HEAVISIDE FUNCTIONS OF A CONFIGURATION OF HYPERPLANES

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UDC 517.58+517.88

1. Introduction

We consider the ring of functions that are defined on the complement M of the union of a finite set of hyperplanes in real affine space, and that have integer or constant values on each connected component. In this ring, which we denote by P, there are distinguished multiplicative generators, namely, Heaviside functions of hyperplanes: for each hyperplane there is a fixed function that is 1 from one side and 0 from the other. Any element of the ring is a polynomial in the Heaviside functions. The filtration of the ring in terms of the degrees of polynomials, denoted by $\{P^k\}$, $k \geq 0$, brings to P properties near to those of the ring of cohomologies of the complement M_C of the union of complexified hyperplanes in the complexified affine space.

The ring $H^*(M_C)$ has been described by Arnol'd [1], Brieskorn [2], and Orlik and Solomon [3]. Orlik and Solomon [3] drew attention to the fact that the dimension of $H^*(M_C)$ is equal to the number of connected components of M_C. In the present article we offer an explanation of this by comparing the rings P and H^*. The ring P is commutative and is equipped with an increasing filtration $\{P^k\}$. The ring H^* is anticommutative and is endowed with a graduation $\{H^k\}$. We give certain properties of P and state the known analogous properties of H^*.

It may be that P and H^* are included in a one-parameter family of rings that has independent interest.

Our study was carried out in connection with an investigation of general hypergeometric functions, and is devoted to geometrical aspects of this theory (see [4-10]).

The authors thank V. I. Arnol'd for useful discussions.

1. Definition. We consider a finite set fo linear functions $\{f_i\}$, $i \in I$, on an n-dimensional affine space V defined over a field R. Let S denote the union of the hyperplanes $A_i = \{v \in V : f_i(v) = 0\}$, $i \in I$. We call S and $\{f_i\}$ a configuration of hyperplanes. We consider the ring P(S, Z) of integer-valued functions on $M = V \setminus S$ that are constant on each connected component. We consider in P certain multiplicative generators, namely, Heaviside functions; these are functions x_i, $i \in I$, defined by $x_i(v) = 1$ if $f_i(v) > 0$, and $x_i(v) = 0$ if $f_i(v) < 0$. Every $x \in P(S, Z)$ can be written as a polynomial in the $\{x_i\}$, $i \in I$, with integer coefficients.

For $x \in P(S, Z)$ we call the minimum of the degrees of polynomials in $\{x_i\}$ that represent x the degree of x.

We define an increasing filtration

$$0 \subset P^0 \subset P^1 \subset \ldots \subset P,$$

where P^k is the subspace of functions representable by polynomials of degree not higher than k. In particular, P^0 consists of the constant functions. Obviously, $P^k \cdot P^\ell \subset P^{k+\ell}$. We call the filtration $\{P^k\}$ a degree filtration.

Example. Consider a configuration of lines $\{A_i\}$, $i \in I$, on a plane. The degree filtration in P consists of three terms: P^0 (the constant functions), P^1 (linear combnations of Heaviside functions), and $P^2 = P$. Assume that any three lines do not intersect at a single point. A basis over Z for the ring consists of the constant function equal to 1, the Heaviside functions, and all monomials $x_i x_j$ for which the lines A_i, A_j intersect. The dimension of P^0 is 1, that of P^1/P^0 is the number of lines, and that of P^2/P^1 is the number of points of intersection of the lines.

I. M. Gubkin Institute of the Petrochemical and Gas Industry. Institute of Applied Mathematics, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 21, No. 4, pp. 1-18, October-December, 1987. Original article submitted May 26, 1987.

0016-2663/87/2104-0255$12.50

We say that a function $x \in P$ has zero index at the intersection of two lines if the antisymmetrized sum of its four values at the four components of the complement that come to the point is equal to zero. A function $x \in P$ has degree ≤ 1 if and only if it has zero index at every point of intersection.

2. Properties of the Ring P. THEOREM 1. $P^n = P$, that is, any piecewise constant function on the complement of a union of hyperplanes in n-dimensional affine space is a polynomial of degree no higher than n in the Heaviside functions of the hyperplanes.

Let V_C denote the complexification of the space V, $A_{i,C}$ denote the complexification of the hyperplanes A_i, $i \in I$, S_C denote the union of the hyperplanes $\{A_{i,C}\}$, $i \in I$, and $M_C = V_C \setminus S_C$. Theorem 1 is the analogue of the assertion: $H^k(M_C) = 0$ for $k > n$.

A configuration S on V naturally cuts a new configuration S_U on each affine subspace $U \subset V$. S_U consists of the hyperplanes $\{A \cap U : A \in S, U \not\subset A\}$ defined by the linear functions $\{f_i|_U\}$, $i \in I$.

If the affine subspace U is not contained in S, then there is defined the natural homomorphism j_U: $P(S) \to P(S_U)$ such that a function from P(S) is restricted to $U \setminus S_U$.

Any nonempty intersection F of the hyperplanes of a configuration is called a rib. We denote the codimension of a rib by r(F). In particular, a hyperplane is a rib of codimension 1. We denote the set of all ribs by $\mathscr{L}$.

We call an affinte subspace $U \subset V$ of dimension d a subspace of general position relative to a configuration S if U is transversal to all ribs, and intersects all ribs of codimension not greater than d.

THEOREM 2. If $U \subset V$ is a subspace of general position, then the homomorphism j_U, restricted to $P^k(S)$, defines for $k \leq d$ an isomorphism of $P^k(S)$ and $P^k(S_U)$.

This is the analogue of Brieskorn's Theorem [2]: if $U_C \subset V_C$ is a sufficiently general affine subspace of dimension d, then for $k \leq d$ the map $H^k(M_C) \to H^k(M_C \cap U_C)$ is an isomorphism.

Let F be a rib of a configuration, $I^F \subset I$ be the set of all indices i for which $F \subset A_i$. We denote by S^F the configuration formed by the hyperplanes $\{A_i\}$, $i \in I^F$, that is, that contain F. We say that S^F is a localization of S at the rib F. We consider the ring $P(S^F, Z)$ of the configuration S^F. There is a natural embedding $P(S^F, Z) \subset P(S, Z)$ defined by the restriction to M of functions from $P(S^F, Z)$. Its image is the subring generated by functions $\{x_i\}$, $i \in I^F$. The embedding preserves the degree filtration.

THEOREM 3. The natural map

$$\bigoplus_{\substack{F \in \mathscr{L} \\ r(F)=k}} P^k(S^F)/P^{k-1}(S^F) \to P^k(S)/P^{k-1}(S)$$

is an isomorphism for any $k > 0$.

COROLLARY 1. If the hyperplanes of a configuration intersect normally, then the number $\dim_Z P^k(S)/P^{k-1}(S)$ is equal to the number of ribs of codimension k.

Let $M_C^F = V_C \setminus \bigcup_{i \in I^F} A_{i,C}$. By Brieskorn's Theorem [2] the natural map $\bigoplus_{\substack{F \in \mathscr{L} \\ r(F)=k}} H^k(M_C^F, \mathbf{Z}) \to H^k(M_C, \mathbf{Z})$ is an isomorphism for any $k > 0$. Theorem 3 is the analogue of Brieskorn's Theorem.

COROLLARY 2. $\dim_Z P^k(S, \mathbf{Z})/P^{k-1}(S, \mathbf{Z}) = \dim_Z H^k(M_C, \mathbf{Z})$ for $k \geq 0$.

The corollary is easily obtained by induction on the dimension of the enveloping space using an observation of Orlik and Solomon, namely, $\dim_Z C_n(S) = \dim_Z H^*(M_C)$, Theorem 3 and Brieskorn's Theorem.

Remark 1. From the corollary it follows, in particular, that the number $\sum_{k \geq 0} (-1)^k \dim_Z P^k(S)/P^{k-1}(S)$ is equal to the number of bounded connected components of the set M (see the combinatorial formulae for $\dim H^k(M_C)$ and for the number of bounded components in [3, 13]).

2. If k and i are positive, then k can be written in a natural way as $k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}$, where $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$. Following [14] we define

$$k^{\langle i\rangle}=\binom{n_i+1}{i+1}+\binom{n_{i-1}+1}{i}+\ldots+\binom{n_j+1}{j+1},\quad 0^{\langle i\rangle}=0.$$

An integer-valued vector $(k_0, k_1, \ldots, k_d)$ is called an M-vector if $k_0 = 1$ and $0 \leq k_{k+1} \leq k_i^{\langle i\rangle}$ for $1 \leq i \leq d-1$. It follows from [15] that the sequence of numbers $\dim_Z P^k(S)/P^{k-1}(S)$, $k \geq 0$, is an M-vector.

We call a monomial $x_{i_1}\cdots x_{i_k} \in P$ admissible if $df_{i_1} \wedge \ldots \wedge df_{i_k} \neq 0$. The support of an admissible monomial is a k-dimensional simplicial cone multiplied by an (n − 1)-dimensional affine space.

COROLLARY 3. The set of admissible monomials generates P as a module over Z.

3. Dual Degree Filtration. We consider the ring P(S, Z) of a configuration of hyperplanes in n-dimensional affine space as a linear space over Z. We define on the dual space P^* the decreasing filtration

$$0\subset P_n^*\subset P_{n-1}^*\subset\ldots\subset P_0^*=P^*$$

by the condition $P_k^* = \operatorname{Ann} P^{k-1}$. We call the filtration $\{P_k^*\}$ a degree filtration.

We give another construction for this filtration. We shall find a finite set of vectors in P^* called flag cochains. Each flag cochain has a degree. Then P_k^* coincides with the linear hull of all flag cochains of degree not less than k. We turn to the construction.

The connected components of the set M are called domains. Domains are n-dimensional polytopes (not necessarily bounded). The open faces of any dimension of these polytopes are called the faces of the configuration. In particular, domains are n-dimensional faces. The zero-dimensional faces are called vertices.

Let $F_{n-k}\subset F_{n-k+1}\subset\ldots\subset F_n=V$ be a sequence of ribs of a configuration S, where the dimension of the rib F_j is j, and the coorientation of F_j in F_{j+1} is defined. We call this sequence a flag of ribs of degree k, and denote it by F. Let Δ be an (n − k)-dimensional face of a configuration S that lies in F_{n-k}. The flag F with the face Δ is called distinguished.

With a distinguished flag there are connected 2^k domains such that Δ is in the closure of each of them. In fact, there are 2^k domains to which one can go from Δ by moving a small distance along F_{n-k+1} on one side or the other, etc. up to a translation from F_{n-1} on one side or the other. To each such domain there corresponds an ordered sequence α of length k that consists of plus or minus signs: at the j-th position there is a + or − depending on whether the translation into F_{n-k+j} is in accordance with or against the coorientation of $F_{n-k+j-1}$ in F_{n-k+j}. We denote the domain with the index α by Δ_α (see Fig. 1).

By the flag cochain of a distinguished flag we mean the vector $\psi_{F,\Delta} \in P^*$ defined by the formula $\Psi_{F,\Delta}(x)=\sum_\alpha(-1)^{\varepsilon(\alpha)}x(\Delta_\alpha)$, for $x \in P$, where ε(α) is the number of minus signs in the sequence α. We call the number k the degree of the flag cochain.

THEOREM 4. The linear hull of the flag cochains of degree not less than k coincides with $\operatorname{Ann} P^{k-1}$.

Theorem 4 can be used to define the degree of a given function $x \in P$; see the example in Sec. 1.1.

4. Relations between Heaviside Functions. In this section we assume that V is an n-dimensional linear space, $\{f_i\}$, $i \in I$, are linear functions on V, and all the hyperplanes $\{A_i\}$ pass through the origin.

Let $\alpha_1 f_{j_1}+\ldots+\alpha_s f_{j_s}$ be a linear relation, J_+ be the numbers of all the linear functions occurring in the relation with positive coefficients, and J_- be the numbers with negative coefficients.

THEOREM 5. The Heaviside functions of a configuration $\{f_i\}$, $i \in I$, satisfy the following relations:

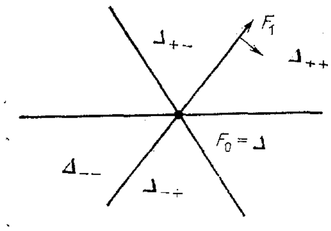

Fig. 1

$$x_i^2 - x_i = 0,\ i \in I \tag{1}$$

for any linear dependence $\alpha_1 f_{j_1} + \ldots + \alpha_s f_{j_s} = 0$,

$$\prod_{j \in J_+} x_j \prod_{k \in J_-} (x_k - 1) - \prod_{j \in J_+} (x_j - 1) \prod_{k \in J_-} x_k = 0. \tag{2}$$

If in a linear relation all the coefficients are nonzero, then in (2) there is a polynomial of degree s − 1 that has exactly s monomials of degree s − 1. The relation (2) is the even analogue of a relation of Orlik and Solomon [3] for differential forms. Namely, we consider the differential forms $\omega_i = df_i/2\pi\sqrt{-1}f_i$. If $f_{j_1}, \ldots, f_{j_s}$ are linearly dependent, then

$$\sum_{l=1}^{s} (-1)^{l-1} \omega_{j_1} \wedge \ldots \wedge \hat{\omega}_{j_l} \wedge \omega_{j_s} = 0.$$

THEOREM 6. The relations (1), (2) determine P. More precisely, if y is an ideal in the ring of polynomials $Z[X_i;\ i \in I]$ generated by the left-hand sides of the relations in Theorem 5, then the natural homomorphism $Z[X_i;\ i \in I]/\mathcal{Y} \to P$ is an isomorphism.

Theorem 6 is the even analogue of the theorem of Orlik and Solomon [3] that describes the ring $H^*(M_C, Z)$. Namely, we consider the exterior algebra of the vector space with the basis e_i, $i \in I$; we consider the ideal generated by the elements $\sum_{l=1}^{s} (-1)^{l-1} e_{j_1} \ldots \hat{e}_{j_l} \ldots e_{j_s}$, where $(f_{j_1}, \ldots, f_{j_s}) \subset \{f_i\},\ i \in I$, is an arbitrary subset of linearly dependent elements. Then the factor-algebra of the exterior algebra with respect to this ideal is naturally isomorphic to $H^*(M_C, Z)$. Under isomorphism an element e_i goes into the cohomology class of the form ω_i.

We consider in Euclidean space with the coordinates $\{x_i\}$, $i \in I$, the cube bounded by the hyperplanes $x_i = 0$, $x_i = 1$, $i \in I$. With the configuration $\{f_i\}$ there is connected a subset of vertices of this cube. Namely, for each connected component M_0 of the set M we distinguish the vertex of the cube at which $x_i = 1$ if $f_i(M_0) > 0$, $x_i = 0$ if $f_i(M_0) < 0$, $i \in I$. Let X denote the set of all such vertices of the cube. It is clear that the ring of integer-valued functions on X is isomorphic to P. Theorem 5 shows the system of equations that distinguish X as a subset of Euclidean space. Namely, if we consider in Euclidean space the system of equations in Theorem 5, then the set of solutions to it coincides with X.

We give a basis of P over Z. A subset $J = (j_1, \ldots, j_k) \subset I$ is called a cycle if the covectors $f_{j_1}, \ldots, f_{j_k}$ are linearly dependent, but this is not so for any proper subset of J.

We fix a linear ordering on I. A subset $J \subset I$ is called an open cycle if there is an index $J_0 \in I$ such that $(j_0, j_1, \ldots, j_k)$ is a cycle and j_0 is less than any element in J.

With each subset $J \subset I$ we associate a monomial $x_{j_1} \ldots x_{j_k} \in P$. To the empty subset there corresponds $1 \in P$.

THEOREM 7. The system of all monomials that correspond to subsets of I not containing open cycles is a basis over Z for P. Moreover, the system of all distinguished monomials of degree not higher than k is a basis in P^k, $k \geq 0$.

Theorem 7 is the analogue of Theorem A.1 in [7], which goes back to [11]. By Theorem A.1, the system of differential forms $\omega_{j_1} \wedge \ldots \wedge \omega_{j_k}$ for all subsets $J \subset I$ not containing open cycles, is a basis in $H^*(M_C, Z)$. See also this theorem in [12].

6. Comparison with Cohomologies. In this section we define a noncanonical linear map $\pi_k: P^k \to H^k(M_C, Z)$ whose kernel coincides with P^{k-1}. π_k is defined by the choice of the coorientation of all ribs of codimension k.

We fix the coorientation of all ribs of codimension k. We define the image of a monomial $x = x_{i_1} \ldots x_{i_\ell}, \ell \leq k$. We put $\pi_k(x) = 0$ if $\ell < k$ or if $\ell = k$ and $df_{i_1} \wedge \ldots \wedge df_{i_k} = 0$. If $l = k$ and $df_{i_1} \wedge \ldots \wedge df_{i_k} \neq 0$, then we set $\pi_k(x) = \pm[\omega_{i_1} \wedge \ldots \wedge \omega_{i_k}]$, where the plus sign is taken if the fixed coorientation of the rib $f_{i_1} = \ldots = f_{i_k} = 0$ coincides with the coorientation defined by the form $df_{i_1} \wedge \ldots \wedge df_{i_k}$; otherwise the minus sign is taken.

THEOREM 8. π_k can be properly extended to a linear map $P^k \to H^k(M_C, Z)$ whose kernel coincides with P^{k-1}.

We define π_k geometrically. We put $\pi_k(x) = 0$ if $\ell < k$ or if $\ell = k$ and $df_{i_1} \wedge \ldots \wedge df_{i_k} = 0$. If $\ell = k$ and $df_{i_1} \wedge \ldots \wedge df_{i_k} \neq 0$, then we put $\pi_k(x)$ equal to the following linear function on $H_k(M_C, Z)$, namely, the index of the intersection of the classes in $H_k(M_C, Z)$ with a noncompact $(2n - k)$-dimensional cycle $\{v \in M_C \mid f_{i_1}(v) > 0, \ldots, f_{i_k}(v) > 0\}$, whose orientation is defined by the complex orientation on $\{v \in M_C \mid f_{i_1}(v) = 0, \ldots, f_{i_k}(v) = 0\}$ and the fixed earlier coorientation of the rib $\{v \in V \mid f_{i_1}(v) = 0, \ldots, f_{i_k}(v) = 0\}$ in V.

7. Ring of Functions That Are Constant on Faces. Let S be a given configuration of hyperplanes in n-dimensional real affine space V. We consider the ring Q(S, Z) of integer-valued functions on V that are constant on each face of S. We consider in the ring Q the multiplicative generating functions (Heaviside functions) x_i, X_i, $i \in I$, defined by the condition: $x_i(v) = 1$ if $f_i(v) > 0$, $x_i(v) = 0$ if $f_i(v) \leq 0$; $X_i(v) = 1$ if $f_i(v) = 0$, $X_i(v) = 0$ if $f_i(v) \neq 0$.

In other words, the $\{x_i\}$ are the functions defined by the conditions of subsection 1.1; the $\{X_i\}$ are the characteristic functions of the hyperplanes of the configuration. Every $x \in Q(S, Z)$ can be written as a polynomial in the $\{x_i, X_i\}$, $i \in I$, with integer coefficients. By the degree of a function $x \in Q(S, Z)$ we mean the minimum of the degrees of the polynomials in $\{x_i, X_i\}$ that represent x. We define the degree filtration $0 \subset Q^0 \subset Q^1 \subset \ldots \subset Q$, where Q^k is the subspace of functions representable by polynomials of degree not higher than k.

The properties of the ring Q and its filtration are analogous to those of the ring P. We discuss in detail the analogues for Q of Theorems 1-4. It is not difficult to produce the analogues of Theorems 5-8.

8. Chains. An integer linear combination of faces is called an integer chain. An integer linear function on the linear space of chains is called an integer cochain. The functions in Q(S, Z) are in (1, 1) correspondence with the chains of a configuration: $x \in Q \to \Sigma x(\Delta)\Delta$, where the summation is over all faces Δ of the configuration. The functions in P(S, Z) are in (1, 1) correspondence with the n-dimensional chains of a configuration; this correspondence is linear.

Our account is presented in the geometrical language of chains and cochains.

Let $C_k(S)$ be the linear space of integer k-dimensional chains over Z, $C^k(S)$ be the linear space of integer k-dimensional cochains over Z, and $C_k^{comp}(S) \subset C_k(S)$ be the subspace of integer linear combinations of bounded k-dimensional faces.

The degree and dimensional filtrations in the space of chains $C_*(S) = \bigoplus_{k=0}^{n} C_k(S)$ that are defined below occupy a key position in this article.

We set $D_k(S) = C_0(S) \oplus C_1(S) \oplus \ldots \oplus C_k(S)$, $k \geqslant 0$. Then $0 \subset D_0(S) \subset D_1(S) \subset \ldots \subset D_n(S) = C_*(S)$; we call this filtration a dimensional filtration.

We say that a configuration S_1 is embedded in a configuration S_2 if the union of the hyperplanes of S_1 is contained in the union of the hyperplanes of S_2. Any face of S_1 is represented, as a set, in the form of a sum of faces of S_2. Thus there is defined a natural embedding of chains $C_*(S_1) \hookrightarrow C_*(S_2)$ that preserves the dimensional filtration.

Example. Let V be one-dimensional, S_1 be the point a, and S_2 the two points a < b. Then the face $\{v \subset V : a < v\}$ of S_1 is the sum of three faces of S_2, namely, $\{v \in V : a < v < b\} + \{b\} + \{v \in V : b < v\}$.

A configuration for which all the hyperplanes pass through a single point and intersect normally is called elementary.

Let S_1 be elementary and consist of k hyerplanes. Clearly, $k \leq n$. Assume that S_1 is embedded in S_2. The images of the faces of the construction of S_1 in $C_*(S_2)$ are called the elementary chains of degree k of S_2. The elementary chain of degree 0 is the whole space V.

Example. A hyperplane and the open half-space bounded by a hyperplane are elementary chains of degree 1. A vertex of a configuration is an elementary chain of degree n.

We define the subspace $W_k \subset C_*(S)$ as the linear hull of the elementary chains of degree $\leq k$, $0 \leq k \leq n$. Then

$$0 \subset W_0 \subset W_1 \subset \dots \subset W_n \subset C_*(S).$$

We call this filtration a degree filtration.

We set $\mathrm{gr}_l D = D_l / D_{l-1}$, $W_k\, \mathrm{gr}_l D = (W_k \cap D_l + D_{l-1})/D_{l-1}$, $\mathrm{gr}_k W\, \mathrm{gr}_l D = W_k \mathrm{gr}_l D / W_{k-1} \mathrm{gr}_l D$. $\mathrm{gr}_\ell D$ is canonically isomorphic to $C_\ell(S)$, $\{W_k \mathrm{gr}_\ell D\}$, $k \geq 0$, is the degree filtration induced on $\mathrm{gr}_\ell D$, and $\{\mathrm{gr}_k W \mathrm{gr}_\ell D\}$, $k \geq 0$, are its factor-spaces.

It is not difficult to see that Theorems 1-8 are assertions about the induced degree filtration on $\mathrm{gr}_n D$. See Sec. 4 for generalizations of Theorems 1-4; generalizations of THeorems 5-8 are easily produced.

9. Remarks. Theorems 1-4 are proved in Sec. 4, and Theorems 5-8 in Sec. 5.

We give in the supplement a multidimensional generalization of the theorem on the expansion of a rational function into the simplest fractions, which are conceptually connected with the geometrical constructions of this article.

2. Chains of a Configuration

In this section we assemble the combinatorial information in preparation for the proof of the theorems.

1. Cones and Corners. A configuration is called regular if it has at least one vertex, and central if its hyperplanes have a nonempty intersection. Any chain of a regular central configuration is called a linear cone, and any chain of a nonregular central configuration a corner. A corner has the form of the direct product of a line and the chain induced from the given configuration on a hyperplane of general position.

A cone with vertex v in a configuration S is any chain of the configuration S_v regarded as a chain in S. A corner with a rib F of S is any chain of the configuration S^F regarded as a chain in S. We recall that S^v, S^F are localizations of the configuration at the ribs v, F.

2. Linear Function and a Configuration. A face of a configuration is said to be bounded above relative to a linear function ϕ defined on V if the face lies in a suitable half-space $\phi \leq$ const. A skeleton of a configuration S relative to ϕ is the set of all faces that are bounded above; we denote it by S_ϕ. Let $C_*(S_\phi)$ stand for the space of integer linear combinations of faces from S_ϕ.

An affine localization $S^{v,\phi}$ of a configuration S relative to $v \in V$ and a linear function ϕ is the configuration cut out on the level hyperplane $\{x \in V : \phi(x) = \phi(v) - 1\}$ from the configuration S^v (Fig. 2).

Let Γ be a bounded face of $S^{v,\phi}$. We consider the cone with vertex v and direction Γ from which the vertex v is discarded. We denote this set by $K(\Gamma, v)$. The map $p: \Gamma \to K(\Gamma, v)$ defines a monomorphism of the space $C_*^{\mathrm{comp}}(S^{v,\phi})$ into the subspace of chains of S^v that are bounded from above.

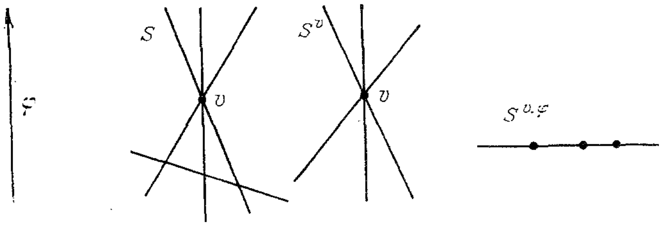

Fig. 2

A linear function on V is said to be a function of general position relative to a vertex v of a configuration S if it is nonconstant on the ribs of positive dimension that pass through v. It is a function of general position relative to S if it is nonconstant on all ribs of positive dimension and has pairwise distinct values at the vertices.

LEMMA 1. Let ϕ be a function of general position relative to a vertex v. Then p defines an isomorphism $C_k^{\text{comp}}(S^{v,\varphi}) \simeq C_{k+1}(S_\varphi)$.

3. Bolted Configurations. A configuration is said to be bolted if each of its hyperplanes intersects each rib of positive dimension. We state the following obvious properties.

LEMMA 2. 1. Let S be a bolted configuration, and U be a hyperplane that intersects all ribs transversally. Then S_U is bolted.

2. Let S be an arbitrary configuration, and ϕ be a function of general position relative to a vertex v. Then $S^{v,\phi}$ is bolted.

3. Let S be a bolted configuration, and F be a rib of S. Then S_F is bolted.

4. Let S be a bolted configuration and ϕ be a linear function that vanishes on a hyperplane $A \in S$. Then ϕ is a function of general position relative to all vertices of S that do not lie in A.

4. Distinguished Substars (cf. [9]). Let $f_1, \ldots, f_N$ be an ordered set of linear functions on an n-dimensional affine space V, and S be the configuration defined by them. We define a linear ordering of the vertices of S as follows: $v < w$ if for some k $(f_j(v))^2 = (f_j(w))^2$ for $j < k$, and $(f_k(v))^2 < (f_k(w))^2$. The set of all bounded faces for which a given vertex is a distinguished vertex is called the distinguished substar of the vertex. The multiplicity of a vertex is the dimension of its distinguished substar. We describe the distinguished substar of a vertex of a bolted configuration.

Let v be a vertex of a configuration S. For any face Γ^v of S^v we can find a unique face Γ of S whose germ at v coincides with the germ of Γ^v at v. The faces Γ^v, Γ are said to be mutually induced at v.

LEMMA 3. Let v be a vertex of an ordered bolted configuration, and let $f_1(v) > 0$. Then a face Γ belongs to the distinguished substar of v if and only if Γ is induced from a face of S^v that is bounded above relative to f_1.

The proof is obvious.

Similarly, let $v \in A_1 \cap A_2 \cap \ldots \cap A_{k-1}$, $f_k(v) > 0$. We consider the configuration $\tilde{S}$ cut by $A_1 \cap \ldots \cap A_{k-1}$ from S.

LEMMA 4. A face Γ belongs to the distinguished substar of a vertex v if and only if Γ is induced from a face of $\tilde{S}^v$ that is bounded above relative to f_k.

COROLLARY. In a bolted regular configuration there is exactly one vertex of zero multiplicity.

5. Euler Characteristic of Certain Chains. Let Γ be a face of a configuration S, Γ^* be the cochain equal to 1 on Γ and to 0 on the remaining faces, and $d(\Gamma)$ be the dimension of the face. The cochain $\chi = \Sigma(-1)^{d(\Gamma)}\Gamma^*$. where the sum is over all faces, is called the cochain of the Euler characteristic.

Let Δ be the chain that is equal to the sum of all the bounded faces of S. Let F be a rib of S, Γ a nonclosed face of S^F, and Δ_F be the chain equal to the sum of all the bounded faces of S that lie in Γ.

THEOREM 9. Let S be a regular bolted configuration. Then $\chi(\Delta) = 1$ and $\chi(\Delta_\Gamma) = 0$.

Proof. This is by induction on the dimension of a configuration. If dim V = 1, then the theorem obviously holds. We assume that the theorem has been proved for bolted configurations in a space of dimension not greater than n − 1, and prove it for dimension n. Here it is enough to analyze the case $d(\Gamma) = n$.

We enumerate the hyperplanes of S so that first come those on which lie the (n − 1)-dimensional faces of the polytope Γ.

We prove that $\chi(\Delta) = 1$. Let $\Delta(v)$ be the distinguished substar of the vertex v. We have $\Delta = \sum_v \Delta(v)$. For the unique vertex v of multiplicity 0 we have $v = \Delta(v)$ and $\chi(\Delta(v)) = 1$. We prove that $\chi(\Delta(v)) = 0$ if the multiplicity of the vertex is positive. In fact, in this case, by Lemmas 1-4 the k-dimensional faces of the distinguished substar for k > 0 are in a (1, 1) correspondence with the (k − 1)-dimensional bounded faces of a suitable bolted configuration in a space of dimension less than n. It follows from the induction hypothesis that the Euler characteristic of the distinguished substar is 0.

We prove that $\chi(\Delta_\Gamma) = 0$. Let $\Delta_\Gamma(v)$ denote the sum of the faces of the distinguished substar of a vertex v lying in Γ. We have $\Delta_\Gamma = \sum_v \Delta_\Gamma(v)$, where the sum is over all vertices lying in the closure of the set Γ. If a vertex v belongs to the interior of Γ, then its multiplicity is positive, its distinguished substar coincides with $\Delta_\Gamma(v)$, and by what was proved earlier, $\chi(\Delta_\Gamma(v)) = 0$. We prove that $\chi(\Delta_\Gamma(v)) = 0$ when v belongs to the boundary of Γ and $\Delta_\Gamma(v)$ is nonempty. In this case v does not belong to the first hyperplanes from S, $\Delta_\Gamma(v)$ consists of faces of positive dimension. Suppose, for definiteness, that $f_1(v) > 0$. By Lemmas 3 and 1 the faces of $\Delta_\Gamma(v)$ are in (1, 1) correspondence with the bounded faces of the affine localization S^{v,f_1} on the hyperplane $H = \{x \in V \mid f_1(x) = f_1(v) - 1\}$ that lie in a certain set $\Gamma(v)$ defined by v and Γ. We describe $\Gamma(v)$ and show that Theorem 1 can be applied to it; thereby we shall prove Theorem 1 for dim V = n.

Suppose that in a small neighborhood of v the face Γ is defined by the inequalities $f_{i_1} > 0,\ f_{i_2} > 0, \ldots, f_{i_k} > 0$. Here the hyperplanes $A_{i_1}, \ldots, A_{i_k} \in S^F$, and v belongs to $A = A_{i_1} \cap \ldots \cap A_{i_k}$. Then $\Gamma(v)$ is distinguished on H by the conditions $f_{i_1} > 0, \ldots, f_{i_k} > 0$. The set A contains F, which lies in A_1. Therefore, $\Gamma(v)$ is bounded by hyperplanes that have a nonempty intersection in H, and so is the union of the nonclosed faces of the configuration $(S^{v,f_1})^{A\cap H}$. It follows from Theorem 1 for $\Gamma(v)$ on H that $\chi(\Delta_\Gamma(v)) = 0$.

3. An Expansion into Cones

1. THEOREM 10. Let ϕ be a linear function of general position relative to a configuration S, and Δ be the chain of the faces that are bounded above. Then Δ can be represented as a sum of the cones of S that are bounded above:

$$\Delta = \sum_v K_v, \tag{3}$$

where the summation is over all the vertices of the configuration. The expansion (3) is unique. In (3) the dimension of each cone is not greater than the dimension of Δ.

Proof. The values of the function at the vertices define an ordering of the vertices. In a neighborhood of the greatest vertex v of the chain Δ the chain Δ has the form of a cone K_v of S^v that is bounded above. We subtract this cone from Δ. We proceed similarly with the remainder of the chain $\Delta - K_v$. The uniqueness of the representation is obvious.

For a description of the cones K_v in (3) in terms of the behavior of the chain Δ in a neighborhood of v see Sec. 3.5.

COROLLARY. The natural map $\bigoplus_{F\in\mathscr{E}} C_*(S^F) \to C_*(S)$ is an epimorphism.

Proof. By using Theorem 10 it is not difficult to show that each face of a configuration S is a sum of cones (if it has a vertex), or a sum of corners.

Let I(S) denote the subspace of chains that is generated by the corners of the configuration. In other words, I(S) is the image of the natural map $\bigoplus_{\substack{F\in\mathcal{L} \\ r(F)<n}} C_*(S^F)\to C_*(S)$. The dimensional filtration induces a filtration on I: $I_\ell(S) = I(S) \cap D_\ell(S)$, $\ell \geq 0$. Let CI(S) denote the factor-space $C_*(S)/I(S)$. The dimensional filtration $CI_\ell(S) = (D_\ell + I)/I$ is defined on CI(S).

THEOREM 11. Let ϕ be a linear function and Δ a chain of dimension not higher than ℓ, $\ell \geq 0$. Then we can find a linear combination $\Sigma\Gamma_\alpha$ of corners, each of which has dimension not higher than ℓ, such that $\Delta - \Sigma\tilde{\Gamma}_\alpha$ is bounded above relative to ϕ. In other words, the natural map $C_\ell(S_\phi) \to CI_\ell(S)$ is an epimorphism.

Proof. Let t_0 be such that for any $t > t_0$ the hyperplane of level t of ϕ intersects all ribs of S transversally. We consider the configuration S_U cut on the hyperplane $U = \{x \in V: \phi(x) = t_0\}$.

Let Γ be the cone of S_U with the vertex v. Then we can find a corner $\tilde{\Gamma}$ of S whose intersection with U gives Γ. A rib of $\tilde{\Gamma}$ is one-dimensional and passes through v. Similarly, let Γ be a corner of S_U. Then we can find a corner $\tilde{\Gamma}$ of S whose intersection with U gives $\tilde{\Gamma}$. The intersection of a rib of $\tilde{\Gamma}$ gives a rib of Γ.

Let Δ be the original chain and $\Delta \cap U$ its intersection with U. By the Corollary to Theorem 10, $\Delta \cap U$ is representable as a linear combination of corners and cones of S_U: $\Delta \cap U = \Delta\Gamma_\alpha$. Let $\tilde{\Gamma}_\alpha$ be a corner of S for which $\Gamma_\alpha = \tilde{\Gamma}_\alpha \cap U$. It is not difficult to see that the chain $\Delta - \Sigma\tilde{\Gamma}_\alpha$ is bounded above.

2. Skew Cochains. A cochain of a configuration is called localized at a given vertex if it is zero on any face not in the star of the vertex. A cochain is said to be skew if it is zero at any corner of the configuration.

Many skew localized cochains can be produced.

Let ϕ be a linear function on V. We say that the cochain $\chi_\varphi = \sum_{\Gamma^*:\, \varphi(\Gamma)\leqslant 0} (-1)^{d(\Gamma)}\Gamma^*$ is associated with ϕ. The value of this cochain on an arbitrary chain is equal to the Euler characteristic of the part of the chain that falls in the half-space $\phi \leq 0$. By the Euler cochain of a configuration S we mean that cochain associated with a linear function of general position. The set of Euler cochains is finite.

THEOREM 12. An Euler cochain is zero at any corner.

Proof. Let χ_ϕ be an Euler cochain and Δ a corner with the rib F. It is enough to consider the case when Δ is a face of dimension n of S^F.

Let Γ be a face of S that lies in $\Delta \cap \{q \leq 0\}$. Suppose that the maximum of ϕ, which is bounded on the closure of Γ, is attained at the vertex v. We denote by $\Delta(v)$ the sum of all such faces. Then $\chi(\Delta) = \sum_v \chi(\Delta(v))$. We prove that $\chi(\Delta(v)) = 0$.

If v belongs to the interior of the set Δ, then $\Delta(v)$ consists of all faces of the star of the vertex v on which ϕ attains its maximum at v. Such faces of positive dimension are in (1, 1) correspondence with the bounded faces of the affine localization $S^{v,\phi}$. Now the equality $\chi(\Delta(v)) = 0$ follows from the first part of Theorem 9.

If v belongs to the boundary of Δ, then $\chi(\Delta(v)) = 0$ follows similarly from the second part of Theorem 9.

3. Linear Combinations of Euler Cochains. Let ϕ be a function of general position, and $t_1 < t_2 < \ldots < t_N$ be its critical values, that is, values at the vertices of S. We consider the Euler cochain $\chi_{\phi-t}$. It is unchanged for $t \in [t_j, t_{j+1})$, $\chi_{\varphi-t} \equiv 0$ for $t < t_1$. If v is a vertex at which $\phi(v) = t$, then we put $\chi_{\varphi, v} = \chi_{\varphi-t+\varepsilon} - \chi_{\varphi-t-\varepsilon}$, where ε is a small positive number. $\chi_{\phi,v}$ is a cochain localized at v. More precisely, $\chi_{\varphi, v} = \Sigma\, (-1)^{d(\Gamma)}\Gamma^*$, where the summation is over all the faces of the star of v on which ϕ attains its maximum at v. That is,

in the sum we take all the faces that go from v to the side of decrease of ϕ. We call the cochain $\chi_{\phi,v}$ the <u>local Euler cochain with center at</u> v.

THEOREM 13. Let u, v be distinct vertices of a configuration S, K_u the cone of S with vertex u, and $\chi_{\phi,v}$ the local Euler cochain with center v. Then $\chi_{\phi,v}(K_u) = 0$.

Proof. In a neighborhood of v the cone K_u looks like a corner. Therefore, by Theorem 12 we have $\chi_{\phi,v}(K_u) = 0$.

4. There Are Sufficiently Many Euler Cochains. THEOREM 14. Let Δ be a nonzero chain of a configuration S that is bounded above relative to a function ϕ of general position. Then we can find a linear combination of Euler cochains whose value on Δ is nonzero.

COROLLARIES. 1. A nonzero chain bounded above cannot be a linear combination of corners, that is, the natural map $C_*(S_\phi) \to CI(S)$ is an isomorphism.

2. Every skew cochain is a linear combination of Euler cochains. In other words, the Euler cochains generate the dual space of CI(S).

Proof. This is by induction on the dimension of the configurations. If dim V = 1, then the theorem obviously holds. We prove the inductive step.

We expand Δ as a sum of cones bounded above relative to ϕ: $\Delta = \sum_v K_v$. We distinguish a vertex v at which a zone is nonzero. By Theorem 13 it is enough to prove the existence of a linear combination of local Euler cochains with center v whose value of K_v is nonzero.

If the vertex v occurs in K_v with the coefficient λ, then $\chi_{-\varphi,v}(K_v) = \lambda$. If $\lambda \neq 0$, then the theorem is proved. Therefore we assume that the coefficient of v is zero.

Let $\chi_{\alpha,v}$ be a local Euler cochain. The isomorphism of Lemma 1 carries it into a certain cochain ψ on the bounded chains of the localizations $S^{v,\phi}$.

LEMMA 5. ψ is the Euler cochain of $S^{v,\phi}$ associated with the linear function $\alpha - \alpha(v)$ that is bounded on the space of the affine localization.

Proof. This is obvious.

We return to the cone K_v. It induces in $S^{v,\phi}$ a nonzero bounded chain K. By the induction hypothesis, for $S^{v,\phi}$ we can find a linear combination of Euler cochains of it that is nonzero on K. This, together with Lemma 5, proves the theorem.

5. Characterization of the Cones in an Expansion (3). Let ϕ be a linear function of general position on S, Δ a chain that is bounded above, and $\Delta = \sum_v K_v$ be an expansion as a sum of cones of S that are bounded above relative to ε.

THEOREM 15. 1. For any local Euler cochain $\chi_{\alpha,v}$ with center v we have $\chi_{\alpha,v}(\Delta) = \chi_{\alpha,v}(K_v)$.

2. Let K be a cone of S that is bounded above relative to ϕ. Assume that $\chi_{\alpha,v}(\Delta) = \chi_{\alpha,v}(K)$ for any local Euler cochain $\chi_{\alpha,v}$ with center v. Then $K = K_v$.

The first part of the theorem follows from Theorem 13, and the second from Theorem 14.

6. Combinatorial Connectedness (a Corollary of Theorem 14). Let ϕ_1, ϕ_2 be linear functions of general position. The next two assertions follow from Corollary 1.

COROLLARY 3. $C_*(S_{\phi_1})$ and $C_*(S_{\phi_2})$ are canonically isomorphic.

Let S be a central regular configuration with vertex v, and ϕ_1, ϕ_2 be linear functions of general position.

COROLLARY 4. $C_*^{\text{comp}}(S^{v,\varphi_1})$, $C_*^{\text{comp}}(S^{v,\varphi_2})$ are canonically isomorphic.

Example 1. Let S be a regular central configuration, and ϕ a function of general position. We give the form of the isomorphism π: $C_*(S_\phi) \to C_*(S_{-\phi})$: a cone going downwards relative to ϕ is turned by the addition of corners into a cone going upwards.

THEOREM 16 (cf. [9, Theorem 7]). Let $\Gamma \in C_*(S_\phi)$ be a k-dimensional face. Then $\pi(\Delta) = (-1)^k\tilde{\Gamma}$, where $\tilde{\Gamma}$ is the closure of the reflection of Γ in a vertex v (Fig. 3).

Example 2. We consider in R^3 the configuration that consists of four planes through the origin and in general position. If ϕ is a function of general position then $S^{v,\phi}$ is a

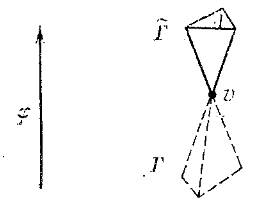

Fig. 3

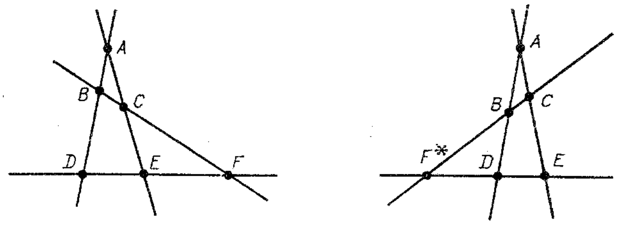

Fig. 4

plane configuration consisting of four lines in general position. For suitable ϕ_1 and ϕ_2 the configurations S^{v,ϕ_1} and S^{v,ϕ_2} are shown in Fig. 4. We indicate the isomorphism of the bounded chains: $F \mapsto -F^*$, $CF \mapsto -CF^* - C$, $EF \mapsto -EF^* - E$, $CEF \mapsto CEF^* - CE$, where CF, CF^*, EF, EF^* are intervals, and CFE and CFE^* are open triangles. The remaining faces go into themselves: $A \to A$, etc.

4. Dimensional and Degree Filtrations

1. Expansion into Simplexes. THEOREM 17. If S is a bolted configuration, then $C_*^{comp}(S)$ has a basis of open simplexes of various dimensions. This basis has the property: the expansion of a given chain of C_*^{comp} of dimension not higher than ℓ in this basis contains simplexes of dimension not higher than ℓ.

Proof. This is by induction on the dimension of the space. For $\dim V = 0$ the unique face is a point. Let $\dim V = n > 0$. We order the hyperplanes of the configuration. Let A be the hyperplane with the smallest number. By the induction hypothesis there exists a basis in $C_*^{comp}(S_A)$. We complete it to a basis for $C_*^{comp}(S)$. Let v be a vertex outside A. We include the 0-dimensional chain v in the basis. Let ϕ be a linear function for which $\phi(A) = 0$ and $\phi(v) > 0$. We consider the affine localization $S^{v,\phi}$. Its hyperplanes are ordered. We choose by the induction construction a simplified basis in $C_*^{comp}(S^{v,\phi})$. With each element Γ of this basis we link the simplex $K(\Gamma)$, which is the cone with vertex v, direction Γ and base lying in A; $K(\Gamma)$ does not include v and the lower base. It is easy to see that the basis in $C_*^{comp}(S_A)$ together with the simplexes constructed for all verticles outside A forms a basis for $C_*^{comp}(S)$ with the property in the theorem (see the proof of Theorem 10).

THEOREM 18. For any configuration S of hyperplanes, $C_*(S)$ has a basis of elementary chains. This basis has the property: a chain of dimension not higher than ℓ is expanded as a sum of elementary chains of dimension not higher than ℓ.

Proof. This is carried out by induction on the dimension of the configuration, and follows easily from Theorem 17 (cf. Theorems 10 and 11 and Lemma 1).

2. Properties of Dimensional and Weighted Filtrations in Chains.

1. $W_n = C_*(S)$.

2. D_ℓ is the linear hull of elementary chains of dimension not higher than ℓ for $\ell \geq 0$.

3. If $S_1 \subset S_2$ and $i: C_*(S_1) \to C_*(S_2)$ is the natural embedding, then $i(W_k(S_1)) \subset W_k(S_2)$ and $i(D_\ell(S_1)) \subset D_\ell(S_2)$ for $k, \ell \geq 0$.

4. $W_k(S)$ coincides with the image of the natural map

$$\bigoplus_{F \in L,\ r(F) \leqslant k} C_*(S^F) \to C_*(S), \tag{4}$$

where S^F is the localization of S at the rib F, and $r(F)$ is the codimension of this rib.

5. If $U \subset V$ is a subspace and j_U: $C_*(S) \to C_*(S_U)$ denotes the natural epimorphism, then $j_U(W_k(S)) = W_k(S_U)$.

Property 3 follows from the definition of a filtration, and Properties 1 and 2 from Theorem 18.

We prove Property 4. Clearly, $W_k(S)$ lies in the image of the map (4). By Property 1, $C_*(S^F) = W_{r(F)}(S^F)$. Hence $W_k(S)$ coincides with the image of the map (4).

Property 4 can serve as a definition of a degree filtration.

We prove Property 5. Every elementary chain of degree k in $C_*(S_U)$ is the image of an elementary chain of degree k in $C_*(S)$. Hence $j_U(W_k(S)) \supset W_k(S_U)$. The image of an elementary chain of degree k is a face of the configuration on U that consists of not more than k hyperplanes. It follows from Property 4 that this image belongs to $W_k(S_U)$.

6. If $\Delta \in W_k \cap D_\ell$, then Δ is representable as a linear combination of elementary chains of degree not higher than k and of dimension not higher than ℓ.

Proof. This is by induction on the dimension of the configuration. For $n = 0$ the property is true for any k and ℓ. Let $n > 0$. Let ϕ be a linear function of general position relative to the configuration S. Let t_0 be such that for any $t \geq t_0$ the hyperplane of level t of ϕ intersects all ribs transversally. We consider the configuration S_U cut on the hyperplane $U = \{x \in V: \phi(x) = t_0\}$. Then the chain $\Delta \cap \overline{U}$ has dimension $\leq \ell - 1$.

Suppose that $k < n$. Then $\Delta \cap U \in W_k(S_U) \cap D_{l-1}(S_U)$. By the induction hypothesis, $\Delta \cap U = \Sigma a_m \Delta_m$, where $a \in \mathbf{Z}$, $\{\Delta_m\} \subset W_k(S_U) \cap D_{l-1}(S_U)$ are elementary chains. For each elementary chain Δ_m there is an elementary chain $\tilde{\Delta}_m \in W_k(S) \cap D_l(S)$ for which $\tilde{\Delta}_m \cap U = \Delta_m$. The intersection of the chain $\tilde{\Delta} = \Delta - \Sigma a_m \tilde{\Delta}_m$ and U is empty. By Theorems 12 and 14, $\tilde{\Delta} = 0$. Thus Property 6 is proved. For $k = n$ this property follows from Theorem 18.

7. $W_k \cap D_\ell = 0$ for $k + \ell < n$.

8. Let $U \subset V$ be a d-dimensional subspace of general position relative to S. Then the homomorphism $j_U: C_*(S) \to C_*(S_U)$ lowers the dimension of every chain by $n - d$, and for any $0 \leq k$ and $\ell \leq d$ defines the isomorphism

$$W_k(S) \cap D_{l+n-d}(S) \to W_k(S_U) \cap D_l(S_U).$$

Proof. It is enough to analyze the case when U is a hyperplane. It clearly follows from the generality of U that $D_{\ell+1}(S) \to D_\ell(S_U)$ and $W_k(S) \to W_k(S_U)$. We prove that $j_U|W_{n-1}$ does not have a kernel. For if $\Delta \in W_{n-1}(S)$ belongs to the kernel of j_U, then $\Delta = \Delta_+ + \Delta_-$, where the $\Delta_\pm$ lie on different sides of U. It follows from Theorems 10, 13 and 14 that $\Delta_+ = \Delta_- = 0$. Thus, j_U defines an embedding $W_k(S) \cap D_{l+1}(S) \hookrightarrow W_k(S_U) \cap D_l(S_U)$ for $k \leq n - 1$. If $\Delta \in W_k(S_U) \cap D_l(S_U)$, then by Property 6 we can find a $\tilde{\Delta} \in W_k(S) \cap D_{l+1}(S)$ for which $j_U(\bar{\Delta}) = \Delta$. This proves Property 8.

9. Under the conditions of Property 8, j_U defines an isomorphism $W_k \operatorname{gr}_{l+n-d} D(S) \to W_k \operatorname{gr}_l D(S_U)$ for any $0 \leq k, \ell \leq d$.

Property 9 follows from Property 8 since $W_k \operatorname{gr}_m D = W_k \cap D_m / W_k \cap D_{m-1}$.

10. For any $k \geq 0$ the natural map

$$\bigoplus_{\substack{F \in \mathscr{L} \\ r(F)=k}} W_k(S^F)/W_{k-1}(S^F) \to W_k(S)/W_{k-1}(S)$$

is an isomorphism.

Proof. Let ϕ be a linear function of general position. By Theorems 10-14 we have $C_*(S_\varphi) = \bigoplus_{\substack{F \in \mathscr{L} \\ r(F)=n}} C_*(S_\varphi^F)$. By Corollary 1 of Theorem 14 this equality implies Property 10 for $k = n$. The case for an arbitrary k reduces to the one we have analyzed by using Property 8. The same argument proves the next property.

11. For any $k, \ell \geq 0$ we set

$$D_l \operatorname{gr}_k W = D_l \cap W_k / D_l \cap W_{k-1}.$$

Then the natural map

$$\bigoplus_{\substack{F \in \mathscr{L} \\ r(F)=k}} D_l \operatorname{gr}_k W(S^F) \to D_l \operatorname{gr}_k W(S)$$

is an isomorphism.

12. A chain of $W_k(S)$ is uniquely defined by its general section of dimension k. More precisely, if $U \subset V$ is a subspace of general position relative to S, dim U = k, and $c \in W_k(S)$ is a chain for which $c \cap U = 0$, then c = 0.

Property 12 follows from Property 8.

13. Let $U_1, U_2 \subset V$ be subspaces of general position relative to a configuration S, and $\dim U_1 = \dim U_2$. Then $C_*(SU_1)$ is canonically isomorphic to $C_*(SU_2)$. The isomorphism is defined by Property 8.

We call this isomorphism a combinatorial connectivity. In [9], combinatorial connectivity is defined in a near situation.

3. Ring P(S, Z) Defined in the Introduction. Properties 1, 9, and 10 give Theorems 1-3.

4. Dual of a Degree Filtration. We define a degree filtration in $C^*(S)$ by the condition $W^k = \operatorname{Ann} W_{k-1}$. We have

$$0 \subset W^n \subset W^{n-1} \subset \ldots \subset W^0 = C^*(S).$$

We give another construction for this filtration. Let $\chi(S) \subset C^*(S)$ be the linear hull of the Euler cochains. Let $U_1, \ldots, U_{n-1}, U_n = V$ be affine subspaces of V of dimension 1, .., n − 1, n, respectively. We assume that each of them is in general position relative to the configuration S. Let $j_k: C^*(SU_k) \to C^*(S)$ be the natural monomorphism.

THEOREM 19. For any $k \geq 0$ we have

$$W^k(S) = \chi(S) + j_{n-1}(\chi(S_{U_{n-1}})) + \ldots + j_k(\chi(S_{U_k})).$$

Theorem 19 follows from the corollary to Theorem 14 and Property 8.

Flag Cochains.

LEMMA 7. An arbitrary flag cochain of degree n is a linear combination of Euler cochains.

Proof. This is by induction on the dimension of the configuration; we have to do down to the affine localization of the 0-dimensional rib of a flag, and to use Lemma 5.

LEMMA 8. The flag cochains of degree n generate the space dual to $C_*(S)/(C_{n-1}(S) + W_{n-1}(C_*(S)))$.

Proof. This is by induction on the dimension of the space. Let ϕ be a linear function of general position. By Theorems 10-14 it is enough to prove that the flag cochains of degree n generate $C^n(S_\phi)$. Moreover, it is enough to consider the case when S is a regular central configuration. Let v be a vertex of it. We go to the affine localization $S^{v,\phi}$. Then $C_n(S_\varphi) \simeq C_{n-1}^{\text{comp}}(S^{v,\varphi})$. The flag cochains of degree n on S go into flag cochains of degree n − 1 on $S^{v,\phi}$. By the induction hypothesis, for a nonzero chain of $C_{n-1}^{\text{comp}}(S^{v,\phi})$ we can find a flag cochain of degree n − 1 that is nonzero on it. This proves the lemma.

Proof of Theorem 4. Theorem 4 for k = n is the same as Lemma 8. The proof for an arbitrary k follows from the lemma and Property 9.

5. COMPARISON WITH COHOMOLOGIES, AND RELATIONS

1. Comparison with Cohomologies. Let S be a configuration of hyperplanes in a real n-dimensional affine space V.

Each oriented n-dimensional face $\Delta \in C_n(S)$ defines a cohomology class $[\Delta] \in H^n(M_C, Z)$ that is equal to the index of the intersection with a noncompact cycle of Δ (we assume that M_C is complex oriented).

We fix an orientation on V, and so on each n-dimensional face. We define a linear map $\pi: C_n(S) \to H^n(M_C, Z)$ that associates with the linear combination $\Sigma a_\alpha \Delta_\alpha$ of faces the class $\Sigma a_\alpha [\Delta_\alpha]$.

To describe the kernel of π we consider the natural isomorphism $i: C_n(S) \to gr_n D$.

THEOREM 20. $\ker \pi = i^{-1}(W_{n-1} gr_n D)$.

Proof. With each flag $F = \{F_0 \subset F_1 \subset \ldots \subset F_n\}$ of degree n we associate the homology class in $H_n(M_C, Z)$ of the torus defined as follows. Let $\varepsilon_1, \ldots, \varepsilon_n > 0$. In C^n with the coordinates $z_1, \ldots, z_n$ we consider the torus $T(\varepsilon) = \{z \in C^n: |z_j| = \varepsilon_j\}$. We fix its orientation. We map R^n with the coordinates $z_1, \ldots, z_n$ affinely onto V so that the standard flag $\{z_1 = \ldots = z_n = 0\} \subset \{z_2 = \ldots = z_n = 0\} \subset \ldots \subset \{z_n = 0\} \subset \mathbf{R}^n$ is mapped onto the flag F. We consider the complexification $C^n \to V_C$ of this map. For $0 < \varepsilon_1 \ll \varepsilon_2 \ll \ldots \ll \varepsilon_n \ll 1$ the torus $T(\varepsilon)$ is mapped into M_C, and defines a homology class that does not depend on ε. We denote it by T_F.

LEMMA 9. We consider the cochain on $C_n(S)$ equal to the index of the intersection with T_F. Then to within a multiplication by ± 1 this cochain is equal to the flag cochain ψ_F.

The proof is obvious.

Let L denote the linear hull of the classes T_F obtained for all possible flags F of degree n. By Lemmas 8 and 9 we have $\dim_Z L = \dim C_n(S)/i^{-1}(W_{n-1} gr_n D)$. In view of Corollary 2, $\dim_Z H_n(M_C, \mathbf{Z}) = \dim_Z C_n(S)/i^{-1}(W_{n-1} gr_n D)$. Hence we obtain Theorem 20 and the assertion $L = H_n(M_C, Z)$.

Example. We consider in R^n the configuration of coordinate hyperplanes $A_j = \{z_j = 0\}$, $j = 1, \ldots, n$. We orient R^n by the form $dz_1 \wedge \ldots \wedge dz_n$. M consists of 2^n octants. The space $C_n(S)/i^{-1}(W_{n-1} gr_n D) = gr_n W gr_n D$ is one-dimensional and generated by the positive octant $\{z_1 > 0, \ldots, z_n > 0\}$. Under the map π the positive octant goes into the cohomology class of the form $(-1)^{n(n-1)/2} \omega_1 \wedge \ldots \wedge \omega_n$, where $\omega_j = dz_j/2\pi\sqrt{-1}z_j$.

We mention a useful consequence of Theorem 20. Let ϕ be a linear function of general position. Then the map π, defined on $C_n(S_\phi)$, gives an isomorphism of the bounded-above n-dimensional chains of the configuration and the spaces $H^n(M_C, Z)$.

Proof of Theorem 8. For k = n Theorem 8 follows from Theorem 20, the preceding example, and Properties 10 and 11. The case k < n follows from the k = n case by using Property 9.

2. Relations. Proof of Theorem 5. The assertion reduces to the case of the cycle $f_1, \ldots, f_{k-1}, f_k = -(f_1 + \ldots + f_{k-1})$ for which we have $x_1 \ldots x_k = 0$ and $(x_1 - 1)\ldots(x_k - 1) = 0$.

Proof of Theorem 6. By Theorem 5 it is enough to prove that $\dim Z[x]/\mathcal{Y} = \dim H^*(M_C)$. The relations $x_i^2 = x_i$ annihilate the monomials in which there is at least one variable of degree greater than 1. We need to prove that all relations on the remaining monomials can be derived from (2). Under the isomorphism $P^s/P^{s-1} \to H^s(M_C)$ the relation (2) goes into a homogeneous relation of degree s − 1:

$$\omega_{i_2} \wedge \ldots \wedge \omega_{i_s} - \omega_{i_1} \wedge \omega_{i_3} \wedge \ldots \wedge \omega_{i_s} + \ldots + (-1)^{s-1} \omega_{i_1} \wedge \ldots \wedge \omega_{i_{s-1}}. \tag{5}$$

By [3], the exterior algebra spanned by the $\{\omega_{i \in I}\}$ and factorized by (5) for all cycles is isomorphic to $H^*(M_C)$. Hence $Z[x]/\mathcal{Y}$ has the required dimension.

Theorem 7 follows from Theorems 6 and 8 and the theorem on bases in [7, 11, 12].

6. SUPPLEMENT. EXPANSION IN THE SIMPLEST FRACTIONS

Theorem 18 on the expansion in simplicial cones has an elementary analytical analogue, namely, a generalization of the theorem on the expansion of a rational function in the simplest fractions.

1. The Expansion. We consider a rational function R = P/Q on an n-dimensional linear space V, where P and Q are polynomials. We assume that Q is expressed as the product of polynomials of degree 1: $Q = \prod_{i \in I} f_i^{k_i}$. Let $z_1, \ldots, z_n$ be linear coordinates in V.

THEOREM 21. R is representable as

$$R=\sum_{t,\alpha}(t_1)^{\alpha_1}\dots(t_n)^{\alpha_n}A_{t,\alpha},\tag{6}$$

where either $\alpha_j \geq 0$ and $t_j = z_j$, or $\alpha_j < 0$ and t_j is a polynomial of the first degree in z_j, z_{j+1}, ..., z_n with 1 as the coefficient of z_j, and the $A_{t,\alpha}$ are numbers. This representation is unique.

Proof. We restrict R to each line parallel to the z_1 axis and expand in the simplest fractions in z_1. This yields a representation

$$R=\sum_{i\in I}\sum_{-k_i\leq p<0} f_i^p B_{p,i}+\sum_l z_1^l C_l,$$

where $B_{p,i}$ and C_ℓ are rational functions of z_2, ..., z_n, and the denominator in each of them is a product of polynomials of the first degree. This representation is unique. Then we proceed similarly for each of the functions $B_{p,i}$, C_ℓ.

Example. $1/(z_1-z_3)(z_1-z_2)(z_1-z_2-z_3)=1/(z_1-z_3)(z_3-z_2)(-z_2)+1/(z_1-z_2)(z_2-z_3)(-z_3)+1/(z_1-z_2-z_3)z_2z_3=1/(z_1-z_3)(z_3-z_2)(-z_3)+1/(z_1-z_3)(-z_2)z_3+1/(z_1-z_2)(z_2-z_3)(-z_3)+1/(z_1-z_2-z_3)z_2z_3$.

Theorem 21 is near to Theorem 5.2 in [7].

Suppose that all the linear functions $\{f_{i\in I}\}$ are homogeneous and the set t_1, ..., t_n occurs in (6) with negative α_1, ..., α_n. We indicate the connection between t_1, ..., t_n and the covectors $\{f_{i\in I}\}$. t_1 is proportional to one of the $\{f_{i\in I}\}$. We project the other covectors in $\{f_{i\in I}\}$ along t_1 onto the hyperplane in V^* orthogonal to the vector (1, 0, ..., 0). We obtain a set of covectors {g}. Then t_2 is proportional to one of them. We project the remaining covectors of {g} along t_2 onto the (n − 2)-dimensional plane orthogonal to the plane in V spanned by (1, 0, ..., 0) and (0, 1, 0, ..., 0). We obtain a set of covectors {h}. Then t_3 is proportional to one of them, and so on.

If the set $\{f_{i\in I}\}$ of covectors is closed under these projection operations, then in (6), for $\alpha_j < 0$ the polynomial t_j is proportional to one of the $\{f_{i\in I}\}$. The following are examples of such families.

1. Type A. Q is the product of powers of the polynomials $z_j - z_k$, $j < k$.
2. Type B. Q is the product of powers of the polynomials z_j, $z_j \pm z_k$, $j \leq k$.

2. Appendix. We consider the linear map h: $R^N \to R^n$ that sends the j-th basis vector into a vector h_j. We assume that all the h_j lie in the half-space $x_1 > 0$, where x_1, ..., x_n are coordinates in R^n. We define on R^n a function U such that U(x) is the (N − n)-dimensional volume of the intersection of a fibre over x and the positive octant in R^N, where the form of the volume is taken to be the special form of the volume on R^N and R^n. As regards this function see [7].

THEOREM 22. For the set $\{h_{j\leq N}\}$ of general position

$$U(x)=\sum_{\substack{1\leq s_1,\dots,s_{n-1}\leq N\\ s_i\neq s_j}}\frac{(h_{s_1},\dots,h_{s_{n-1}},x)^{N-n}\chi_{s_1,\dots,s_{n-1}}(x)}{(N-n)!\prod\limits_{\substack{s_n\notin\{s_1,\dots,s_{n-1}\}\\ 1\leq s_n\leq N}}(h_{s_1},\dots,h_{s_n})},\tag{7}$$

where $(v_1, \dots, v_n)$ is the determinant whose columns are v_1, ..., v_n, where $\chi_{s_1\dots,s_{n-1}}$ is the characteristic function of the simplicial cone generated by the vectors h_{s_1}, $h^n_{s_1,s_2}$, ..., $h_{s_1,\dots,s_n}$, where $h_{s_1,\dots,s_p}$ for $p > 1$ is the projection of h_p along $\{h_1, \dots, h_{p-1}\}$ onto the coordinate plane {(0, ..., 0, 1_p, 0, ..., 0), ..., (0, ..., 0, 1)}, taken with the sign for which the p-th coordinate is positive.

In particular, for n = 2, χ_j is the characteristic function of the cone generated by the vectors h_j, (0, 1).

For an arbitrary n and N = n, (7) becomes

$$U(x)=\sum_{s\in S_n}\chi_{s_1,\ldots,s_{n-1}}(x)/(h_{s_1},\ldots,h_{s_n}).$$

In this case U(x) is piecewise constant and proportional to the characteristic function of the cone generated by the vectors $h_1, \ldots, h_n$.

Remark. If in (7) we discard χ, then the sum vanishes identically. For example, for n = 2, $f_j = (1, a_j)$ we have

$$\sum_{i=1}^{N}\frac{(x_2-a_ix_1)^{N-2}}{(a_1-a_i)\ldots(a_{i-1}-a_i)(a_{i+1}-a_i)\ldots(a_N-a_i)}\equiv 0.$$

To prove the theorem it is enough to consider the Laplace transform of U(x) (see [7]), expand the resulting rational function in the simplest fractions, and to take the inverse Laplace transformation of the terms.

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