition 2. The *center* Z of a PBA is its center as a Lie algebra relative to the operation {,}.

Definition 3. A PBA is said to be *nondegenerate* if its center is one-dimensional.

Definition 4. A *homogeneous Poisson brackets Lie algebra* (HPBA) is a polynomial algebra with the generators $\{x_j\}_1^N$ and with the Poisson bracket

$$\{x_j, x_k\} = C_{jk}(x), \tag{14}$$

where the $C_{jk}(x)$ are homogeneous polynomials of degree n in the generators x_j satisfying $C_{jk}(x) = -C_{kj}(x)$ and the Jacobi identity

$$\{\{x_j, x_k\}, x_l\} + \{\{x_k, x_l\}, x_j\} + \{\{x_l, x_j\}, x_k\} = 0. \tag{15}$$

In particular, a quadratic HPBA is described by the relations

$$\{x_j, x_k\} = \sum_{l, m=1}^{N} c_{jk}^{lm} x_l x_m, \tag{16}$$

where the tensor of the structure constants c_{jk}^{lm} must have the symmetries

$$c_{jk}^{lm} = -c_{kj}^{lm} = c_{jk}^{ml} = -c_{kj}^{ml} \tag{17}$$

and satisfy the system of quadratic equations

$$\sum_{m=1}^{N} c_{jk}^{mn} c_{ml}^{pq} + c_{kl}^{mn} c_{mj}^{pq} + c_{lj}^{mn} c_{mk}^{pq} = 0, \tag{18}$$

which ensures that the Jacobi identity (15) is fulfilled.

Let us give several examples of HPBAs.

1. When n = 0, any HPBA reduces to a linear symplectic structure [12]

$$\{x_j, x_k\} = c_{jk}, \quad c_{jk} = -c_{kj}. \tag{19}$$

2. A linear HPBA is given by relations

$$\{x_j, x_k\} = \sum_{l=1}^{N} c_{jk}^{l} x_l, \tag{20}$$

where c_{jk}^{l} is the tensor of the structure constants of some Lie algebra (the Berezin–Kirillov–Kostant symplectic structure [12]).

Let us return to the Poisson brackets (13).

Proposition 1. Relations (13) define a quadratic HPBA (which we shall denote by $\mathcal{P}$).

To carry out the proof, it suffices to verify that the identities (17) and (18) are satisfied. A straightforward computation shows that (17) always holds, while (18) is valid provided (8) is satisfied, and this is true because of the definition (7) of the $J_{\alpha\beta}$. On the other hand, one can show that the Jacobi identity for (13) is a consequence of the classical Yang–Baxter equation (4). Indeed, take $L'(u_1) = L(u_1) \otimes 1 \otimes 1$, $L''(u_2) = 1 \otimes L(u_2) \otimes 1$, and $L'''(u_3) = 1 \otimes 1 \otimes L(u_3)$, and consider the expression $\{\{L'(u_1), L''(u_2)\}, L'''(u_3)\}$. Applying (3) and (4), we obtain the Jacobi identity for the operators $L'(u_1)$, $L''(u_2)$, and $L'''(u_3)$. Now, taking the residues at the poles of the coefficients $w_\alpha(u)$, we are led to the required Jacobi identity for the variables S_α.

Let $\mathcal{P}^*$ be the extension of algebra $\mathcal{P}$ consisting of analytic functions in the arguments S_α.

THEOREM 1. The center $Z_{\mathcal{P}^*}$ of algebra $\mathcal{P}^*$ consists of functions of the two quadratic central functions K_0 and K_1:

$$K_0 = \sum_{\alpha=1}^{3} S_\alpha^2, \quad K_1 = S_0^2 + \sum_{\alpha=1}^{3} J_\alpha S_\alpha^2 \tag{21}$$

(notice that under a translation $J_\alpha \mapsto J_\alpha + c$ of all J_α's, K_1 changes to $K_1 + cK_0$).

Proof. Let $f \in Z_{\mathscr{P}^*}$. This means that $\{f, S_\alpha\} = 0$, or, by virtue of (13), that

$$\{f, S_0\} = \sum_{\alpha=1}^{3} \frac{\partial f}{\partial S_\alpha} J_{\beta\gamma} S_\beta S_\gamma = 0,$$
$$\{f, S_\alpha\} = -\frac{\partial f}{\partial S_0} J_{\beta\gamma} S_\beta S_\gamma + \frac{\partial f}{\partial S_\beta} S_0 S_\gamma - \frac{\partial f}{\partial S_\gamma} S_0 S_\beta = 0. \tag{22}$$

The general solution of the system of linear homogeneous equations (22) is of the form $df = f_0 dK_0 + f_1 dK_1$ (provided (8) holds), which immediately proves the theorem.

Problem. Show that the center $Z_{\mathscr{P}}$ of the algebra $\mathscr{P}$ consists of polynomials in K_0 and K_1.

COROLLARY. Let $\Gamma(K_0, K_1)$ be the two-dimensional algebraic variety defined by equations (21) for fixed K_0 and K_1. Then the PBA $\mathscr{P}_\Gamma$, defined as the restriction of algebra $\mathscr{P}$ to Γ, is nondegenerate in the sense of Definition 2.

The topology of Γ depends, in an essential way, upon the choice of the parameters K_0 and K_1. Notice that for the parametrization (6) that we selected, one has $J_1 \geqslant J_2 \geqslant J_3$, whence S_0^2 lies on the segment $[K_1 - K_0 J_1, K_1 - K_0 J_3]$.

a) $K_1 \in (K_0 J_1, \infty)$, b) $K_1 \in (K_0 J_3, K_0 J_1)$, в) $K_1 \in (-\infty, K_0 J_3)$.

In cases a) and b), respectively, the variety Γ is homeomorphic to the sphere S^2 and to the disjoint union of two spheres (which merge into a torus for $J_2 = J_3$). In case c) there are no real solutions.

To conclude this section, let us describe the completely integrable dynamical system related to the L operator (10). As the phase space, we take the product of N copies of the variety $\Gamma(K_0, K_1)$. In other words, we consider a ring with N nodes, and associate to each node a quartet of dynamical variables $S_\alpha^{(n)}$ that satisfy constraints (21). Let the Poisson brackets between the variables $S_\alpha^{(n)}$ corresponding to a fixed node be given by relation (13), and let the Poisson brackets between variables corresponding to distinct nodes be equal to zero. We introduce the L operator in the n-th node by the formula $L_n(u) \equiv L(u, S_\alpha^{(n)})$, where $L(u, S_\alpha)$ is given by (10). Construct the monodromy matrix [1, 2] $T(u) = L_N(u) L_{N-1}(u) \ldots L_1(u)$. By virtue of (3), the traces $t(u) \equiv \operatorname{tr} T(u)$ of the monodromy matrix are in involution: $\{t(u), t(v)\} = 0$, and this allows us to take $\ln t(u)$ as a generating function for the integrals of motion. In a manner analogous to [9], the local integrals of motion $H^{(j)}$ are derived from the expansion of $\ln |t(u)|^2$ in powers of $(u - u_0)$,

$$H^{(j)} = \frac{\partial^j}{\partial u^j} \ln |t(u)|^2 \bigg|_{u=u_0}$$

at the point u_0 determined from the nondegeneracy condition for the L operator

$$\det L(u_0) \equiv K_1 + K_0 (w_1^2(u_0) - J_1) = 0.$$

In particular, the simplest two-point Hamiltonian $H^{(0)}$ has the form

$$H^{(0)} = \sum_n H_{n+1,n},$$
$$H_{n+1,n} = \ln \operatorname{tr} L_{n+1}(u_0) L_n(u_0) = \ln \Big(S_0^{(n+1)} S_0^{(n)} + \sum_{\alpha=1}^{3} \Big(\frac{K_1}{K_0} - J_\alpha \Big) S_\alpha^{(n+1)} S_\alpha^{(n)} \Big). \tag{23}$$

The Hamiltonian (23) is nothing but the discrete variant of the Landau–Lifshits Hamiltonian from ferromagnetism theory [11]. As in the continuous case, the discrete nonlinear Schrödinger equation [8, 9] and the sine–Gordon equation [2, 8] are degenerate cases of (23). A more detailed investigation of the model described by the Hamiltonian (23) in the framework of the inverse problem method will be published separately.

2. Quantum Case

Let R(u) be the solution to Eq. (2) discovered by R. Baxter [10]:

$$R(u) = 1 + \sum_{\alpha=1}^{3} W_\alpha(u)\,\sigma_\alpha \otimes \sigma_\alpha, \tag{24}$$

Here the coefficients $W_\alpha(u)$

$$W_1(u) = \frac{sn\,(i\eta, k)}{sn\,(u+i\eta, k)}, \quad W_2(u) = \frac{dn}{sn}(u+i\eta, k)\frac{sn}{dn}(i\eta, k),$$
$$W_3(u) = \frac{cn}{sn}(u+i\eta, k)\frac{sn}{cn}(i\eta, k) \tag{25}$$

lie on the algebraic curve

$$\frac{W_\alpha^2 - W_\beta^2}{W_\gamma^2 - 1} = \mathbf{J}_{\alpha\beta}. \tag{26}$$

We shall assume that the constants η and k are real. Then it is easily seen that the constants $J_{\alpha\beta}$ are real, too. Notice that among the equations (26) only two are independent, because the $J_{\alpha\beta}$ are related via

$$\mathbf{J}_{12} + \mathbf{J}_{23} + \mathbf{J}_{31} + \mathbf{J}_{12}\mathbf{J}_{23}\mathbf{J}_{31} = 0, \tag{27}$$

This represents the quantum analogue of the classical relation (8). As in the classical case, it is convenient to express $J_{\alpha\beta}$ in the form

$$\mathbf{J}_{\alpha\beta} = -\frac{\mathbf{J}_\alpha - \mathbf{J}_\beta}{\mathbf{J}_\gamma}, \tag{28}$$

where the constants J_α are uniquely defined modulo the transformation $J_\alpha \mapsto cJ_\alpha$.

In full analogy with the classical case, we shall seek the solution L(u) to Eq. (1) in the form

$$\mathbf{L}(u) = \mathrm{S}_0 + \sum_{\alpha=1}^{3} W_\alpha(u)\,\mathrm{S}_\alpha, \tag{29}$$

where the $\mathbf{S}_\alpha$ are temporarily unknown quantities. Insert (24) and (29) in (1), and apply the identities

$$W_\beta(u-v)\,W_\gamma(u) - W_\alpha(u-v)\,W_\gamma(v) + W_\alpha(u)\,W_\beta(v) - W_\gamma(u-v)\,W_\beta(u)\,W_\alpha(v) = 0, \tag{30}$$

$$\frac{W_\beta(u-v)\,W_\gamma(u)\,W_\beta(v) - W_\gamma(u-v)\,W_\beta(u)\,W_\gamma(v)}{W_\alpha(u) - W_\alpha(u-v)\,W_\alpha(v)} = \mathbf{J}^{\beta\gamma}. \tag{31}$$

Recall that in (30) and (31), as well as in the sequel, we use the convention on the indices $\alpha\beta\gamma$ mentioned in the introduction. As in the classical case, identity (30) is equivalent to the Yang—Baxter equation.

As a result, we obtain the following commutation relations for the variables S_α:

$$[\mathrm{S}_\alpha, \mathrm{S}_0]_- = -i\mathbf{J}_{\beta\gamma}[\mathrm{S}_\beta, \mathrm{S}_\gamma]_+,$$
$$[\mathrm{S}_\alpha, \mathrm{S}_\beta]_- = i[\mathrm{S}_0, \mathrm{S}_\gamma]_+. \tag{32}$$

Relations (32) generate a two-sided ideal I in a standard manner in the free associative algebra $\mathcal{A}$ with the four generators $\mathbf{S}_\alpha$. The quotient algebra $\mathcal{F} = \mathcal{A}/I$ corresponding to the ideal I is the basic object of investigation in the present section.

First, let us clarify the relationship between the algebra $\mathcal{F}$ and the quadratic Poisson brackets algebra $\mathcal{P}$ described in the previous section. To this end, we introduce Planck's constant h and apply the well-known principle of correspondence [12] between classical and quantum mechanics. According to the latter, the commutator of two observables becomes the Poisson bracket when $h \to 0$:

$$[,]_- \sim -ih\,\{,\}. \tag{33}$$

Setting $\eta = \rho h$ in (25) and passing to the limit $h \to 0$, we obtain the following relations:

$$W_\alpha(u) = ihw_\alpha(u) + O(h^2), \quad R(u) = 1 + ihr(u) + O(h^2),$$
$$\mathbf{J}_{\alpha\beta} = h^2 J_{\alpha\beta} + O(h^4), \quad \mathbf{J}_\alpha = 1 - h^2 J_\alpha + O(h^4). \tag{34}$$

Now, using expansion (34) and assuming that the quantum quantities $\mathbf{S}_\alpha$ become the corresponding classical quantities S_α when $h \to 0$, according to the rule $\mathbf{S}_0 \sim hS_0$, $\mathbf{S}_\alpha \sim S_\alpha$, it is not hard to convince ourselves that the quantum equalities (1), (2), and (32) are transformed into the classical ones (3), (4), and (13), respectively. At the same time one has the relation $\mathbf{L}(u) \sim hL(u)$. Thus, the quadratic PBA $\mathcal{P}$ is the classical limit of the algebra $\mathcal{F}$.

It is instructive to compare the quantization of the quadratic HPBA $\mathcal{P}$ with that of an HPBA of degree $n < 2$. For $n = 0$ and $n = 1$, the classical relations (19) and (20) become, when quantized, the relations: $[x_j, x_k]_- = -ihc_{jk}$ and $[x_j, x_k]_- = -ih \sum_{l=1}^{N} c_{jk}^l x_l$, respectively (see [12]); i.e., the tensor of structure constants of the HPBA is simply multiplied by $(-ih)$. This allows us to obtain strong results concerning the relation between a given classical HPBA and the corresponding quantum algebra [13].

When $n = 2$ the picture is significantly more intricate, because the simple correspondence rule $c \mapsto -ihc$ is no longer available. Indeed, Planck's constant enters in an essentially nonhomogeneous way in the quantum structure constants $\mathbf{J}_{\alpha\beta}$. This can be seen at least from the relation (27), which becomes the classical relation (8) only at the limit $h \to 0$. This circumstance, as well as the absence (in the generic case) of a continuous symmetry, strongly encumbers the investigation of algebra $\mathcal{F}$.

Since for $\mathbf{J}_{\alpha\beta} = 0$ the algebra $\mathcal{F}$ degenerates into the trivial deformation of the universal Lie enveloping algebra $\mathfrak{U}(so(3))$, it is natural to expect a certain similarity between the properties of the algebras $\mathcal{F}$ and $\mathfrak{U}(so(3))$. In particular, it would be desirable to obtain for $\mathcal{F}$ some sort of analogue of the well-known Poincaré–Birkhoff–Witt theorem [12] for universal enveloping algebras.

Problem. Prove (or disprove) that the dimension of the subspace of homogeneous polynomials of degree p in the variables $\mathbf{S}_\alpha$ in the algebra $\mathcal{F}$ is $(p + 3)(p + 2)(p + 1)/6$, i.e., coincides with the dimension of the space of polynomials of degree p in four commuting variables $\mathbf{S}_\alpha$.

The author succeeded in proving this statement for $p = 3$ by direct calculations. In so doing, it turned out, as expected, that in order to count the linearly independent monomials one must make essential use of relation (27), which plays, for the algebra $\mathcal{F}$, the same role as the Jacobi identity plays for the structure constants of a Lie algebra.

The next important problem is that of analyzing the structure of the center $\mathbf{Z}_\mathcal{F}$ of the algebra $\mathcal{F}$.

THEOREM 2. The quadratic operators $\mathbf{K}_0$ and $\mathbf{K}_1$

$$\mathbf{K}_0 = \sum_{\alpha=0}^{3} \mathbf{S}_\alpha^2, \quad \mathbf{K}_1 = \mathbf{S}_0^2 + \sum_{\alpha=1}^{3} (1 - \mathbf{J}_\alpha)\mathbf{S}_\alpha^2 \tag{35}$$

belong to $Z_\mathcal{F}$.

The fact that $\mathbf{K}_0$ and $\mathbf{K}_1$ commute with $\mathbf{S}_\alpha$ is straightforward, choosing some basis in the subspace of cubic polynomials in the variables $\mathbf{S}_\alpha$. It also follows from the following lemma, given in [1]:

LEMMA. The quantum determinant D(u), defined by the formula

$$D(u) \equiv \operatorname{tr} P\_\mathbf{L}(u) \otimes \mathbf{L}(u - 2i\eta) = \mathbf{S}_0^2 - \sum_{\alpha=1}^{3} W_\alpha(u) W_\alpha(u - 2i\eta)\mathbf{S}_\alpha^2, \tag{36}$$

where $P_- = (1 - \sigma_\alpha \otimes \sigma_\alpha)/4$ is the antisymmetrization operator, belongs to $Z_\mathcal{F}$.

The operators $\mathbf{K}_0$ and $\mathbf{K}_1$ are derived from D(u) as the coefficients of its expansion into linearly independent elliptic functions of argument u.

In the limit $h \to 0$, $\mathbf{K}_0$ and $\mathbf{K}_1$ become, respectively, K_0 and K_1 in (21). Notice that here, in contrast to the case of Lie algebras [13], the centers of the classical and quantum algebras do not coincide.

Problem. Prove (or disprove) that the center of the algebra $\mathcal{F}$ consists of polynomials $\mathbf{K}_0$ and $\mathbf{K}_1$.

In conclusion, let us approach the problem of finding the representations of the algebra $\mathcal{F}$. By a representation φ of $\mathcal{F}$ in a linear space V we mean any homomorphism $\varphi: \mathcal{F} \to \mathrm{End}\, V$, as is natural. A linear representation will be called self-adjoint if V is endowed with a φ Hermitian bilinear form relative to which the operators $\varphi(S_a)$ are self-adjoint. The reducible and decomposable representations are defined in the standard way [12]. Obviously, every reducible self-adjoint representation of $\mathcal{F}$ is decomposable.

The following are examples of irreducible finite-dimensional self-adjoint representations of algebra $\mathcal{F}$:

1. The two-dimensional representation by Pauli matrices:

$$S_0 = 1,\ S_\alpha = \sigma_\alpha.$$

2. The three-dimensional representation

$$S_0 = \begin{pmatrix} J_3 & 0 & J_1 - J_2 \\ 0 & J_1 + J_2 - J_3 & 0 \\ J_1 - J_2 & 0 & J_3 \end{pmatrix}, \quad S_1 = \sqrt{2J_2J_3}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$S_2 = \sqrt{2J_3J_1}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = 2\sqrt{J_1J_2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Notice that this representation is self-adjoint only when $J_\alpha > 0$.

Presently, there are no other examples of representations of the algebra $\mathcal{F}$ in the generic case. It is highly probable that one could succeed in finding new representations of $\mathcal{F}$ using the construction of "multiplication" of R matrices proposed in [1, 4]. Some series of representations of algebra $\mathcal{F}$ for the degenerate cases $k = 0$ and $k = 1$ are given in [2, 8, 9].

3. Possible Applications and Generalizations

The main field to which the algebraic structures described in the present work find themselves applicable is the theory of classical and quantum completely integrable systems. Relations (1) and (3) provide a systematic procedure for obtaining completely integrable lattice approximations to various continuous completely integrable systems.

In the derivation of relations (13) and (32), a great role was played by the successful choice of the substitutions (10) and (29) for the L operator. The problem of enumerating all L operators (possibly with a more complicated functional dependence upon the spectral parameter for a given R matrix is still open. A possible approach to the solution of this problem was suggested in [14].

One of the possible directions in which the examples given in the present paper could be further generalized is to search for quadratic PBAs and the corresponding quantum algebras for other known examples of solutions to the Yang—Baxter equations (2) and (4).

In addition, one can consider other real forms of the same algebra $\mathcal{F}$ that we studied here, and look for their (possibly infinite-dimensional) representations.

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STRUCTURE OF TRANSVERSAL LEAVES IN MULTIDIMENSIONAL SEMIDISPERSING BILLIARDS

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1. Introduction

An important class of dynamical systems is made up of the systems of billiard type (billiards), which are generated by the motion of a particle in a d-dimensional domain of Euclidean space or a d-dimensional Euclidean torus T^d, out of which an open subset U is excised, while the boundary of the domain is piecewise smooth. We shall denote this domain by Q. A billiard particle moves rectilinearly with unit speed inside Q and is reflected from the boundary ∂Q in accord with the law "angle of incidence equals angle of reflection."

The so-called dispersing billiards, i.e., those such that the smooth components of the boundary ∂Q are strictly convex inside the domain Q, have the strongest ergodic properties. By virtue of this property of the boundary, close trajectories of the system diverge with exponential speed, as also in the case of smooth hyperbolic systems. For such billiard systems, the properties of ergodicity and mixing, and the K property have been proved (cf. [1, 2, 10]); also, in a series of cases the rate of decrease of correlations has been investigated with the help of the method of Markov partitions (cf. [11, 12]).

In the present paper we study billiard systems for which the boundary ∂Q is not strictly convex from within, i.e., for which subspaces of flat directions with zero curvature are possible. Such billiards are called semidispersing. A typical example is the model of a gas of rigid spheres. For this system the configuration space is a dN-dimensional cube V^N (dN-dimensional torus T^{dN}), from which cylinders in R^{dN} are excised (cf. [5, 6]). Each cylinder is the direct product of a (d — 1)-dimensional sphere and a Euclidean space $R^{d(N-2)}$. One can get other examples by considering a system of rigid spheres, split up into several noninteracting groups, and also systems of rigid spheres on a torus.

The ergodic properties of semidispersing billiards can depend on properties of the boundary; more precisely, the ergodic properties are determined by the nonparallelness of the flat directions of the boundary of the billiard at its various points. For example, the system of

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