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ESTIMATE OF THE NUMBER OF ZEROS OF AN ABELIAN INTEGRAL DEPENDING ON A PARAMETER AND LIMIT CYCLES

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1. Introduction

We consider in the plane a polynomial function and a polynomial differential 1-form ω. Let us assume that with each number t from a certain interval there is associated a nonsingular compact connected component $\delta(t)$ ("oval") of the t level line of the polynomial, which depends continuously on t. We consider the function on the interval defined by the formula $t \to \int_{\delta(t)} \omega$. It is proved in this paper that the number of isolated zeros of this function and the multiplicities of the zeros can be estimated above in terms of the degree of the polynomial and the differential form.

We consider a Hamiltonian system on the plane with polynomial-Hamilton and its polynomial nonconservative deformation depending analytically on a parameter. As a consequence of the estimate of the number and multiplicity of the zeros we get that the number of limit cycles of the deformation which arise from nonsingular ovals of level lines of the Hamiltonian can be estimated above in terms of the degree of the Hamiltonian and the deformation.

The problem of estimating the number of zeros of an Abelian integral which depends on a parameter in connection with estimating the number of limit cycles was repeatedly posed by V. I. Arnol'd starting in 1976, cf., e.g., [1, 2]. The author thanks V. I. Arnol'd, R. I. Bogdanov, A. M. Gabriélov, Yu. S. Il'yashenko, and A. G. Khovanskii for many helpful discussions. The results of the paper were announced at the International Congress of Mathematicians in Warsaw.

1. Formulation of Results. We define the integral whose zeros we shall investigate. Let $P(x_1,\dots,x_n)$ be a polynomial,

$$\omega = \sum_{j=1}^{n} P_j(x_1, \dots, x_n)\, dx_1 \wedge \dots \widehat{dx_j} \dots \wedge dx_n \qquad (1)$$

be a polynomial differential (n − 1)-form with real coefficients. We consider them as a function and a differential form on $\mathbf{C}^n$. Let us assume that some interval $(a, b) \subset \mathbf{R}$ has the following property: The function P is bounded on the preimage of some neighborhood in $\mathbf{C}$ of the interval (a, b), and is a topological (trivial) fibration. We consider the associated fibration over (a, b) with fiber $H_{n-1}(P^{-1}(t), \mathbf{R})$. All its fibers are canonically isomorphic. In each fiber there is defined the automorphism of complex conjugation, induced by the map $(x_1,\dots,x_n) \mapsto (\bar{x}_1,\dots,\bar{x}_n)$. This automorphism commutes with the canonical isomorphism. Let us assume that there is given a section $\delta\colon t \mapsto \delta(t) \in H_{n-1}(P^{-1}(t), \mathbf{R})$ of the homology bundle,

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where all the values of the section are canonically isomorphic. In addition, let us assume that all values of the section are invariant or all values are antiinvariant with respect to the automorphism of complex conjugation. The object of investigation is the function I:$(a, b) \to \mathbf{C}$, defined by the formula $I(t)=\int_{\delta(t)} \omega$. We note that the integral is well-defined (since the form restricted to a hypersurface is bounded). If the values of the section are invariant, then the values of the function are real, and if the values of the section are antiinvariant, then the values of the function are purely imaginary. The integral mentioned earlier of a 1-form over an oval is an example of such a function.

It is well known (cf., e.g., [3-5]) that this function is analytic at interior points of the interval (a, b), and at its boundary points (finite or infinite) it splits into a series of the form

$$\sum_{\alpha} \sum_{k=0}^{k(n)} \sum_{p=0}^{\infty} a_{k,\alpha,p} (\ln t)^k t^{\alpha+p}, \tag{2}$$

where t is a local parameter at the boundary point, k(n) is a natural number which depends only on n, α runs through a finite set of rational numbers.

BASIC THEOREM. For any natural number N there exist a natural number C(N) and a finite set S(N) of rational numbers with the following properties: if the degrees of the polynomials $P, P_1, \ldots, P_n$ are not gerater than N, then

1. The number of isolated zeros of the function I on the interval (a, b) does not exceed C(N).
2. The multiplicity of each isolated zero is not greater than C(N).
3. If the function I is not identically zero, then in the expansion (2) of the function I at a boundary point of the interval (a, b), the minimal exponent $\alpha + p$ among the nonzero summands is between $-C(N)$ and C(N). Each α belongs to S(N) up to an integral summand.

The theorem is proved in Secs. 2, 3.

Remarks. 1) Point 1 is the basic point of the theorem. Its proof makes essential use of the ideas of the proof of Khovanskii's theorem on small terms [6, 7] and is based on a theorem of Gabriélov [8] on the boundedness above of the number of connected components of a semianalytic set depending on a parameter. To prove points 2 and 3 one uses the following argument. A solution of an ordinary linear homogeneous differential equation cannot have an isolated zero of order higher than the order of the equation. One proves that the functions under investigation are solutions of differential equations whose orders are bounded above.

Throughout the entire proof we do not consider integral-functions separately but all integral-functions together, and the coefficients of the forms and polynomials are considered parameters. 2) The basic theorem is the transcendental analog of Bezout's theorem, cf. Secs. 3 and 4.

2. Limit Cycles.

We consider a Hamiltonian system on the plane with polynomial Hamiltonian H and a deformation of it which is analytic with respect to the parameter and polynomial in the phase variables

$$\begin{cases} \dot{x} = -H_y(x, y) + \varepsilon P(x, y, \varepsilon); \\ \dot{y} = \quad H_x(x, y) + \varepsilon Q(x, y, \varepsilon). \end{cases}$$

Here H, P, Q are polynomials in x, y, the coefficients of the polynomials P, Q depend analytically on the parameter $\varepsilon \in (-1, 1)$.

By a family of ovals of a Hamiltonian, parametrized by the interval (a, b), we mean a map δ of the interval which assigns to the number t an oval (i.e., a nonsingular compact connected component) of the t-level line of the Hamiltonian, which depends continuously on t. Each oval can be included in a family. For a polynomial-Hamiltonian there are a finite number of families, containing all ovals of all level lines of the Hamiltonian. Moreover, the number of families can be estimated above in terms of the degree of the Hamiltonian, cf. [9].

A deformation of a Hamiltonian system is said to be nonconservative along the family of ovals δ, parametrized by the interval (a, b), if the function $I:(a, b) \to \mathbf{R}$, defined by $I(t) = \int_{\delta(t)} P\,dy - Q\,dx$, is not identically equal to zero.

An oval $\delta(t)$ of a family is called a generating limit cycle, if in any neighborhood of it, for arbitrarily small ε one can find limit cycles of the deformation corresponding to ε.

THEOREM ON LIMIT CYCLES. For any natural number N there exists a natural number C(N) with the following property: if the degrees of the polynomials H, P, and Q are not greater than N and any family of ovals of the Hamiltonian parametrized by an interval is chosen, along which the deformation is nonconservative, then the number of generating ovals of the family does not exceed C(N), and each generating oval gives rise to no more than C(N) limit cycles of the deformation, corresponding to small values of the parameter ε.

As far as I know this is the first general result on the finiteness of the set of limit cycles. Cf. [1, 10-16, 24] for information on limit cycles.

Proof. Let $\delta(t)$ be an arbitrary oval of the family. We consider, in a transversal to $\delta(t)$, the Poincare map $(z, \varepsilon) \to (\Phi(z, \varepsilon), \varepsilon)$, where z is a local parameter on the transversal, ε is the parameter of the deformation, cf. [10, 24]. As parameter on the transversal we choose the Hamiltonian decreased by t, i.e., we set $z = H - t$. According to Theorem 77 of [10] and [24], $\Phi(z, \varepsilon) = z + \varepsilon I(z + t) + \varepsilon^2\psi(z, \varepsilon)$, where the function I is defined above, the function ψ is analytic in a neighborhood of the point $z = \varepsilon = 0$. The limit cycles of the deformation given rise to by $\delta(t)$ are small solutions of the equation $z = \Phi(z, \varepsilon)$ for fixed ε. Hence if $\delta(t)$ is a generating oval, then $I(t) = 0$; if $I(z + t) = az^k + o(z^k)$, where $a \neq 0$, then $\delta(t)$ gives rise to no more than k limit cycles. Now the theorem follows from points 1 and 2 of the basic theorem.

Remark. This argument does not estimate the number of limit cycles which arise from singular ovals which correspond to boundary points of the interval parametrizing the family.

2. Integrals Depending on Parameters

In this section we recount the elementary properties of integrals of a rational differential form over cycles lying on algebraic varieties, depending algebraically on parameters, and the basic theorem is proved in Secs. 2 and 3.

1. Properties of Integrals. We consider complex quasiprojective nonsingular algebraic sets X, Λ, and a rational map π of the set X onto the set Λ, which is regular on X. Let us assume that the map is a topological, locally trivial fibration. We denote by $X(\lambda)$ the fiber over $\lambda \in \Lambda$. For a natural number r, we consider over the base Λ the associated bundle π_* with fiber $H_r(X(\lambda), \mathbf{C})$ and the associated bundle π^* with fiber $H^r(X(\lambda), \mathbf{C})$. These bundles are naturally dual, have canonical holomorphic structures and holomorphic dual Gauss–Manin connections, cf. [3].

We denote by Ω^r the space of rational regular differential r-forms on X. We consider a form $\omega \in \Omega^r$, which is closed on any fiber of the bundle π (i.e., is relatively closed). The form defines a holomorphic section of the cohomology bundle π^*: $s[\omega]: \lambda \mapsto [\omega|_{X(\lambda)}] \in H^r(X(\lambda), \mathbf{C})$. This section is called the geometric section of the form.

Let Λ' be the closure of the set Λ in the corresponding projective space. We shall describe the behavior of geometric sections as the discriminant $\Sigma = \Lambda' \setminus \Lambda$ is approached.

Let $D = \{t \in C \mid |t| < 1\}$ be the disk, $D^\cdot = D \setminus \{0\}$ the punctured disk, $\gamma: D \to \Lambda'$ be a holomorphic map carrying the punctured disk into Λ. For any relatively closed form $\omega \in \Omega^r$ and any covariantly constant section δ of the homology bundle π_*, we consider the multivalued holomorphic function $I: D^\cdot \to \mathbf{C}$, induced by the map γ from the function $\langle s[\omega], \delta\rangle = \int_\delta \omega$, defined on Λ (here $<\ \ >$ is the natural pairing).

THEOREM (cf. [3, 17]). There exists a positive number N with the following property: in any sector $a < \arg t < b$ for any branch of the function I one has the estimate $I = o(|t|^{-N})$ as $t \to 0$.

In the standard way one can derive the corollary formulated below from the theorem (cf., e.g., Sec. 11 of [26]). Let $D^m = \{(t_1, \ldots, t_m) \in \mathbf{C}^m \mid |t_j| < 1,\ \ j = 1, \ldots, m\}$ be a polydisk, $\gamma: D^m \to \Lambda'$ be a holomorphic mapping for which $\gamma((D^\cdot)^m) \subset \Lambda$. On $(D^\cdot)^m$ we consider the multivalued holomorphic function I, induced by the map γ from the function $<s[\omega], \delta>$.

COROLLARY. In any polysector $a_j < \arg t_j < b_j$, $j = 1,\ldots,m$, any branch of the function I can be expanded in a series

$$I = \sum_{k,\alpha} a_{k,\alpha} t_1^{\alpha_1} (\ln t_1)^{k_1} \ldots t_m^{\alpha_m} (\ln t_m)^{k_m}. \tag{3}$$

In this series $\alpha = (\alpha_1,\ldots,\alpha_m)$; $k = (k_1,\ldots,k_m)$; for any j the exponent α_j runs through a finite set of numbers with the property: $\exp(2\pi i\alpha_j)$ is an eigenvalue of the linear operator M_j defined below; $k_1,\ldots,k_m$ are nonnegative integers; each k_j is less than the size of the Jordan block of the operator M_j, which is maximal among the blocks which correspond to the eigenvalue $\exp(2\pi i\alpha_j)$; the coefficients $a_{k,\alpha}:D^m \to \mathbf{C}$ are functions which are holomorphic in the polydisk D^m.

Definition of the Operator M_j. We consider a closed real curve in $(D^\cdot)^m$, which goes around the hyperplane $t_j = 0$ counterclockwise. The map γ carries this curve into a curve which lies in Λ. To the new curve corresponds the monodromy operator of the Gauss–Manin connection of the bundle π_*. This is also M_j.

Remarks. 1) The eigenvalues of the indicated monodromy operators are roots of 1, the size of the Jordan blocks does not exceed constants which depend on the dimension of the fibers of the map π [3, 18-20]. Hence in (3), all α_j are rational numbers, and all k_j are less than the constants mentioned.

2) If the differential form ω depends holomorphically on parameters, then the coefficients $a_{k,\alpha}$ also depend holomorphically on the parameters.

2. Algebraic Sections. In this paragraph and the next we recount the information used in the proof of the basic theorem in Secs. 2 and 3.

A holomorphic section s of the bundle π^* is said to be algebraic, if for any covariant constant section δ of the bundle π_* and any analytic curve $\gamma:(D, D^\cdot) \to (\Lambda', \Lambda)$ the multivalued function $\langle s, \delta\rangle \circ \gamma : D^\cdot \to \mathbf{C}$ in each sector of the disk D grows no faster than a suitable power of the function as the variable tends to the center of the disk.

Theorem 2.1 asserts that geometric sections are algebraic. It is easy to see that for algebraic sections the assertion of Corollary 2.1 about expansion in a series (3) holds.

A collection of algebraic sections is said to be an algebraic frame, if one can find a point of the base at which the values of the sections form a basis for the fiber of the bundle.

By a resolution of singularities of the pair Λ', Σ is meant a proper morphism $\varphi : \Lambda'' \to \Lambda'$ of a nonsingular Λ'', which is an isomorphism over Λ and for which $\varphi^{-1}(\Sigma)$ is a divisor with normal intersections. According to [21] a resolution of singularities exists. We call a meromorphic function on Λ quasirational, if, lifted to a resolution of singularities of the pair Λ', Σ, i.e., Λ'', it becomes a meromorphic function on Λ''.

In this paragraph and the following we assume that Λ is irreducible and there exists an algebraic frame in the bundle π^*.

LEMMA 1. Let s be an algebraic section, $s_1,\ldots,s_\mu$ be an algebraic frame, $a_1,\ldots,a_\mu$ be coordinates of the section s in the frame. Then the coordinates are quasirational functions.

Proof. That the coefficients are meromorphic on Λ is obvious. Let $\varphi : \Lambda'' \to \Lambda'$ be a resolution of singularities, $t_1,\ldots,t_m$ be the coordinates of a local chart on Λ'', in which $\varphi^{-1}(\Sigma) \subset \{t_1 \ldots t_m = 0\}$. We shall prove that the functions $a_1 \circ \varphi, \ldots, a_\mu \circ \varphi$ are meromorphic in this chart. Let $e_1,\ldots,e_\mu$ be a covariant constant (multivalued) frame of the bundle π^*. In this frame the sections s, $s_1,\ldots,s_\mu$ acquire (multivalued) coordinates, which, on being lifted to Λ'', can be expanded in the chosen chart in a series of the form (3). We calculate $a_j \circ \varphi$ according to Cramer's rule as particular suitable determinants. Each of the determinants can be expanded in a series of the form (3), and upon passage around a coordinate hyperplane it is multiplied by the determinant of the corresponding monodromy operator, so to a suitable power it is an analytic function. This proves the lemma.

LEMMA 2. Let Π be the closure in Λ' of the zeros (or poles) of a quasirational function. Then Π is an algebraic set.

In fact, by Hironaka's theorem [21] there exists a resolution of singularities $\varphi: \Lambda'' \to \Lambda'$, in which Λ'' is projective. Then $\varphi^{-1}(\Pi)$ is algebraic. A proper morphism carries algebraic sets into algebraic sets (cf. [22, Chap. AG]).

LEMMA 3. Let $s_1,\ldots,s_k$ be algebraic sections and let there exist a point of the base at which the values of the sections are linearly independent. Then the set on which the values of the sections are linear dependent is contained in a proper algebraic subset of the base.

The lemma is an easy consequence of Lemmas 1, 2.

3. Differential Equations. Let Λ be imbedded in $\mathbf{C}P^M$, v be a rational vector field in $\mathbf{C}P^M$, which is regular on Λ and tangent to Λ, s be a holomorphic section of the bundle π^*. We associate a differential equation with the pair s, v. Namely, we consider on Λ the following section of the bundle π^*:s, $\nabla_v s$, $(\nabla_v)^2 s,\ldots$, where ∇ is differentiation in the Gauss–Manin connection. Let r be the maximal number with the property: One can find a point of the base at which the values of the sections s, $\nabla_v s,\ldots,(\nabla_v)^{r-1}s$ are linearly independent. For this r there exist unique meromorphic functions $f_1,\ldots,f_r$ on Λ for which

$$(\nabla_v)^r s + f_1(\nabla_v)^{r-1}s + \ldots + f_r s = 0. \tag{4}$$

We say this is the equation associated with s, v. By the singular set of an equation we mean the union of the discriminant Σ and the closure of the subset of Λ on which the sections s, $\nabla_v s,\ldots,(\nabla_v)^{r-1}s$ are linearly dependent. It is easy to see that the poles of the functions $f_1,\ldots,f_r$ belong to the singular set of the equation.

LEMMA 1. Let us assume that s is a nonzero algebraic section. Then the singular set of (4) is contained in a proper algebraic subset of Λ'.

Proof. Each section $(\nabla_v)^j s$ is algebraic. This can be derived easily from Corollary 2.1 with the help of the resolution of singularities. Now cf. Lemma 2.2.3.

Let us assume that there is given a regular rational map $\psi:\Lambda' \to \Gamma$, whose fibers are nonsingular curves. We denote by $\Lambda(\gamma)$ the fiber over the point $\gamma \in \Gamma$. Let us assume that the rational vector field v is tangent to the fibers of the map ψ. We consider Eq. (4) associated with the field v and the holomorphic section s of the bundle π^*. We denote by $\Pi(\gamma)$ the intersection of the singular set of (4) with the fiber $\Lambda(\gamma)$.

LEMMA 2. Let $\Pi(\gamma)$ be a proper subset of $\Lambda(\gamma)$. We consider on the curve $\Lambda(\gamma)$ the ordinary differential equation

$$(L_v)^r y + f_1|_{\Lambda(\gamma)}(L_v)^{r-1}y + \ldots + f_r|_{\Lambda(\gamma)}y = 0, \tag{5}$$

where y is the unknown function, L_v is differentiation along v. Then all solutions of this equation consist of (multivalued) functions of the form $\langle s, \delta\rangle|_{\Lambda(\gamma)}$, where δ is a covariant constant section of the bundle π_*.

The lemma is obvious.

If s is an algebraic section, then all singular points of (5) are regular.

LEMMA 3. Let s be a nonzero algebraic section. Then under the hypotheses of Lemma 2 there exist a proper algebraic subset $\Delta \subset \Gamma$, a finite set S of rational numbers, and a natural number C, which have the following properties:

1. $\Pi(\gamma) \subset \Lambda(\gamma)$ is a proper subset for any point $\gamma \in \Gamma \setminus \Delta$.

2. Let $\gamma \in \Gamma \setminus \Delta$, $\lambda \in \Pi(\gamma)$, t be a local parameter on the curve $\Lambda(\gamma)$ at the point λ, δ be a covariant constant section of the bundle π_*. Then in each sector $a < \arg t < b$ each branch of the function $\langle s, \delta\rangle|_{\Lambda(\gamma)}$ can be expanded in a series

$$\sum_{\alpha\in S}\sum_{p=0}^{\infty}\sum a_{k,\alpha,p}t^{\alpha+p}(\ln t)^k, \text{ where } a_{k,\alpha,p}\in\mathbf{C}. \tag{6}$$

Moreover, if the branch is not identically equal to zero, then in this series the exponent $\alpha + p$, which is minimal among the nonzero summands, lies between $-C$ and C.

Proof. Let Π' be a proper algebraic subset of Λ' containing Π, $\Pi'(\gamma) = \Pi' \cap \Lambda(\gamma)$. According to [9], there exists a proper algebraic subset $\Delta \subset \Gamma$ with the property: the map ψ, restricted to the pair of sets $\psi^{-1}(\Gamma \setminus \Delta)$, $\psi^{-1}(\Gamma \setminus \Delta) \cap \Pi'$, is a topological locally trivial bundle pair over $\Gamma \setminus \Delta$ with fiber $\Lambda(\gamma)$, $\Pi'(\gamma)$. We shall prove the lemma for this Δ. Point 1 is obvious. Under change of the point $\gamma \in \Gamma \setminus \Delta$ the points of the set $\Pi'(\gamma)$ change continuously without merging. To passage around a fixed point of the set $\Pi'(\gamma)$ there corresponds precisely one monodromy operator of the Gauss–Manin connection on the bundle π_* (up to linear equivalence). According to Corollary 2.1, this gives the existence of the set S. We shall prove the existence of the number C. The singular points of (5) belong to $\Pi'(\gamma)$ and are regular. At an arbitrary point of $\Pi'(\gamma)$ we form the characteristic equation of the singular point of (5) to determine the minimal exponent in (6). The roots of the characteristic equation are rational numbers; hence under change of γ and motion of the point of $\Pi'(\gamma)$ they are unchanged. Consequently, the C sought exists.

LEMMA 4. Under the hypotheses of Lemma 3, the function $\langle s, \delta\rangle|_{\Lambda(\gamma)}$ cannot have an isolated zero of multiplicity greater than $r - 1$ outside $\Pi(\gamma)$.

The proof is obvious.

4. Proof of Points 2 and 3 of the Basic Theorem. We describe a family of algebraic varieties, to subfamilies of which we apply the results of Paragraphs 2.1-2.3. This family will be denoted by $\pi : X \to D$.

We consider the space of polynomials in $x_1,\dots,x_n$, having complex coefficients and degree no higher than N. We denote the affine space of their coefficients by $\mathbf{C}A$. We denote the polynomial corresponding to $a \in \mathbf{C}A$ by P_a. We consider the space of polynomial differential $(n - 1)$-forms in $x_1,\dots,x_n$, whose coefficients $P_1,\dots,P_n$ [cf. (1)] are polynomials of degree no higher than N with complex coefficients. We denote the affine space of the collection of coefficients of the polynomials $P_1,\dots,P_n$ by $\mathbf{C}B$. We denote the form corresponding to $b \in \mathbf{C}B$ by ω_b. In the space $\mathbf{C}^n \times \mathbf{C}B \times \mathbf{C}A \times \mathbf{C}B \times \mathbf{C}$ we consider the algebraic subset $X = \{(x, b', a, b, t) | P_a(x) = t, b' = b\}$ together with its natural projection π onto the set $D = \mathbf{C}A \times \mathbf{C}B \times \mathbf{C}$. We denote the fiber over $d = (a, b, t)$ by $X(d)$; $X(d)$ is the level t hypersurface of the polynomial P_a.

Let $E \subset \mathbf{C}A \times \mathbf{C}B$ be an irreducible algebraic set. According to [9], there exists a nonsingular quasiprojective $\Lambda \subset E \times \mathbf{C}$, which is dense in $E \times \mathbf{C}$ and has the following properties: The map π restricted to $\pi^{-1}(\Lambda)$ is a topological locally trivial bundle and the set $\pi^{-1}(\Lambda)$ is nonsingular.

Let us assume that the fiber of this bundle is not empty (i.e., the polynomials corresponding to E are not constant). We consider over Λ the associated bundles π_*, π^* with fibers $H_{n-1}(X(d), \mathbf{C})$, $H^{n-1}(X(d), \mathbf{C})$, respectively. We single out the section s of the bundle π^*, defined by the formula $s\colon d = (a, b, t) \to [\omega_b|_{X(d)}] \in H^{n-1}(X(d), \mathbf{C})$. It is easy to see that s is a geometric section. In the bundle π^* there exists a geometric frame. In fact, an arbitrary polynomial differential $(n - 1)$-form in $\mathbf{C}^n$, considered as a form on $\mathbf{C}^n \times \mathbf{C}B \times \mathbf{C}A \times \mathbf{C}B \times \mathbf{C}$, becomes relatively closed upon restriction to X. Consequently, such a form defines a geometric section of the bundle π^*. From such sections one can compose an algebraic frame, since by Grothendieck's theorem [23] the cohomology of a nonsingular affine variety is generated by closed polynomial differential forms.

Let E' be a projective algebraic set containing E as an open, everywhere dense subset. Then $\Lambda' = E' \times \mathbf{C}P^1$ is the closure of the set Λ. We denote by v the rational vector field on Λ', equal to $\partial/\partial t$, where t is a coordinate on the affine part of the second factor. Let $\psi\colon \Lambda' \to E'$ be the natural projection.

Obviously the lemmas of Paragraph 2.3 are applicable to the objects Λ, π^*, π_*, s, Λ', ψ, v. As a consequence of the lemmas we get points 2 and 3 of the basic theorem.

3. Boundedness Above of the Number of Zeros

In this section we prove the following theorem. Let us assume that there is given a system of equations depending on parameters on a real cube. Let us assume that the functions appearing in the system can be expanded in series of the form (3) and depend analytically on the parameters. Let us assume that the set of parameters is compact. Then the number of connected components of the set of solutions of the system is uniformly bounded above. From this theorem one derives point 1 of the basic theorem. Generalizations are given in Paragraph 3.4.

1. Formulation. We set $I = [0, 1]$, $I' = (0, 1]$. Let $\alpha_1, \dots, \alpha_s \in \mathbf{R}$. On the set $(I') \times I^m$ with coordinates $t_1,\dots,t_n$, $a_1,\dots,a_m$ we consider the system of equations

$f_j(t_1^{\alpha_1}, \ldots, t_1^{\alpha_s}, \ln t_1, t_1, \ldots, t_n^{\alpha_1}, \ldots, t_n^{\alpha_s}, \ln t_n, t_n, a_1, \ldots, a_m) = 0$, $j = 1, \ldots, r$, where $f_j(u_1^1, \ldots, u_1^s, v_1, t_1, \ldots, u_n^1, v_n, t_n, a_1, \ldots, a_m)$ is a polynomial in the variables u, v, whose coefficients are analytic functions of the variables t, a, defined in a neighborhood of the set $I^n \times I^m \subset \mathbf{R}^n \times \mathbf{R}^m$. Considering a as a parameter, we get the system of equations f(t, a) = 0 on $(I')^n$, which depends on the parameter a.

THEOREM. There exists a positive number C with the property: For any $a \in I^m$ the number of connected components of the set of solutions of the system f(t, a) on $(I')^n$ is not greater than C.

The proof is analogous to the proof of Khovanskii's theorem on small terms [6, 7], which Khovanskii explained to me. Only at the concluding step, instead of Bezout's theorem (in the case of small terms) one uses the theorem of Gabriélov [8] formulated below. When I acquainted Khovanskii with the theorem, he informed me that several years before he already knew a generalization of it, cf. [25].

2. Proof of the Theorem.

It suffices to prove the following lemma.

LEMMA 1. Let r (the number of solutions) be equal to n. Then there exists a positive C with the property: For any $a \in I^m$ the number of nondegenerate solutions of the system of equations f(t, a) is not greater than C.

In fact, we consider the analytic set

$$X_{a,\varepsilon,\delta} = \left\{(t, u) \in (I')^n \times I' \,\middle|\, \sum_{j=1}^{r} f_j^2(t, a) = \varepsilon,\ \delta^2 + u^2 = \prod_{j=1}^{n} (t_j - t_j^2)\right\},$$

where $a \in I^m$, $\varepsilon, \delta \in \mathbf{R}$ are parameters. It suffices to prove that for all a, ε, δ the number of connected components of the set $X_{a,\varepsilon,\delta}$ is uniformly bounded. We consider the function $g_{a,\varepsilon,\delta,b}: X_{a,\varepsilon,\delta} \to \mathbf{R}$, equal to the square of the distance to the point $b \in I^n \times I$. We consider b as a parameter. It is easy to see (with the help of Morse's theorem) that the boundedness of the number of components follows from the assertion: There exists a positive number C with the property: for any a and almost all ε, δ, b, the number of nondegenerate critical points of the function $g_{a,\varepsilon,\delta,b}$ is not greater than C. It is easy to see that a nondegenerate critical point is a nondegenerate solution of a system of equations of the same form as the system f(t, a) if r = n.

To prove Lemma 1 it suffices to prove the following Lemma 2. On the set $\mathbf{R}^k \times I^n \times I^m$ with coordinates $u_1, \ldots, u_k$, $t_1, \ldots, t_n$, $a_1, \ldots, a_m$, we consider the system of equations $h_j(u_1, \ldots, u_k, t_1, \ldots, t_n, a_1, \ldots, a_m) = 0$, j = 1,...,n, where h_j is a polynomial in the variables u, whose coefficients are analytic functions of the variables t, a, defined in a neighborhood of the set $I^n \times I^m \subset \mathbf{R}^n \times \mathbf{R}^m$. Considering a as a parameter, we get the system of equations h(u, t, a) on $\mathbf{R}^k \times I^n$, depending on the parameter a.

LEMMA 2. There exists a positive number C with the following property: For any $a \in I^m$ the number of connected components of the set of solutions of the system of equations h(u, t, a) = 0 is not greater than C.

Lemma 1 is reduced to Lemma 2 by the process indicated in [6], considering the fact that the functions t^α, ln t satisfy the equations t dy = αy dt, t dy = dt, respectively. Lemma 2 in its own right can be derived easily from the following theorem of A. M. Gabriélov.

THEOREM (cf. [8]). Suppose given a semianalytic subset $X \subset I^n \times I^m$. We denote by X_a the set $X \cap (I^n \times \{a\})$. Then there exists a positive number C with the following property: For any $a \in I^m$ the number of connected components of the set X_a is not greater than C.

3. Proof of Point 1 of the Basic Theorem.

We reduce the assertion of point 1 to the form: the number of solutions of a suitable system of equations depending on a parameter is uniformly bounded above. We begin with constructions analogous to the constructions in Paragraph 2.3.

We consider the space of polynomials in $x_1, \ldots, x_n$, having real coefficients and degree no higher than N. We denote the affine space of their coefficients by A. We consider the space of polynomial differential (n − 1)-forms in $x_1, \ldots, x_n$, whose coefficients $P_1, \ldots, P_n$ [cf. (1)] are polynomials of degree no higher than N with real coefficients. The affine space of the collection of coefficients of the polynomials $P_1, \ldots, P_n$ will be denoted by B. A and B are the real parts of the spaces **C**A, **C**B, respectively, defined in Paragraph 2.3.

In $\mathbf{C}^n \times \mathbf{C}A \times \mathbf{C}$ we consider the algebraic subset $X = \{(x, a, t) | P_a(x) = t\}$ and its natural projection $\pi: X \to \mathbf{C}A \times \mathbf{C}$.

Let $E \subset A$ be an arbitrary real irreducible algebraic subset. By $\mathbf{C}E$ we denote the smallest complex algebraic subset of $\mathbf{C}A$, containing E. According to [9], there exists a nonsingular quasiprojective $\mathbf{C}\Lambda \subset \mathbf{C}E \times \mathbf{C}$, dense in $\mathbf{C}E \times \mathbf{C}$ and having the properties that the map π restricted to $\pi^{-1}(\mathbf{C}\Lambda)$ is a topological locally trivial bundle, the set $\pi^{-1}(\mathbf{C}\Lambda)$ is nonsingular, the set $\Lambda = \mathbf{C}\Lambda \cap (E \times \mathbf{R})$ is nonsingular. Let us assume that the fibers $X(\lambda)$ of this bundle are not empty.

We consider the bundle $\pi |_{\pi^{-1}(\Lambda)}: \pi^{-1}(\Lambda) \to \Lambda$ and the associated bundles π_*, π^* with fibers $H_{n-1}(X(\lambda), \mathbf{C})$, $H^{n-1}(X(\lambda), \mathbf{C})$, respectively. An arbitrary form $\omega \in \mathbf{C}B$ and covariant constant section δ of the bundle π_* define on Λ a function $I_{\omega,\delta}: \lambda \mapsto \langle s[\omega](\lambda), \delta(\lambda)\rangle$, where $s[\omega]$ is the geometric section of the form ω. This function is generally multivalued and complex-valued.

For any point $\lambda = (a, t) \in \Lambda$ the fiber $X(\lambda)$ is invariant with respect to complex conjugation $(x_1,\ldots,x_n, a, t) \to (\bar{x}_1,\ldots,\bar{x}_n, \bar{a}, \bar{t})$. The involution of conjugation induces an involution in $H_{n-1}(X(\lambda), \mathbf{R})$. The invariant and antiinvariant subspaces (Inv and Ant) of this involution are preserved under parallel translations in the Gauss–Manin connection.

Let us assume that $\omega \in B$ and the values of the covariant constant section δ at all points belong to the invariant or antiinvariant subspaces Inv or Ant. In this case we call the function $I_{\omega,\delta}$ an Abelian function on Λ. An Abelian function assumes only real or pure imaginary values and is an analytic function on Λ. We shall indicate its behavior upon approaching the boundary of the set Λ.

Let E' be a real projective algebraic set containing E as an everywhere dense subset. Then Λ is everywhere dense in $\Lambda' = E' \times \mathbf{R}P$. We set $\Sigma = \Lambda' \setminus \Lambda$. Let $\varphi: \Lambda'' \to \Lambda'$ be a resolution of singularities of the pair Λ', Σ; λ be a point of the divisor $\varphi^{-1}(\Sigma)$. We consider a local analytic chart with center at λ and coordinates $t_1,\ldots,t_m$ in which $\varphi^{-1}(\Sigma) \subset \{t_1 \ldots t_m = 0\}$. We single out an octant in this chart (i.e., we fix the signs of the coordinates). We lift an arbitrary Abelian function to Λ'' and we fix an analytic branch of it in the chosen octant.

LEMMA 1. The branch can be expanded in a series

$$I_{\omega,\delta}\circ\varphi = \sum_{k,\alpha} a_{k,\alpha} |t_1|^{\alpha_1} (\ln|t_1|)^{k_1} \ldots |t_m|^{\alpha_m} (\ln|t_m|)^{k_m}.$$

In this series $\alpha = (\alpha_1,\ldots,\alpha_m)$, $k = (k_1,\ldots,k_m)$; the exponents $\alpha_1,\ldots,\alpha_m$ run through a finite set of rational numbers, which does not depend on the choice of Abelian function; $k_1,\ldots,k_n$ are nonnegative integers, which do not exceed constants depending only on n; the coefficients $a_{k,\alpha}$ are analytic functions in a neighborhood of the point $(0,\ldots,0)$, which depend analytically on the form $\omega \in B$ and invariant section δ.

The lemma is a consequence of (3).

On Λ' we consider the family L of all lines of the form $(a, \mathbf{R}P)$, where $a \in E'$. It is easy to see that to prove point 1 of the basic theorem it suffices to prove the following lemma.

LEMMA 2. There exists a positive number C, depending in general on E, which has the following property. Let $l \in L$ be a line which does not lie in Σ, I be an Abelian function. On $l \setminus (l \cap \Sigma)$ we single out a (single-valued) analytic branch of the function I. Then this branch has no more than C isolated zeros.

Proof of the Lemma. We must show that the restriction to a line depending on a parameter of a function depending on a parameter has small zeros. A line depending on a parameter can be given as the set of zeros of a system of rational functions depending on a parameter. Thus, the lemma asserts that a system of equations depending on a parameter has small zeros if the functions which appear in the system are rational or Abelian. To prove this it suffices to lift all the functions of the system to a resolution of singularities $\varphi: \Lambda'' \to \Lambda'$ and to prove that locally on Λ'' the lifted system of equations has small zeros.

One makes the local estimate as follows. We cover Λ'' by a finite number of local analytic charts $\{U_j\}$, having the properties:

1) Each of the charts is a compact semianalytic set;

2) each of the charts either does not intersect $\varphi^{-1}(\Sigma)$, or the intersection of the chart with the set $\varphi^{-1}(\Sigma)$ belongs to a union of coordinate hyperplanes of this chart.

We divide the charts into charts of the first or second kind according to the second property. By Gabriélov's theorem there exists a natural number N with the following properties:

3) For any line $l \in L$ and any chart U of the first kind, the set $\varphi^{-1}(l) \cap U$ has no more than N connected components;

4) let U be an arbitrary chart of the second kind, $t_1, \ldots, t_m$ be its local coordinates; then for any line $l \in L$ the set $\varphi^{-1}(l) \cap \{(t_1, \ldots, t_m) \in U \mid t_1 \cdot \ldots \cdot t_m \neq 0\}$ has no more than N connected components.

Now for any line $l \in L$, any chart U of the covering, on each of the connected components mentioned the number of isolated zeros of an arbitrary Abelian function is bounded uniformly with respect to the parameters according to Gabriélov's theorem if this is a chart of the first kind, and according to Theorem 3.1 and Lemma 1, if this is a chart of the second kind. This proves the local boundedness of the number of zeros.

Refinement. For the reference to Gabriélov's theorem or Theorem 3.1, we need the compactness of the space of parameters. In our case the space of parameters is the direct product of the space of differential forms and the space of sections. Multiplication of a section or form by a nonzero number does not decrease the number of zeros of the corresponding Abelian function. Hence as space of parameters it suffices to take the direct product of spheres with center at the origin in the space of forms and in the space of sections. This proves the basic theorem.

4. Abelian Functions.

By Bezout's theorem on an algebraic variety the number of isolated zeros of a system of rational functions of fixed degree is uniformly bounded above with respect to the coefficients of the functions.

Principle. Assertions of the type of Bezout's theorem hold on a real algebraic variety for systems of equations in which rational and Abelian functions appear. Here one should consider the degree of an Abelian function to be the degree of the polynomial differential form, the integration of which gives the function.

The basic theorem is an example of such an assertion. The problem (due to which the principle, interpreted literally, is not true) is the possible infinite-valuedness of Abelian functions. For the validity of an assertion of the type of Bezout's theorem it is necessary that only a finite number of branches of each of the Abelian functions should participate in the system of equations. For example, one branch of the Abelian function participates in the equations of the basic theorem, since the Abelian function is considered on an interval and an interval is simply-connected. Another method to avoid infinite-valuedness is to take real components of the Abelian functions.

We consider a real projective irreducible algebraic set Λ and a real quasiprojective nonsingular connected set X, a rational map $\pi: X \to \Lambda$ which is regular on X. Let us assume that the image $\pi = \pi(X)$ is open in Λ and nonsingular, and the map $\pi: X \to \Pi$ is a smooth bundle with compact n-dimensional fibers. We single out in one of the fibers one of the connected components. Let us assume that it is orientable. We orient the component and extend the orientation by continuity to an orientation of the neighboring components of neighboring fibers. We get a multivalued finitely-sheeted family $\gamma: \lambda \to \gamma(\lambda)$ of oriented nonsingular connected components of fibers depending continuously on a parameter $\lambda \in \Pi$. We consider on X a rational regular differential n-form ω. We call the function $I_\omega: \lambda \mapsto \int_{\gamma(\lambda)} \omega$ an Abelian function (of a component) on Λ, and the map $\pi: X \to \Pi$ we call the normalization (of the function). The integral which occurs in the definition of nonconservativity in Paragraph 1.2 is an example of an Abelian function.

By the degree of the Abelian function I_ω we mean the degree of the form ω. By the degree of the form we mean the minimum of the degrees of rational forms in the ambient projective space to X whose restrictions to X are equal to ω. By the degree of a rational form in projective space, written in terms of homogeneous ordinates, we mean the maximum of the degrees of the denominators of its coefficients. For example, the form xdx/y^2 has degree 2.

We give an example of an assertion of the type of Bezout's theorem.

THEOREM. Let us assume that on Λ there are given normalizations π_j: $j \to \Pi_j \subset \Lambda$, $j = 1,\ldots,k$, and natural numbers p, q, N. Then there exists a positive number C with the following property: if I_j is an Abelian function of degree no higher than p with framing $\pi_j: X_j \to \Pi_j$, $j = 1,\ldots,k$ and $P_1,\ldots,P_N$ are polynomials in $x_1,\ldots x_k$ with real coefficients of degree not higher than q, then the set of solutions of the system of equations

$$P_j(I_1,\ldots,I_k) = 0,\ j = 1,\ldots,N,$$

defined on $\Pi = \bigcap_{j=1}^{k} \Pi_j$, has no more than C connected components.

We call a point a solution, if suitable values of the functions $I_1,\ldots,I_k$ at the point are a solution of the system of equations $P(x) = 0$.

The proof is analogous with the proof of point 1 of the basic theorem: One resolves the singularities of the set Π and its boundary, and then applies Lemma 3.3.1 [i.e., (3)], Theorem 3.1 or Gabriélov's theorem to the resolution.

Remark. As A. G. Khovanskii noted, in this argument it is not important that the functions $I_1,\ldots,I_k$ are integrals; for example, they could be solutions of systems of differential equations with regular singularities [for which (3) is valid].

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INTEGRAL FORMULAS AND INTEGRAL GEOMETRY FOR $\bar{\partial}$-COHOMOLOGIES IN $\mathbf{CP}^n$

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Let D be an (n − q)-linearly concave domain in the complex projective space $\mathbf{CP}^n$, that is, through every point of D there passes a complex q-plane lying entirely in the domain. In [1] there is constructed the Radon integral transform of the q-dimensional cohomologies $H^q(D, \mathcal{O}(-n-1)) \cong H^{(n,q)}(D)$ (integration over q-planes in D) (see also [2]*). Its real analogue was studied in [3] (see also [4]). An inversion formula for this transformation was obtained, and it was shown that for q = n − 1 (such domains are called linearly concave) the Radon transform coincides with the Fantappie indicatrix. The inversion formula has an important corollary: There exists a fibering $\hat{F}(D) \to D$ such that the restrictions of holomorphic (n + q)-forms on $\hat{F}(D)$ to a cross section of the fibering give representatives of all the cohomology classes $H^{(n,q)}(D)$. This reduction to holomorphic forms is very useful in applications.

The present paper is devoted to the derivation of integral formulas for $H^q(D, \mathcal{O}(-s))$, $1 \leqslant s \leqslant n$, to the construction of an analogue of Fantappie's indicatrix for these cohomologies (for q = n − 1 it coincides with the classical analogue), to proving that it coincides with the Radon transform, and to the derivation of an inversion formula from these results. The useful modifications of the Cauchy–Fantappie formula are oriented towards bounded domains in which there is a fixed affinization. In [1] the need to work with these formulas led to a whole series of technical difficulties, since for q > 0 the (n − q)-linearly concave domains are unbounded. In the global formula constructed here the affinization varies from point to point of the boundary. It is important that this formula gives immediately a holomorphic continuation of the class of cohomologies to the fibering of $\hat{F}(D)$ over D. The formulas we obtain appear fairly unwieldy, but if we segregate the explicit expression for the coefficients (with numerous factorials and powers of π), then their structure is sufficiently obvious: an addition of terms connected with the variation of the affinization along the boundary and necessary for the invariance of the closedness of the kernel. It is essential that the kernels are the residues of forms of a simple form, and only in the calculation of a residue do the additional structures (an affinization variable) appear. In the majority of arguments it is convenient to consider forms up to taking a residue. It should be borne in mind that the case s = n is simpler than the general one: in this case the kernel does not depend on the affinization (since the pole singularity is of the first order).

I wish to thank G. M. Khenkin for many discussions of questions touched on in this paper. From him I understood the idea of representing the kernel of an integral formula in the form of a residue form; this eliminated certain additional structures. We obtained

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*The choice s = n + 1 in the coefficient bundle $\mathcal{O}(-s)$ is not essential.

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