TWO-DIMENSIONAL "INVERSE SCATTERING PROBLEM" FOR NEGATIVE ENERGIES AND GENERALIZED-ANALYTIC FUNCTIONS.

I. ENERGIES BELOW THE GROUND STATE

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INTRODUCTION

The two-dimensional inverse problem of reconstructing the general (without the hypothesis of continuous symemetry) Schrödinger operator $L = -\sum_{\alpha} (\partial_\alpha - iA_\alpha)^2 + u$ on the basis of data, "collected" from the family of eigenfunctions of one energy level $L\Psi = \varepsilon_0\Psi$, was first considered in 1976 in [1] for the periodic case. From [2] the idea arose of a profound connection of this problem with integrable problems of the theory of solitons in dimension 2 + 1. [3-5] are devoted to the development of this approach in the periodic case and later [6, 7] in the rapidly decreasing one. The contemporary stage of research began with [3, 4] in 1984, where, in the periodic case, the group of reductions $\mathbf{Z}_2 \times \mathbf{Z}_2$ was explicitly found, singling out those data of the inverse problem from which one gets purely potential self-adjoint operators (1)

$$n = 2, \quad L = -\Delta + u, \quad u = \bar{u}, \quad L\Psi = \varepsilon_0\Psi, \quad A_\alpha = 0. \tag{1}$$

The analog of these results, it is true only for rapidly decreasing potentials of sufficiently small norm (of type $\int |u(z)|^{1+\varepsilon_1}(1+|z|)^{\varepsilon_2}\, dz\, d\bar{z},\ \varepsilon_2 > 2\varepsilon_1$)), was found in [4] together with the group of reductions $\mathbf{Z}_2 \times \mathbf{Z}_2$, singling out operators of the form (1), where $u(x, y) \to x^2 + y^2 \to \infty$. Here [6, 7] used a number of important technical considerations from [8]. In both problems, the periodic and the rapidly decreasing, at the base lie the manifolds F and DKN of eigenfunctions of the following form:

$$L\Psi = \varepsilon\Psi, \quad \Psi(\vec{x}, \vec{k}) = e^{i(k, x)}(1 + O(r^{-\alpha_n})),$$
$$\vec{k} = (k_1, \ldots, k_n) = \vec{k}_R + i\vec{k}_I, \vec{k}_I \neq 0, \quad \alpha_n = \frac{n-1}{2} \tag{2}$$

(in the rapidly decreasing case, the manifold of Faddeev functions, "the family F" of [8])

$$L\Psi = \varepsilon\Psi, \ \Psi(\ldots, x_j + T_j, \ldots) = e^{ip_jT_j}\Psi(\vec{x}) \tag{3}$$

(in the periodic case, the complete complex collection of functions of Bloch–Floquet type, first studied in [1], "the family DKN" of Dubrovin–Krichever–Novikov).

The family F is nonanalytic in the variable k_j for $n > 1$. Only the one relation

$$k_{I\alpha}\frac{\partial\Psi(\vec{x}, \vec{k})}{\partial\vec{k}_\alpha} \equiv 0 \tag{4}$$

holds, from which the analyticity in the family of one-dimensional directions follows. Most recently the characterization of analytic properties of the family F $\Psi(\vec{x}, \vec{k})$ is finished [9, 20], but only using large energies $\varepsilon = k^2$ for all $n \geqslant 2$ (cf. also [15, 16]).

The family DKN is holomorphic in all variables p_j, but multivalued: the complete collection of these functions is formed of a single function $\Psi(\vec{x}, \vec{\mathscr{P}})$, meromorphic on a complex n-dimensional "manifold of quasi-impulses" $\mathscr{P} \in W$ first introduced in [1] [locally $\vec{\mathscr{P}} = (p_1, \ldots, p_n)$] together with a "scattering law," a meromorphic function

$$\varepsilon\colon W \to \mathbf{C}. \tag{5}$$

Some theorems (however insufficient for inverse problems), justifying the existence of the manifold W, were proved in [19]. The analytic properties with respect to $\vec{k}$ or $\vec{\mathscr{P}}$ of complete

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n-dimensional complex families of functions F and DKN contain strongly redundant information about the operator L for $n \geqslant 2$; a detailed description of this information would be quite useful.

Problem. How can one single out a minimal collection of information about the analytic properties of the families F and DKN, independent and sufficient for reconstructing the operator L with the help of an effective procedure?

For n = 2 this problem, as already said above, can be solved by collecting data from one energy level $\varepsilon = \varepsilon_0$ for both families F and DKN in a large class of operators L; apparently this class is dense among all potential operators L with smooth periodic real potential, according to a conjecture of Novikov.

The present paper is devoted to the solution of the following problem.

Problem. How can one characterize the collection of data of the inverse problem for one energy level $\varepsilon = \varepsilon_0$, in order that the value of ε_0 be below the ground state (lower boundary of the spectrum $\varepsilon_{\min}$), $\varepsilon_0 < \varepsilon_{\min}$? We do not assume the norm is small.

For the periodic case in the admissible class of "algebrogeometric" or "finite-zoned for one energy" operators L a sufficient test for the inequality $\varepsilon_0 < \varepsilon_{\min}$ was found in [3], but the idea of the proof (still unpublished) was unclear and has nothing in common with the idea of this paper (below), which is general for the periodic and rapidly decreasing cases. In the rapidly-decreasing case $\varepsilon_{\min} \leqslant 0$ always; [7] concentrated only on "physical" energies $\varepsilon_0 > 0$ for potentials of sufficiently small norm. The only paper [6] devoted to the case $\varepsilon_0 < 0$, only discussed a collection of special examples of potentials, where always $\varepsilon_0 = \varepsilon_{\min}$ and the rate of decrease is sluggish ($\sim r^{-1}$).

1. Formulation of Results. Data of the Inverse Problem

In what follows let n = 2 and $\varepsilon = \varepsilon_0$. We denote by Γ the collection of functions of the families F or DKN of one energy $\varepsilon = \varepsilon_0$. The data of the inverse problem are the following:

A. Periodic Case. Let the genus $g(\Gamma)$ be finite and Γ be nonsingular. Then the surface Γ has two distinguished "infinite" points ∞_1 and ∞_2 with local parameters $w_1 = k_1^{-1}$ and $w_2 = k_2^{-1}$, group of reductions $\mathbf{Z}_2 \times \mathbf{Z}_2$ with generators: σ (holomorphic) and τ (antiholomorphic) and collection of poles, a divisor $\mathcal{D}$ of degree g, $\mathcal{D} = Q_1 + \ldots + Q_g$, $Q_j \in \Gamma$.

The collection $(\Gamma, \infty_1, \infty_2, w_1, w_2, \mathcal{D}, \sigma, \tau)$ is called the "data of the inverse problem." It satisfies the requirements

$$g = 2h,\ \sigma(\infty_\beta) = \infty_\beta,\ \sigma(k_\alpha) = -k_\alpha,\ \tau(\infty_1) = \infty_2,$$
$$\tau(k_1) = \bar{k}_2,\quad \tau(\mathcal{D}) = \mathcal{D},\quad \sigma\mathcal{D} + \mathcal{D} \sim K + \infty_1 + \infty_2,\ \beta, \alpha = 1, 2, \tag{6}$$

where the symbol ~ means linear equivalence of divisors, K is the divisor of zeros of holomorphic 1-forms. The function $\Psi(x, y, \mathcal{P})$ is normalized by the condition $\Psi(0, \mathcal{P}) \equiv 1$, it has $\mathcal{D}$ as the divisor of its poles, and asymptotics at the points ∞_α

$$\Psi = e^{k_1 z}(1 + O(w_1)),\qquad \mathcal{P} \to \infty_1,\quad z = x + iy,$$
$$\Psi = e^{(-1)^{\alpha+1} k_2 \bar{z}}(1 + O(w_2)),\quad \mathcal{P} \to \infty_2,\quad \bar{z} = x - iy. \tag{7}$$

A collection of data satisfying (6) and (7) defines exactly two real potentials $\neg\alpha$ = 1, 2

$$L\Psi = \varepsilon_0\Psi,\quad L = -\partial\bar{\partial} + u_\alpha,\quad \varepsilon_0 = \varepsilon_0(\Gamma, \sigma),$$
$$u_1 = -2\partial\bar{\partial} \ln \theta(uz + \bar{u}\bar{z} + \xi_0(\mathcal{D})), \tag{8}$$
$$u_2 = -2\partial\bar{\partial} \ln \theta(uz - \bar{u}\bar{z} + i\xi_0(\mathcal{D})),$$

here $\theta(\eta)$ is the Prym function without characteristics, depending on h = g/2 variables (cf. [3, 5]). The most useful formulas for $\varepsilon_0(\Gamma, \sigma)$ are found in [10], using nonlinear equations with respect to a scheme of the type of [11]. In [3, 4] only one potential $u_2(z, \bar{z})$ is given explicitly. Taimanov and Natanzon gave a refinement. Necessary and sufficient conditions for the smoothness of the potentials u_1, u_2 have not yet been found (they can have a pole).

For the distinguished case $\varepsilon_0 < \varepsilon_{\min}$, i.e., for $\varphi \in \mathcal{L}_2(R^2)$

$$((L - \varepsilon_0)\varphi, \varphi) \geqslant C\,|\,\varphi\,|^2,$$

it is necessary to take the data [4]: the antiholomorphic involution τ: $\Gamma \to \Gamma$ has exactly g + 1 = 2h + 1 fixed ovals $a_1, \ldots, a_h, \ldots, a_{2h}, b$ $\tau/a_j = 1$, $\tau/b = 1$ such that

$$\sigma(a_j) = a_{j+h}, \quad j = 1, \ldots, h, \quad \sigma(b) = b. \tag{9}$$

Here the divisor of poles $\mathscr{D} = Q_1 + \ldots + Q_{2h}$ is such that (9) lies on the oval a_j, j = 1,..., 2h.

THEOREM 1 [4]. If the collection of data satisfies (6), (7), and (9), then the potential u_2 is smooth (without poles) and $\varepsilon_0 < \varepsilon_{min}$.

Probably these conditions are also necessary. The proof which the authors of [3, 4] had in mind was unclear and based on the connection of this family for given g; one can get the rest by deformation with respect to the parameters from finite-zoned potentials of the form u(x) + v(y). A simple explicit proof will be given below.

B. Rapidly Decreasing Potentials. Let $\varepsilon = \varepsilon_0 < 0$ and suppose given a family F of functions $\Psi(z, \bar{z}, k)$, $k^2 = \varepsilon_0 = -4\varkappa^2$. We introduce the parameter λ by the formula $k = (k_1, k_2)$, $z = x + iy$

$$k_1 = i\varkappa(\lambda + 1/\lambda), \quad k_2 = -\varkappa(\lambda - 1/\lambda). \tag{10}$$

$\chi(z, \bar{z}, \lambda, \bar{\lambda}) = e^{-i(\vec{k}\vec{x})}\Psi$ is bounded as $|z| \to \infty$. We introduce

$$U(\lambda, \bar{\lambda}) = -\frac{i}{2}\iint e^{\varkappa[(1/\bar{\lambda}-\lambda)z+(\bar{\lambda}-1/\lambda)\bar{z}]}u(z, \bar{z})\chi(z, \bar{z}, \lambda, \bar{\lambda})\,dz\,d\bar{z}, \tag{11}$$

where $L = -\partial\bar{\partial} + u(z, \bar{z})$, $L\Psi = \varepsilon_0\Psi$.

Suppose further $\varkappa^2 = 1$, $\varepsilon_0 = -4$. The quantities (Ψ, T, χ) have the properties (cf. [7] for $\varepsilon_0 > 0$; passage to negative ε_0 does not change the arguments of [7] in this point)

$$\frac{\partial\Psi(z, \bar{z}, \lambda, \bar{\lambda})}{\partial\bar{\lambda}} = T(\lambda, \bar{\lambda})\Psi(z, \bar{z}, 1/\bar{\lambda}, 1/\lambda), \tag{12}$$

where

$$T(\lambda, \bar{\lambda}) = \frac{\operatorname{sgn}(|\lambda|^2 - 1)}{\pi\bar{\lambda}}U(\lambda, \bar{\lambda}).$$

The group of reductions is as follows

$$U(1/\bar{\lambda}, 1/\lambda) = \bar{U}(\lambda, \bar{\lambda}), \quad U(-1/\bar{\lambda}, -1/\lambda) = U(\lambda, \bar{\lambda}). \tag{13}$$

One of the reductions differs in sign from [7], since $\varepsilon_0 < 0$. Obviously we have

$$\chi(z, \bar{z}, 1/\bar{\lambda}, 1/\lambda) = \bar{\chi}(z, \bar{z}, \lambda, \bar{\lambda}), \quad \chi \to 1 \text{ as } \lambda \to 0, \infty. \tag{14}$$

Let

$$\tilde{T}(\lambda, \bar{\lambda}, z, \bar{z}) = T(\lambda, \bar{\lambda})\exp[(\lambda - 1/\bar{\lambda})z + (1/\lambda - \bar{\lambda})\bar{z}].$$

The equation of generalized analyticity for χ on the sphere $\mathbf{C}P^1$

$$\frac{\partial\chi(z, \bar{z}, \lambda, \bar{\lambda})}{\partial\bar{\lambda}} = \tilde{T}(\lambda, \bar{\lambda}, z, \bar{z})\bar{\chi}(z, \bar{z}, \lambda, \bar{\lambda}) \tag{15}$$

follows from (12) and (14). Following [7], one can justify (12)-(15) for potentials of small norm.

The function Ψ has asymptotics at the points $(\infty_1, \infty_2) = (0, \infty)$ completely analogous to the periodic case (7) for the choice of local parameters $\lambda = k_1^{-1}$, $1/\lambda = k_2^{-1}$.

The relations of (13) are the analog of the group of reductions $\mathbf{Z}_2 \times \mathbf{Z}_2$ of [3]. The family F is nonanalytic with respect to $\lambda \in \mathbf{C}P^1$: instead of analyticity on the surface for DKN-families we have (12) on the Riemann sphere. (T is something like continuous "density of handles.")

Definition. The function $T(\lambda, \bar{\lambda})$ or $U(\lambda, \bar{\lambda})$ is called "the collection of data of the inverse problem" for $\varepsilon_0 < 0$.

THEOREM 2. Suppose given a continuous function $U(\lambda, \bar{\lambda})$ on $\mathbf{C}P^1$ such that

1) $$\frac{1}{\lambda}T(\lambda, \bar{\lambda}) \in \mathscr{L}_p(D), \quad p > 2, \tag{16}$$

D here is the unit disc $|\lambda| \leqslant 1$;

2) $U(\lambda, \bar{\lambda})$ has the properties of reduction (13).

Then there exists a unique function $\Psi(z, \bar{z}, \lambda, \bar{\lambda})$, satisfying (12) with the conditions (14), and an operator $L = -\partial\bar{\partial} + u(z, \bar{z})$ such that

a) $L\Psi = \varepsilon_0\Psi$, $\varepsilon_0 < \varepsilon_{\min} \leqslant 0$;

b) $u(z, \bar{z})$ is a continuous potential, while $u(z, z) \to 0$ as $|z| \to \infty$;

c) if $U(\lambda, \bar{\lambda})$ decreases faster than any power of λ as $\lambda \to 0$, then the potential $u(z, \bar{z})$ is infinitely differentiable;

d) if the potential $u(z, \bar{z})$ found by solving the inverse scattering problem according to the scheme indicated, decreases as $r = |z| \to \infty$ faster than $r^{-2-\varepsilon}$, then the following relation holds on the unit circle

$$U(\lambda, \bar{\lambda}) - h_1(\lambda, \bar{\lambda}) = C = \text{const}, \tag{17}$$

where $|\lambda| = 1$,

$$h_1(\lambda, \bar{\lambda}) = \frac{1}{2\pi i} \iint \bar{U}(\mu, \bar{\mu})\, T(\mu, \bar{\mu})\, (\mu - \lambda)^{-1}\, d\mu\, d\bar{\mu}.$$

Theorem 2 does not assume any smallness of the norm.

Remark 1. If $u(z, \bar{z})$ decreases faster than $r^{-2-n-\varepsilon}$ as $r \to \infty$, then n more relations appear for $|\lambda| = 1$. These relations are found by the same scheme as (17) (cf. below). For example, for $n = 1$ we have

$$-\lambda^2 \frac{\partial}{\partial\lambda} U(\lambda, \bar{\lambda}) - h_2(\lambda, \bar{\lambda}) = A + B\lambda^2, \quad |\lambda| = 1, \tag{18}$$

where A and B are constants, $|\lambda| = 1$, for h_2 one has the formula [the difference on the right side of (18) is continuous for $|\lambda| = 1$]

$$h_2(\lambda, \bar{\lambda}) = \frac{1}{2\pi i} \iint \frac{[-\mu^2\partial_\mu T(\mu, \bar{\mu})]\, U(1/\bar{\mu}, 1/\mu) + T(\mu, \bar{\mu})\, T(1/\bar{\mu}, 1/\mu)(h_1(\mu, \bar{\mu}) + C)}{\mu - \lambda}\, d\mu\, d\bar{\mu}.$$

Problem. Study the singularities of $T(\lambda, \bar{\lambda})$ for $\varepsilon_0 > \varepsilon_{\min}$. What do they look like upon motion with respect to the energy ε upon passage through a point of the discrete spectrum? What are the special properties of the singular level $\varepsilon_0 = 0$?

The answers to these questions will be given in a following paper. Some formulations are given at the end of this paper.

2. Proofs of the Basic Theorems

The idea of the proof of both Theorems 1 and 2 is quite transparent: the main point in it is the observation that that the eigenfunctions of both families F and DKN, $\Psi(z, \bar{z}, \mathcal{P})$, for points $\mathcal{P}$ on the contour $b \subset \Gamma$ (Theorem 1) and the function $\Psi(z, \bar{z}, \lambda, \bar{\lambda})$ for points λ on the contour $|\lambda| = 1$ (Theorem 2) are real and strictly positive (below)

$$\Psi \in \mathbf{R},\ \Psi > 0,\ x, y \in \mathbf{R}^2. \tag{19}$$

Using (19), one can extract the proof of the theorems from the following lemmas, reproducing some elementary arguments from the famous book of Courant and Hilbert [12], although the lemmas in [12] are not formulated in this form.

LEMMA 1. Let $u(x_1, \ldots, x_n)$ be a smooth real potential in R^n such that $u >$ const for $|x|^2 > r_0$ and one can find a smooth real positive solution Ψ of the equation

$$L\Psi = -\Delta\Psi + u\Psi = \varepsilon_0\Psi,\ \Psi > 0.$$

Then for $\varepsilon_{\min}$ one has (21) and $\varepsilon_0 \leqslant \varepsilon_{\min}$, always, where $\varepsilon_{\min}$ is the level of the ground state of the operator L in $\mathcal{L}_2(\mathbf{R}^n)$.

Proof. Let S be a domain in $\mathbf{R}^n$, containing a ball of large radius $r \to \infty$. The energy $\varepsilon_{\min}$ is defined as the limit of the minima of the functionals

$$\min_{S, \varphi} \int_S \!\!\cdots\!\! \int (|\nabla\varphi|^2 + u\varphi^2)\, d^nx = \min_S \varepsilon_{\min}(S),\quad \int_S \!\!\cdots\!\! \int \varphi^2\, d^nx = 1,\quad \varphi|_{\partial S} = 0,\quad \varphi \geqslant 0$$

as $r \to \infty$. Let $\varphi = \eta\Psi$, where $\eta = 0$ on the boundary ∂S, $\eta \geqslant 0$. We have the chain of equalities

$$\int_S \!\!\cdots\!\! \int (|\nabla\varphi|^2 + u\varphi^2)\, d^nx = \int_S \!\!\cdots\!\! \int (-\varphi\Delta\varphi + u\varphi^2)\, d^nx = \int_S \!\!\cdots\!\! \int [-\eta\Psi\Delta(\eta\Psi) + \eta^2\Psi^2 u]\, d^nx =$$

$$=\int_S\cdots\int[\eta\Delta\eta\Psi^2-2(\eta\nabla\eta\Psi\nabla\Psi)-\eta^2\Psi\Delta\Psi+u\eta^2\Psi^2]\,d^nx=$$

$$=\int_S\cdots\int[-\eta\nabla(\nabla\eta\Psi^2)+\eta^2\Psi(-\Delta\Psi+u\Psi)]\,d^nx=\int_S\cdots\int|\nabla\eta|^2\,\Psi^2\,d^nx+\varepsilon_0\int_S\cdots\int\eta^2\Psi^2\,d^nx. \tag{20}$$

From this we get

$$\varepsilon_{\min}(S)=\int_S\cdots\int\left|\nabla\frac{\varphi}{\Psi}\right|^2\Psi^2\,d^nx+\varepsilon_0, \tag{21}$$

where φ is an eigenfunction of the ground state of the domain S. Lemma 1 is proved.

LEMMA 2. Suppose under the hypotheses of Lemma 1 the potential u is rapidly decreasing in $\mathbf{R}^n$ and Ψ does not coincide with an eigenfunction of the ground state in $\mathbf{R}^n$. Then $\varepsilon_0 < \varepsilon_{\min}$.

Proof. Under these conditions one can take $S=\mathbf{R}^n$; let φ be an eigenfunction of the ground state $L\varphi=\varepsilon_{\min}\varphi$. We know that $\varphi>0$ and $\varphi\in\mathcal{L}_2(\mathbf{R}^n)$. From the chain of equalities (20) we get (21)

$$(L\varphi,\varphi)=\varepsilon_{\min}(\varphi,\varphi)=\varepsilon_{\min}=\varepsilon_0+\int_{\mathbf{R}^n}\cdots\int\Psi^2\nabla\left(\frac{\varphi}{\Psi}\right)^2d^nx.$$

Lemma 2 follows from this.

LEMMA 3. Under the hypotheses of Lemma 1 let the potential u be periodic in $\mathbf{R}^n$ and Ψ be a smooth positive eigenfunction $L\Psi = \varepsilon_0\Psi$, which is not a ground state. Then $\varepsilon_0 < \varepsilon_{\min}$.

Proof. Let S_m be a parallelogram in $\mathbf{R}^n$ with sides which are multiples of the basis vectors of the lattice $T_1,\ldots,T_n$ with multiplicities $m_1,\ldots,m_n$, $\min(m_1,\ldots,m_n)\to\infty$. A function φ of the ground state in $\mathbf{R}^n$ is periodic with periods $T_1,\ldots,T_n$, $\varphi>0$ and $L\varphi=\varepsilon_{\min}\varphi$. Let S_1 be an elementary cell; for the area S_m we have $|S|=m_1\cdot\ldots\cdot m_n\,|S_1|$.

The ground state is common for all the domains S_m and coincides with φ, while it is a ground state in $\mathbf{R}^n$. Since φ is peridic, in the chain of equalities (20) there are also no boundary terms for the domains S_m. From this we have for S_1

$$\varepsilon_{\min}=\varepsilon_0+\int_{S_1}\cdots\int\Psi^2\left(\nabla\left(\frac{\varphi}{\Psi}\right)\right)^2d^nx>\varepsilon_0.$$

Here φ is normalized in the domain S_1.

Lemma 3 is proved.

LEMMA 4. Let the potential u be real, smooth, and quasiperiodic, and φ be a ground state of the operator L with the same group of periods as u. If there exists a smooth positive solution $\Psi > 0$ of the equation $L\Psi = \varepsilon_0\Psi$, where Ψ does not coincide with φ, then $\varepsilon_0 < \varepsilon_{\min}$.

The proof also follows quickly from the chain of equalities (20), (21), where instead of the integral over the cell S_1 one takes the Bohr mean.

Now we proceed to the proof of Theorems 1 and 2. One has the following

Basic Lemma. If the hypotheses of Theorems 1 and 2 hold for the data of the inverse problem, eigenfunctions $\Psi(z,\bar z,\mathcal{P})$ and $\Psi(z,\bar z,\lambda,\bar\lambda)$ of both families F and DKN always exist, have no zeros and poles for all $\mathcal{P}$ outside the ovals a_j and all λ; for $\mathcal{P}\in b$ and $|\lambda|=1$ these functions are real and of fixed sign.

Proof. Under the hypotheses of Theorem 1 (periodic and quasiperiodic case) the function Ψ is given by an explicit formula of [3], and under these conditions the divisors D are nonspecial; in complete analogy with n = 1, for real (x, y) the function Ψ has zero and poles only for $\mathcal{P}\in a_j$ (for n = 1 the ovals a_j correspond to finite gaps and the oval b to an infinite gap for $L = -\partial_x^2 + u$). The function Ψ is real for all $\mathcal{P}\in a_j$ or $\mathcal{P}\in b$, as follows from the reductions $\mathbf{Z}_2\times\mathbf{Z}_2$. The zeros of Ψ have the form $\gamma_j(x, y)$ and run through the ovals a_j for all x, y, if $\Psi(0,0,\mathcal{P})$. The situation here is identical to the one-dimensional case (cf. [13]).

The proof in the rapidly decreasing case, where there are no obvious one-dimensional analogs, is more original. The equations of the $\bar\partial$-problem (12), (15) in our case, by virtue of the reductions (13), (14) are local in λ without translations of a point and coincide precisely with the Beltrami equation for generalized analytic functions on CP^1. The problem

(15), as follows from the theory of generalized analytic functions [14], has a unique smooth solution χ, for all $z = x + iy$. Moreover, the function χ has no singularities on CP^1. It follows from this by the argument principle for the number of zeros of analytic functions without singularities, that χ has no zeros on CP^1 with respect to λ. By the uniqueness of the solution, χ satisfies the reduction (14). Consequently, χ, Ψ are real for $|\lambda| = 1$. The basic lemma is proved.

Remark. The functions $\oint_b \Psi(z, \bar{z}, \mathcal{P}) \mid d\mathcal{P} \mid$ and $\oint_{|\lambda|=1} \Psi(z, \bar{z}, \lambda, \bar{\lambda}) |d\lambda|$ grow in all directions $|x|^2 + |y|^2 \to \infty$.

The proof of Theorem 1 follows quickly from the basic lemma with Lemmas 1-4.

From the basic lemma together with Lemma 2 all the points of Theorem 2 except the smoothness and rate of decrease of the potential $u(z, \bar{z})$ also follow.

Proof of Point c) of Theorem 2. We see that all the derivatives of $\chi(z, z, \lambda, \bar{\lambda})$ with respect to z and $\bar{z}$ are continuous and bounded functions of the variable λ. We shall verify this fact by induction on the order of the derivative.

The function χ satisfies the integral equation

$$\chi(z, \bar{z}, \lambda, \bar{\lambda}) = 1 + \bar{\partial}_\lambda^{-1} \tilde{T}(\lambda, \bar{\lambda}, z, \bar{z}) \chi(z, \bar{z}, 1/\bar{\lambda}, 1/\lambda), \tag{22}$$

where

$$(\bar{\partial}_\lambda^{-1} f)(\lambda) = \frac{1}{2\pi i} \iint \frac{f(\mu, \bar{\mu})\, d\mu\, d\bar{\mu}}{\mu - \lambda}. \tag{23}$$

We differentiate (22) n_1 times with respect to z and n_2 times with respect to $\bar{z}$. As a result we get

$$\partial_z^{n_1} \partial_{\bar{z}}^{n_2} \chi(z, \bar{z}, \lambda, \bar{\lambda}) = \bar{\partial}_\lambda^{-1} \tilde{T}(\lambda, \bar{\lambda}, z, \bar{z}) \partial_z^{n_1} \partial_{\bar{z}}^{n_2} \chi\left(z, \bar{z}, \frac{1}{\bar{\lambda}}, \frac{1}{\lambda}\right) + F_{n_1 n_2}(z, \bar{z}, \lambda, \bar{\lambda}), \tag{24}$$

$$F_{n_1 n_2}(z, \bar{z}, \lambda, \bar{\lambda}) = \sum_{\substack{k_1, k_2 \\ k_1 + k_2 \neq 0}} \bar{\partial}_\lambda^{-1} \tilde{T}(\lambda, \bar{\lambda}, z, \bar{z}) (\lambda - 1/\bar{\lambda})^{k_1} (1/\lambda - \bar{\lambda})^{k_2} \partial_z^{n_1 - k_1} \partial_{\bar{z}}^{n_2 - k_2} \chi(z, \bar{z}, 1/\bar{\lambda}, 1/\lambda). \tag{25}$$

All the derivatives of χ which appear in (25) have lower order. Since $\tilde{T}(\lambda, \bar{\lambda}, z, \bar{z})$ decreases rapidly as $\lambda \to 0, \infty$, the operator $\bar{\partial}_\lambda^{-1} \tilde{T}(\lambda, \bar{\lambda}, z, \bar{z})(\lambda - 1/\bar{\lambda})^{k_1} (1/\lambda - \bar{\lambda})^{k_2}$ is compact for all k_1, k_2 in the space of continuous functions of λ. Thus, the function $F_{n_1 n_2}(z, \bar{z}, \lambda, \bar{\lambda})$ is continuous and bounded in λ. Using the absence of nontrivial solutions of the homogeneous equation, we get the continuity and boundedness in λ of $\partial_z^{n_1} \partial_{\bar{z}}^{n_2} \chi$.

Thus we have proved that $\chi(z, \bar{z}, \lambda, \bar{\lambda})$ is a smooth function of z. Using the absence of zeros of χ, proved earlier, we get the smoothness of $u(z, \bar{z})$.

Proof of Point d). Now we give the proof of point d). Let the assumptions of point a) hold. Then $\chi(z, \bar{z}, \lambda, \bar{\lambda})$ is bounded on the whole space of z, λ and for large z, tends to 1 uniformly in λ. We introduce the additional function

$$h(l, \bar{l}, \lambda, \bar{\lambda}) = \frac{i}{2} \iint e^{-[lz - \bar{l}\bar{z}]} u(z, \bar{z}) \chi(z, \bar{z}, \lambda, \bar{\lambda})\, dz\, d\bar{z}. \tag{26}$$

The function $h(l, \bar{l}, \lambda, \bar{\lambda})$ is continuous, bounded, and even Lipschitz in l, λ with exponents ε_1, ε_2, where $\varepsilon_1 < \varepsilon$, $\varepsilon_2 < 1$. For $l = \lambda - 1/\bar{\lambda}$ we get the function $U(\lambda, \bar{\lambda})$

$$U(\lambda, \bar{\lambda}) = h(\lambda - 1/\bar{\lambda}, \bar{\lambda} - 1/\lambda, \lambda, \bar{\lambda}). \tag{27}$$

The following equation for h follows from (12):

$$\frac{\partial h(l, \bar{l}, \lambda, \bar{\lambda})}{\partial \bar{\lambda}} = T(\lambda, \bar{\lambda}) h\left(l + \frac{1}{\bar{\lambda}} - \lambda, \bar{l} + \frac{1}{\lambda} - \lambda, \frac{1}{\bar{\lambda}}, \frac{1}{\lambda}\right) \tag{28}$$

(the analogs of (28) for the multidimensional case were actively used in [9, 15, 16]).

In particular, for the function $\tilde{h}_1(\lambda, \bar{\lambda}) = h(0, 0, \lambda, \bar{\lambda})$ we get

$$\frac{\partial \tilde{h}_1(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} = T(\lambda, \bar{\lambda}) U(1/\bar{\lambda}, 1/\lambda). \tag{29}$$

On the other hand, for $|\lambda| = 1$ $U(\lambda, \bar{\lambda}) = \tilde{h}_1(\lambda, \bar{\lambda})$. Integrating (29), we get

$$U(\lambda, \bar{\lambda}) - \frac{1}{2\pi i} \iint \frac{T(\mu, \bar{\mu}) U(1/\bar{\mu}, 1/\mu)\, d\mu\, d\bar{\mu}}{\mu - \lambda} = \text{const}, \quad |\lambda| = 1. \tag{30}$$

The proof of Theorem 2 is finished.

Remark 3. Suppose the data of a rapidly decreasing inverse problem $T(\lambda, \bar{\lambda})$ depend on an infinite collection of additional variables $t_1, \ldots, t_n, \ldots$, where the dependence is given by the formula

$$T(\lambda, \bar{\lambda}, t_1, \ldots, t_n, \ldots) = T(\lambda, \bar{\lambda}) \exp\left[\sum_{j=1}^{n} (2\varkappa)^{2j+1} \left(\lambda^{2j+1} + \frac{1}{\lambda^{2j+1}} - \bar{\lambda}^{2j+1} - \frac{1}{\bar{\lambda}^{2j+1}}\right) t_j\right]. \tag{31}$$

Then the function $v(z, \bar{z}, t_1, \ldots, t_n, \ldots) = u(z, \bar{z}, t_1, \ldots, t_n, \ldots) - \varepsilon_0$ satisfies a hierarchy of (2 + 1)-dimensional nonlinear equations found by Veselov and one of the authors (cf. [5]). In particular, the first of these equations has the form

$$\begin{cases} v_t = \partial^3 v + \bar{\partial}^3 v + \partial(vw) + \bar{\partial}(v\bar{w}), \\ \bar{\partial} w = -3\partial v, \quad v = \bar{v}. \end{cases} \tag{32}$$

We note that the reductions (13) and relations (17) are invariant with respect to transformation (31).

Supplement 1. Two limiting cases of the scheme recounted are of interest.

I. Scattering Data for Zero Energy. Let $\varepsilon_0 \to 0$, $\mu_+ = \varkappa\lambda$, $\mu_- = 1/(\varkappa\lambda)$, $\varepsilon_0 = -4\varkappa^2$. As $\varepsilon_0 \to 0$ the contour $|\lambda| = 1$ contracts to a point, which we denote by B_0, and the Riemann sphere of the spectral parameter splits into two S_+ and S_- with coordinates μ_+ and μ_-, respectively, which intersect at the point B_0, $\mu_+ = \mu_- = 0$. The reductions (13) become the relations $U(-\mu) = \bar{U}(\mu)$ and $U(\mu_+) = \bar{U}(\mu_-)$. The equations of the inverse problem acquire the form

$$\frac{\partial \Psi(z, \bar{z}, \mu_\pm, \bar{\mu}_\pm)}{\partial \bar{\mu}_\pm} = T(\mu_\pm, \bar{\mu}_\pm) \Psi(z, \bar{z}, \bar{\mu}_\mp, \mu_\mp), \tag{33}$$

where

$$\begin{gathered} T(\mu_\pm, \bar{\mu}_\pm) = \pm U(\mu_\pm, \bar{\mu}_\pm)/(\pi\bar{\mu}_\pm), \\ \Psi(z, \bar{z}, \mu_\pm, \bar{\mu}_\pm) = e^{\mp\mu_\pm z_\pm}, \quad \mu_\pm \to \infty, \quad z_- = \bar{z},\ z_+ = z. \end{gathered} \tag{34}$$

If one requires the $\mathscr{L}_p$-integrability of the function $T(\mu, \bar{\mu})$ with $p > 2$ in a neighborhood of 0 along with sufficiently rapid decrease at infinity, then the proofs of the nonsingularity of the potential and the absence of a discrete spectrum proceed exactly like the corresponding arguments in Theorem 2. However here we get potentials which are not in general position.

II. Limit Passage to a Parabolic Operator. Let $\varepsilon_0 = -4\varkappa^2 \to -\infty$, the function $T(\lambda, \bar{\lambda})$ be concentrated in a neighborhood of the points $\lambda = \pm i$ of size of order $1/\varkappa$, $X = x$, $Y = y/\varkappa$, $\mu = \varkappa(\lambda + i)$. Then as $\varkappa \to \infty$ the function $\Psi_1(X, Y, \mu, \bar{\mu}) = e^{2\varkappa^2 Y} \Psi(x, y, \lambda, \bar{\lambda})$ goes to an eigenfunction of the parabolic operator

$$L_1 = -\partial_X^2 + 4\partial_Y + u(X, Y), \tag{35}$$

$$L_1 \Psi_1(X, Y, \mu, \bar{\mu}) = 0. \tag{36}$$

The inverse scattering problem for the operator L_1 was actively studied in connection with the theory of the KP II equation (cf. [17, 18], etc.]. The function Ψ_1 satisfies the $\bar{\partial}$-equation

$$\frac{\partial \Psi_1(X, Y, \mu, \bar{\mu})}{\partial \bar{\mu}} = T_1(\mu, \bar{\mu}) \Psi_1(X, Y, \bar{\mu}, \mu). \tag{37}$$

The reduction $U(1/\bar{\lambda}, 1/\lambda) = \bar{U}(\lambda, \bar{\lambda})$ goes to the reduction $T_1(\mu, \bar{\mu}) = -T_1(\bar{\mu}, \mu)$, which corresponds to the reality of the potential u(X, Y), the reduction $U(-1/\bar{\lambda}, -1/\lambda) = U(\lambda, \bar{\lambda})$ goes to a relation connecting the scattering data of the operator L_1 and its adjoint. Relations of this type in the theory of KP II were discussed in [18].

Just as in the theory of the Schrödinger operator, application of the theory of generalized-analytic functions lets one prove the nonsingularity of the potential u(X, Y) without assumptions about the smallness of the norm $T_1(\mu, \bar{\mu})$. This fact was not previously noted in the literature although the local form of Eq. (37) of type (15) was used in [18].

If the potential $u(X, Y)$ decreases sufficiently rapidly at infinity, then there arise relations of the type of (17), (18) on the data of the inverse problem. These relations for KP are now being investigated by V. Bakurov in the L. D. Landau Institute of Theoretical Physics (ITF).

Supplement 2. Negative energies of higher ground state. For energies of higher ground state the situation is more complicated. Let the energy ε_0 not be a level of the discrete spectrum of the operator $L = -\Delta + u$. The eigenfunctions $\Psi(z, \bar{z}, \lambda, \bar{\lambda})$ defined above of the family F for $\varepsilon = \varepsilon_0 < 0$ and "data of the inverse problem" $T(\lambda, \lambda)$ are introduced as before and have group of reductions (13), (14), however both functions T and Ψ are no longer smooth in the whole λ-plane. As comparison with [9, 20] shows, these functions in the situation of general position have a collection of poles along the curves Γ_j:

$$f_j(\lambda, \bar{\lambda}) = 0, \quad T = T_{-1}/f_j(\lambda, \bar{\lambda}) + T_0, \quad f_j = \bar{f}_j, \quad \Psi = \Psi_{-1}/f_j(\lambda, \bar{\lambda}) + \Psi_0, \tag{38}$$

where T_{-1}, T_0, Ψ_{-1}, Ψ_0 are smooth functions.

As before, wherever T is smooth, the function Ψ satisfies (15), i.e., is generalized analytic

$$\partial\Psi/\partial\bar{\lambda} = T\bar{\Psi}. \tag{39}$$

The collection of curves Γ_j in the λ-plane is invariant with respect to the group of reductions (13). In the case of general position we have

$$|\partial f_j/\partial\bar{\lambda}| \neq 0, \quad \text{if} \quad f_j = 0. \tag{40}$$

For $\varepsilon > \varepsilon_{min}$, by virtue of Sec. 2 of this paper, the function $\Psi(z, \bar{z}, \lambda, \bar{\lambda})$ necessarily has zeros for $|\lambda| = 1$ and some values of z. For all z the zeros of $\Psi(z, \bar{z}, \lambda, \bar{\lambda})$ lie in a compact part of the λ-plane. It is easy to see that the manifold of poles $\Gamma = \cup\Gamma_j$ is compact and independent of z.

It follows from (15) that all the zeros of Ψ, situated outside the manifold of poles Γ, have positive multiplicity. Hence, for those z for which Ψ has zero outside of Γ, the function Ψ_{-1} necessarily has zeros on curves of Γ of positive multiplicity. As $z \to \infty$ the zeros of Ψ of positive multiplicity go to zeros of Ψ_{-1} of negative multiplicity. We denote the zeros by $\gamma_j^{\pm}(z, \bar{z})$ (their dependence on z will be investigated in later papers):

$$\Psi_{-1} = 0 \quad \text{for } \lambda = \gamma_j(z, \bar{z}), \quad f_j = 0.$$

From (15) and the expansion (38) the following equation T_1 follows easily:

$$\frac{1}{T_{-1}}\frac{\partial f_j}{\partial\bar{\lambda}} = -\frac{\bar{\Psi}_{-1}}{\Psi_{-1}} = -e^{-2ig(\lambda, \bar{\lambda})}.$$

It follows from this in particular that one has the restriction

$$|T_{-1}| = \partial f_j/\partial\bar{\lambda} \neq 0, \ f_j = 0. \tag{41}$$

The most natural position is a "nest"

$$[0 \subset \Gamma_{-N} \subset \Gamma_{-N+1} \subset \ldots \subset \Gamma_{-1} \subset S^1 \subset \Gamma_1 \subset \ldots \subset \Gamma_N],$$

where S^1 is the unit circle $|\lambda| = 1$, all the Γ_j are invariant with respect to the transformation $\lambda \to \lambda$, Γ_{-j} is obtained from Γ_j by the transformation $\lambda \to 1/\bar{\lambda}$. Apparently the rotation number of the function $T_{-1} \neq 0$ along the contour Γ_j is equal to 1. In a following paper we publish the results of investigation of the inverse problem, based on the solvability of (15) under the conditions (38), (41), and the supplementary condition (43).

One has the following

LEMMA 5. In order that (15), in a neighborhood of each point of the contour Γ_j have a collection of solutions locally, depending on two real valued functions of one variable (a point of the contour), it is necessary and sufficient that in addition to (41) the following relations (43) hold.

Suppose given in a neighborhood of the curve Γ_j a semigeodesic coordinate system (α, β), where α is the distance to Γ_j, β is the natural parameter on Γ_j, the lines β = const are line segments perpendicular to Γ_j. Considering (38) and (41) we get

$$\Psi = \exp[ig(\beta)][\varphi_{-1}(\beta)/\alpha + \varphi_0(\beta) + \varphi_1(\beta)\alpha + O(\alpha^2)],$$

$$T = \exp[i(2g(\beta) + h(\beta))][-1/2\alpha + \tau_0(\beta) + \tau_1(\beta)\alpha + O(\alpha^2)],$$

$$d\lambda = \exp[ih(\beta)]\cdot(d\alpha + i\,d\beta) \text{ for } \alpha = 0, \tag{42}$$

where $g(\beta)$, $h(\beta)$, and $\varphi_{-1}(\beta)$ are real functions. Then in order that an arbitrary solution of (15) in a neighborhood of each point of the contour Γ_j depend on two real functions of one variable, it is necessary to require that one have

$$\operatorname{Re}\tau_0(\beta) = K(\beta)/4, \quad K(\beta) = -(\partial e_\beta/\partial\beta, e_\alpha),$$
$$\operatorname{Im}\tau_1(\beta) = \frac{1}{2}\frac{\partial^2 g(\beta)}{\partial\beta^2} + \frac{1}{4}\frac{\partial K(\beta)}{\partial\beta}, \tag{43}$$

e_β being a tangent vector to Γ_j, e_α a normal vector, $e_\alpha \times e_\beta = 1$. Arbitrary functions on which a local solution depends are $\varphi_{-1}(\beta)$ and $\operatorname{Im}\varphi_1(\beta)$. If (43) does not hold then a local solution of (15) depends on only one real-valued function.

It is apparently necessary that (43) hold for a solution of the inverse problem. If (43) does not hold, then the inverse problem will generally not be solvable for energy levels outside the discrete spectrum.

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