# TWISTED PICARD–LEFSCHETZ FORMULAS

A. B. Givental'

UDC 513.836

The Picard–Lefschetz theory [17, 4] describes the monodromy of vanishing cycles in versal deformations of isolated singularities of complex hypersurfaces. The monodromy group turns out to be the group of reflections with respect to the intersection form in the middle dimensional homology space. An example is the group of permutation of the roots of a polynomial in a family of polynomials of one variable. In this theory, along with Euclidean groups of reflections there also arise their "odd" analogs. Thus, upon replacing a family of polynomials by a family of hyperelliptic curves, the group of permutations is replaced by the group of symplectic "reflections" in the one-dimensional homology space of the curve (cf. [22-24]).

The original idea of the present paper is to consider instead of cycles on a hypersurface, cycles on its complement, endowed with a local system of coefficients which transform nontrivially upon circuit around the hypersurface. We describe the monodromy group in the homology space with the so "twisted" coefficients. It turns out that twisted Picard–Lefschetz formulas interpolate between the symmetric and skew-symmetric versions of the classical theory. As noted by Shekhtman [11], the group algebra of the monodromy group turns out to be the analog of the Hecke algebra, a well-known deformation of the group algebra of the group of permutations.

We formulate the result of our calculation for simple singularities of hypersurfaces. As is known [1], such singularities are classified by the Dynkin diagrams $A_\mu$, $D_\mu$, $E_6$, $E_7$, $E_8$. For a suitable basis of cycles, the matrix of the symmetric intersection form of a simple singularity coincides (up to sign) with the Cartan matrix C of the corresponding type. We represent C as a sum $V + V^t$ of an upper triangular matrix with ones on the diagonal and the transpose of one, and then we define in $\mathbf{C}^\mu$ a bilinear form $(\cdot, \cdot)$ with matrix $(e_j, e_i) = qV + V^t$, where q is a nonzero complex number. The "reflections" $M_i$: $h \to h - (h, e_i)e_i$ then generate the monodromy group of twisted cycles. For q = 1 one gets the Weyl group corresponding to the original Cartan matrix, for q = −1, its "odd" analog. For arbitrary q the "reflections" $M_i$ satisfy the defining relations of the Artin–Brieskorn braid group, and also the additional relations $(M + q)(M - 1) = 0$, which define the Hecke algebra of the corresponding Weyl group.

I sincerely thank V. I. Arnol'd, A. N. Varchenko, O. Ya. Viro, V. A. Vasil'ev, I. M. Gel'fand, V. A. Ginzburg, A. B. Goncharov, S. K. Lando, D. B. Fuks, V. M. Kharlamov, and V. V. Shekhtman for interest in the work, which served as a good stimulus for me in writing this paper.

## 1. Monodromy of Vanishing Cycles

We collect here information from the ordinary Picard–Lefschetz theory (cf. [4]).

Let $f\colon \mathbf{C}^n, 0 \to \mathbf{C}, 0$ be a germ of a holomorphic function at an isolated critical point of multiplicity μ. The germ $F\colon \mathbf{C}^n \times \mathbf{C}^\mu, 0 \times 0 \to \mathbf{C}, 0$ is a <u>miniversal deformation</u> of it, if $f(x) = F(x, 0)$ and the germs of the functions $\partial F/\partial\lambda_1(\cdot, 0), \ldots, \partial F/\partial\lambda_\mu(\cdot, 0)$ $(\lambda_1, \ldots, \lambda_\mu$ being coordinates in the space of parameters $\mathbf{C}^\mu)$ are a basis in the local algebra $Q = \mathbf{C}[[x]]/(\partial f/\partial x)$. One can understand this definition as follows: the family F is transverse to the orbit of the germ f under the action of the "group" of diffeomorphisms of the space $\mathbf{C}^n$. Example: n = 2, $f = x_1^{\mu+1} - x_2^2$, $F = x_1^{\mu+1} + \lambda_1 x_1^{\mu-1} + \ldots + \lambda_\mu - x_2^2$.

Let $F_\lambda = F(\cdot, \lambda)$, $V_\lambda = F_\lambda^{-1}(0)$ be the zero level of the function $F_\lambda$ from a miniversal family. In the base of the family we define the <u>discriminant</u> $\Delta \subset \mathbf{C}^\mu$ of those values of the parameter λ, for which $V_\lambda$ is singular. More precisely, we choose a representative of the germ F and a ball $B_\varepsilon$ in $\mathbf{C}^n$ of small radius ε such that the boundary of the ball is transverse to $V_0$. Then

Institute of Chemical Physics, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 22, No. 1, pp. 12-22, January-March, 1988. Original article submitted May 20, 1987.

 0016-2663/88/2201-0010$12.50 

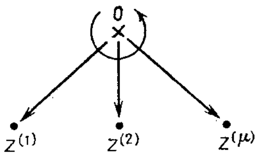


Fig. 1

choose a ball $D_\delta$ in $\mathbf{C}^\mu$ with $\delta$ so small that the fibers $V_\lambda$ with $\lambda \in D_\delta$ will also be transverse to $\partial B_\varepsilon$. Then the nonsingular fibers $V_\lambda \cap B_\varepsilon$ form a locally transverse fibration over the complement of the discriminant in $D_\delta$. In what follows we shall assume that all these $B_\varepsilon$ and $D_\delta$ are chosen once and for all, but for the sake of simplicity we shall not remark on this, denoting $B_\varepsilon$ by $\mathbf{C}^n$, $D_\delta$ by $\mathbf{C}^\mu$, $V_\lambda \cap B_\varepsilon$ by $V_\lambda$, $\Delta \cap D_\delta$ by $\Delta$, and by f and F we shall mean functions representing the corresponding germs.

It is known [16] that a nonsingular fiber $V_\lambda$ has the homotopy type of a bouquet of $\mu$ spheres of dimension $n - 1$. Hence the homology group $H_k(V_\lambda; \mathbf{Z})$ (for $n = 1$, reduced) is zero for $k \neq n - 1$ and isomorphic to $\mathbf{Z}^\mu$ for $k = n - 1$. In the homology fibration of $H_{n-1}(V_\lambda; \mathbf{Z})$ over $\mathbf{C}^\mu \setminus \Delta$ there is a flat <u>Gauss–Manin connection</u>, induced by identifying the homology groups for a local trivialization of the fibration of the hypersurfaces $V_\lambda$. In other words, cycles can be carried from a fiber $V_\lambda$ to a neighboring one by continuity. Carrying cycles from a distinguished fiber $V_\lambda$ along closed curves, avoiding the discriminant, we get a representation of the fundamental group $\pi_1(\mathbf{C}^\mu \setminus \Delta)$ of the base of the fibration by transformations of the lattice $\mathbf{Z}^\mu \simeq H_{n-1}(V_\lambda; \mathbf{Z})$. The image of this representation in $GL_\mu(\mathbf{Z})$ is called the <u>monodromy group</u>.

The index of intersection of cycles of middle dimension of the manifold $V_\lambda$ (equal to $n - 1$) defines an integral bilinear <u>intersection form</u> on the group $H_{n-1}(V_\lambda; \mathbf{Z})$, which is symmetric for odd n and skew-symmetric for even n. The monodromy group lies in the isotropy group of this form, since under transfer of cycles their index of intersection is preserved.

The Picard–Lefschetz formula describes the generators of the monodromy group in a <u>distinguished basis</u> of the lattice $\mathbf{Z}^\mu$. We consider the function $F_\lambda\colon \mathbf{C}^n \to \mathbf{C}$ in general position from a versal family. It has $\mu$ nondegenerate critical points $x^{(1)}, \ldots, x^{(\mu)}$ with different nonzero critical values $z^{(1)}, \ldots, z^{(\mu)}$. We fix a system of disjoint paths on the complex plane $\mathbf{C}$ from the point $z = 0$ to the points $z^{(1)}, \ldots, z^{(\mu)}$ (cf. Fig. 1). We assume that the $z^{(k)}$-s are indexed in the order of the paths leaving zero counterclockwise.

The fiber $F_\lambda^{-1}(z^{(k)})$ has a nondegenerate singular point $x^{(k)}$. In the homology of a nearby nonsingular fiber there is defined uniquely (up to change of sign) an integral cycle $e_k$, which "vanishes" at the singular point $x^{(k)}$, as $z \to z^{(k)}$. This cycle, transferred to $F_\lambda^{-1}(0) = V_\lambda$ along the chosen path, is called a <u>vanishing cycle</u>. One can show that the vanishing cycles $e_1, \ldots, e_\mu$ form a basis in $H_{n-1}(V_\lambda; \mathbf{Z})$. It is called a distinguished basis.

<u>THEOREM (cf. [4]).</u> The monodromy group is generated by $\mu$ reflections $M_k\colon h \mapsto h + (-1)^{n(n+1)/2}\langle h, e_k\rangle e_k$ with respect to the intersection form $\langle \cdot, \cdot \rangle$ on $H_{n-1}(V_\lambda; \mathbf{Z})$ in the vectors $e_1, \ldots, e_\mu$ of a distinguished basis.

We note that the generators indicated correspond to circuits of z around the critical values $z^{(1)}, \ldots, z^{(\mu)}$, generating the fundamental group of the complement of the discriminant. For an odd number of variables n the self-intersection index of a vanishing cycle is equal to $2(-1)^{n(n-1)/2}$, so that the transformation $M_k$ is really a reflection: it has a fixed hyperplane and changes the sign of $e_k$.

<u>Suspension</u> of a singularity f consists of adding the square of a new variable: $F(x, \lambda) \rightsquigarrow F(x, \lambda) + y^2$, $(x, y) \in \mathbf{C}^{n+1}$, $\lambda \in \mathbf{C}^\mu$. Under suspension the critical points and values of the functions $F_\lambda$ are unchanged, but the dimension of the fiber $V_\lambda$ and the parity of the intersection form change.

The two intersection forms, symmetric and skew-symmetric, are connected as follows. Let $\tilde{e}_1, \ldots, \tilde{e}_\mu$ be a distinguished basis of vanishing cycles, constructed from the suspended function $F_\lambda + y^2$ and the same system of paths as $e_1, \ldots, e_\mu$. Then (cf. [4]) $\langle \tilde{e}_j, \tilde{e}_i \rangle = \operatorname{sign}(i - j) \times (-1)^n \langle e_j, e_i \rangle$. Thus, in a general distinguished basis, the matrices of the two intersection forms have either identical superdiagonal and opposite subdigonal parts, or the opposite.

## 2. Twisted Vanishing Homology

The abstract construction which leads to the Gauss–Manin connection refers to a space E on which there is given a locally constant sheaf $\mathcal{F}$. The map $p\colon E \to \Lambda$ lets one define the direct image $p_*\mathcal{F}$ of the sheaf $\mathcal{F}$. The direct image is the sheaf on $\Lambda$ whose stalk over $\lambda \in \Lambda$ is the cohomology space $H^*(p^{-1}(\lambda); \mathcal{F})$ of the stalk $p^{-1}(\lambda)$ with coefficients in the sheaf $\mathcal{F}$, restricted to $p^{-1}(\lambda)$. Suppose over an open contractible $U \subset \Lambda$, $p\colon p^{-1}(U) \to U$ is a trivial bundle. Then the restriction map

$$H^*(p^{-1}(U); \mathcal{F}) \to H^*(p^{-1}(\lambda); \mathcal{F})$$

is an isomorphism for all $\lambda \in U$. This identification of stalks of the direct image over nearby points defines the flat Gauss–Manin connection in the cohomology bundle over the complement to the discriminant of the map p.

In our situation, as E we take the complement in $\mathbf{C}^m \times \mathbf{C}^\mu$ to the zero level of F, as $p\colon E = \mathbf{C}^n \times \mathbf{C}^\mu \setminus F^{-1}(0) \to \mathbf{C}^\mu$ the projection to the space of parameters. The fiber of p over $\lambda$ is $E_\lambda = \mathbf{C}^n \setminus V_\lambda$. It remains to define the sheaf.

Let $\alpha$ be a complex number. The function $F^\alpha$ is multivalued in E. Any two branches of it differ by a constant factor, the degree $q = e^{2\pi i\alpha}$. We define a locally constant sheaf $\mathbf{C}(q)$ on E, whose stalk is the one-dimensional space of germs of functions of the form const$\cdot$ $F^\alpha$. Up to equivalence of sheaves, $\mathbf{C}(q)$ is determined by the number $q \neq 0$, by which its stalk is multiplied upon one positive circuit around the hypersurface $F^{-1}(0)$.

It will be more convenient for us to modify this analytic definition so as to consider all sheaves $\mathbf{C}(q)$ simultaneously.

On E we define a locally constant sheaf $\mathcal{L}(q)$ with stalk $\mathbf{Z}[q, q^{-1}]$ (here q is a formal variable), defining the action of the fundamental group $\pi_1(E) = \mathbf{Z}$ on the stalk as follows: a generator acts by multiplication by q.

Giving the complex value of q, we get the integral subsheaf $\mathbf{Z}(q)$ in $\mathbf{C}(q)$.

One can define the (co)homology of $E_\lambda$ with coefficients in $\mathcal{L}(q)$ with the help of the complex of singular chains or the Cech complex. By definition, an elementary singular chain is a pair $(\varphi, s)$, where $\varphi\colon \nabla \to E_\lambda$ is a continuous map of a simplex, s is a univalent section of a sheaf over $\nabla$. Such a section is defined by its value over one point of the simplex and extends to the whole simplex by continuity, uniquely, in view of the simple connectedness of $\nabla$. One gets the space of chains by factoring the free $\mathbf{Z}[q, q^{-1}]$-module generated by elementary chains by the relations $q(\varphi, s) = (\varphi, qs)$. The boundary operator $\partial$ defined in the standard way turns the space of chains into a complex, whose graded homology group we denote by $H_*(E_\lambda; \mathcal{L}(q))$. It supports the structure of a $\mathbf{Z}[q, q^{-1}]$-module.

Obviously the complex of chains on E so defined is isomorphic to the complex of integral singular chains on the universal covering $\tilde{E}$, in which multiplication by q is defined by the covering transformation R: $\tilde{E} \to \tilde{E}$.

With the help of the intersection index $\varnothing(\varphi, \psi)$ of integral cycles on the covering $(\tilde{E})_\lambda$ we define a $\mathbf{Z}[q, q^{-1}]$-bilinear pairing

$$(\cdot, \cdot)\colon H_*(E_\lambda; \mathcal{L}(q)) \otimes H_{n-*}(E_\lambda; \mathcal{L}(q^{-1})) \to \mathbf{Z}[q, q^{-1}], \ (\varphi, \psi) = \sum_{k=-\infty}^{\infty} \varnothing(R^k\varphi, \psi)\, q^{-k}.$$

Together with the isomorphism $\mathcal{L}(q) \leftrightarrow \mathcal{L}(q^{-1})$, defined by the conjugation automorphism $\mathbf{Z}[q, q^{-1}] \leftrightarrow \mathbf{Z}[q, q^{-1}]$: $\bar{q} = q^{-1}$, the form $(\cdot, \cdot)$ defines the intersection index of cycles of complementary dimension in $H_*(E_\lambda; \mathcal{L}(q))$ with values in $\mathbf{Z}[q, q^{-1}]$, which is a (skew)-Hermitian form $\langle\cdot, \cdot\rangle$ for (odd) even n:

$$\langle q\varphi_1, \varphi_2\rangle = \langle\varphi_1, q^{-1}\varphi_2\rangle = q\langle\varphi_1, \varphi_2\rangle, \quad \langle\varphi_1, \varphi_2\rangle = (-1)^n \overline{\langle\varphi_2, \varphi_1\rangle}.$$

Our next goal is to calculate $H_*(E_\lambda; Z(q))$ for $\lambda \notin \Delta$.

<u>Proposition.</u> Let $SV_\lambda$ be the space obtained from the product $V_\lambda \times S^1$ by contracting the fiber $V_\lambda$ over the distinguished point of the circle $S^1$, $\hat{\mathcal{L}}(q)$ be the sheaf on $SV_\lambda$ with stalk $\mathbf{Z}[q, q^{-1}]$, which undergoes multiplication by q upon traversing a circle and constant on the fibers $V_\lambda$. Then the pair $SV_\lambda$ is homotopy equivalent to $\hat{\mathcal{L}}(q)$ if $E_\lambda, \mathcal{L}(q)$, $\lambda \notin \Delta$.

<u>Proof (S. K. Lando).</u> We contract the null fiber of the function $f = F_0\colon \mathbf{C}^n \to \mathbf{C}$, leaving the rest nonsingular. The complement in $\mathbf{C}^n$ to $f^{-1}(\lambda)$, $\lambda \neq 0$, contracts to the preimage of a circle in $\mathbf{C}$, passing through 0 and going once around the point $\lambda$.

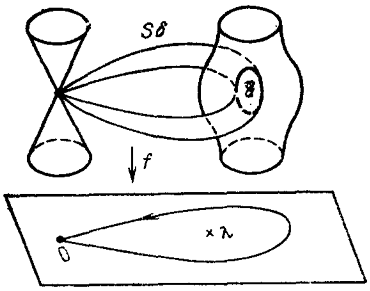

Fig. 2

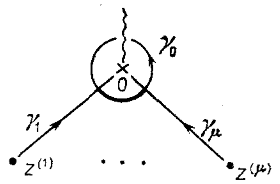

Fig. 3

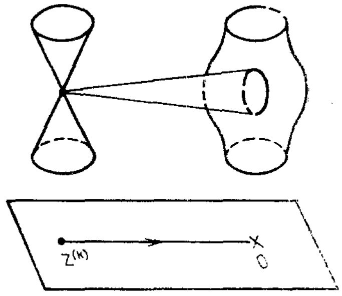

Fig. 4

COROLLARY. For n > 1 the middle dimensional homology group $H_n(E_\lambda; \mathscr{L}(q))$ is a free $\mathbf{Z}[q, q^{-1}]$-module of rank μ. Let $\delta_1, \ldots, \delta_\mu$ be a basis of cycles in $H_{n-1}(V_\lambda; \mathbf{Z})$, $S\delta_1, \ldots S\delta_\mu$ be the basis of suspended cycles in $H_n(SV_\lambda; \mathbf{Z})$ (cf. Fig. 2), s be a basic locally constant section of the sheaf $\hat{\mathscr{L}}(q)$ over $S^1$, single-valued everywhere except the distinguished point. Then $(S\delta_1, s), \ldots, (S\delta_\mu, s)$ is a basis in $H_n(E_\lambda; \mathscr{L}(q))$ $(= H_n(SV_\lambda; \hat{\mathscr{L}}(q))$.

Remark. If $q \neq 1$, the modules $H_k(E_\lambda; \mathbf{Z}(q))$ are zero, if $k = n$. For $q = 1$, $H_k(E_\lambda; \mathbf{Z}(q))$ is the ordinary integral homology group. It is equal to $\mathbf{Z}$ for $k = 0$ and $k = 1$, $n > 1$, and for $k = 1$, $n = 1$ this group is isomorphic to $\mathbf{Z}^{\mu+1}$. Thus the module $H_*(E_\lambda; \mathscr{L}(q))$ in dimensions $* = 0, 1$ has torsion in the form of the $\mathbf{Z}[q, q^{-1}]$-module $\mathbf{Z}$ with identity multiplication by q, and modulo this torsion is concentrated in dimensions $* = n$, where (it) has rank μ.

## 3. Picard-Lefschetz Theorem

Let $\partial E_\lambda$ be the intersection of a tubular neighborhood of $V_\lambda$ in $\mathbf{C}^n$ with $E_\lambda$. Along with $H_n(E_\lambda; \mathscr{L}(q))$ we consider the relative homology space $H_n(E_\lambda, \partial E_n; \mathscr{L}(q^{-1}))$. Between these spaces there is defined a $\mathbf{Z}[q, q^{-1}]$-bilinear pairing $(\cdot, \cdot)$, the index of intersection of an absolute cycle with a relative one.

Below we define compatible distinguished bases $(e_1, \ldots, e_\mu)$ and $(\hat{e}_1, \ldots, \hat{e}_\mu)$ of absolute and relative cycles, corresponding to a distinguished basis of vanishing cycles $(\delta_1, \ldots, \delta_\mu)$ in $H_{n-1}(V_\lambda; \mathbf{Z})$.

THEOREM. 1) Let V be an upper triangular matrix with diagonal elements $(-1)^{n(n-1)/2}$ and $\langle\delta_j, \delta_i\rangle$ at the intersection of row i with column j for $i < j$. Then the matrix $(e_j, \hat{e}_i)$ of the $\mathbf{Z}[q, q^{-1}]$-bilinear intersection form in compatible distinguished bases $\{e_k\}$, $\{\hat{e}_k\}$ is $qV - (-1)^n V^t$.

2) The monodromy group in $H_n(E_\lambda; \mathscr{L}(q))$ is generated by reflections $M_k\colon \mapsto h + (-1)^{n(n+1)/2}(h, \hat{e}_k)\, e_k$.

COROLLARY 1. $(M_k - (-1)^n q)(M_k - 1) = 0$.

In fact, $M_k - 1$ maps the hyperplane orthogonal to $\hat{e}_k$ to zero and has one-dimensional image generated by $e_k$. Moreover, $M_k e_k = (-1)^n q e_k$, since $(e_k, \hat{e}_k) = (-1)^{n(n-1)/2}(q - (-1)^n)$.

Definition of the Bases $\{e_k\}$, $\{\hat{e}_k\}$. We choose a Morse function $F_\lambda$ from a miniversal family and a system of paths $\{\gamma_k\}$ as in Sec. 1. We orient the circle $\gamma_0$ of small radius with center at z = 0 counterclockwise. We fix a basic section s of the sheaf $\mathscr{L}(q)$, which is trivial over the z-plane with a slit (Fig. 3) issuing from 0 between $\gamma_\mu$ and $\gamma_1$. Thus, the section s is single-valued over the union of the paths $\gamma_1 \cup \ldots \cup \gamma_\mu$ and the arcs $[\gamma_0 \cap \gamma_1, \gamma_0 \cap \gamma_\mu]$ of the circle $\gamma_0$ between its points of intersection with $\gamma_1$ and $\gamma_\mu$.

The cycles $\delta_1, \ldots, \delta_\mu$, which vanish at the critical points over $z^{(1)}, \ldots, z^{(\mu)}$, upon transport along the paths $\gamma_1, \ldots, \gamma_\mu$, respectively, sweep out relative cycles in $(E_\lambda, \partial E_\lambda)$ (Fig. 4). We endow them with the section s of the sheaf $\mathscr{L}(q^{-1})$, and we denote the relative twisted cycles so obtained by $\hat{e}_1, \ldots, \hat{e}_\mu$.

Analogously, we define the cycle $e_k$ in the complement to $F_\lambda^{-1}(0)$, by carrying the vanishing cycle $\delta_k$ from the fiber over $z^{(k)}$ along the path $\gamma_k$ to the intersection with the circle $\gamma_0$, then along $\gamma_0$, and again along $\gamma_1$, in the opposite way to the fiber over $z^{(k)}$ (Fig. 2), and choosing a section of the sheaf $\mathscr{L}(q)$, equal to s under motion from $z^{(k)}$ along $\gamma_k$ and extended by continuity along all trajectories of transport of the cycle (so that upon returning to the point $z^{(k)}$ this section is qs).

As is explained in Sec. 2, $\{e_k\}$ is a basis in $H_n(E_\lambda; \mathscr{L}(q))$ (modulo torsion).

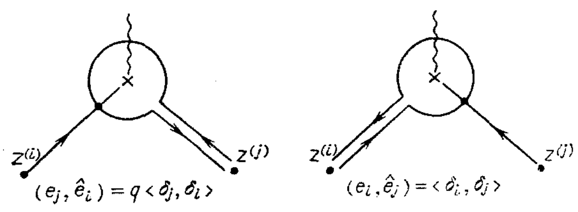


Fig. 5

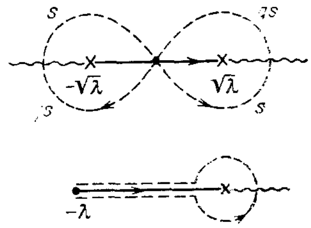


Fig. 6

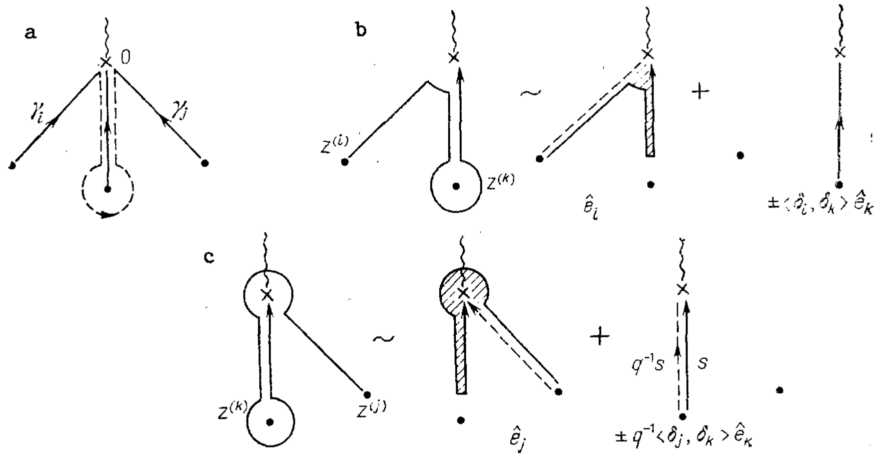


Fig. 7

<u>COROLLARY 2.</u> The sesquilinear intersection form in $H_n(E_\lambda; \mathcal{L}(q))$ is (skew)-Hermitian for (odd) even n and with respect to the distinguished basis $\{e_k\}$ is given by the matrix $\langle e_j, e_i\rangle = (q-1)V + (-1)^n (q^{-1}-1)V^t$.

In fact, under the natural map $H_n(E_\lambda; \mathcal{L}(q)) \to H_n(E_\lambda, \partial E_\lambda; \mathcal{L}(q))$ the cycle $e_k$ goes into the relative cycle conjugate to $(1-q^{-1})\hat{e}_k$, since $F_\lambda^{-1}(\gamma_0) \subset \partial E_\lambda$.

<u>Proof of the Theorem.</u> 1°. For i < j the intersection indices $(e_j, \hat{e}_i)$, $(e_i, \hat{e}_j)$ can be calculated starting from Fig. 5. Over a point of intersection of corresponding paths vanishing cycles must intersect: in the first case $-q\delta_j$ with $\delta_i$, and in the second $-\delta_i$ with $\delta_j$.

2°. To find the index of "self-intersection" $(e_k, \hat{e}_k)$ it suffices to calculate it for a singularity f with nondegenerate critical point. For clarity we restrict ourselves to the case n = 1. Let $f = x^2$, $F_\lambda = x^2 - \lambda$. Cycles e, ê in the x-plane are illustrated in Fig. 6. It is clear from it that the index of intersection of the dashed cycle with the solid one is equal to 1 + q, as was needed.

3°. The monodromy group is generated by circuits of the point z = 0 about the critical values of the function $F_\lambda$. Under a circuit about the critical value $z^{(k)}$ the vanishing cycle $\delta_k$ goes into $(-1)^n\delta_k$. Moreover, the point $z^{(k)}$ here finishes one revolution about the point 0. Hence

$$e_k \mapsto (-1)^n q e_k, \quad \hat{e}_k \mapsto (-1)^n q^{-1}\hat{e}_k.$$

4°. Now we find out how the cycles $\hat{e}_i$, $\hat{e}_j$, i < k < j are transformed upon circuit of z = 0 about $z^{(k)}$ (the cycles $e_i$, $e_j$ are transformed in the same way as the duals to $\hat{e}_i$, $\hat{e}_j$, i.e., in the formulas found below it is necessary to replace $q^{-1}$ by q). We represent cycles by paths in the z-plane. In Fig. 7a the dashed one represents the circuit of z = 0 around $z^{(k)}$. In Fig. 7b, c on the left it is shown what the paths $\gamma_i$, $\gamma_j$, along which it is necessary to transport the vanishing cycles $\delta_i$, $\delta_j$ turn into. Upon circuit around $z^{(k)}$ one adds to $\delta_i$, $\delta_j$ cycles proportional to $\delta_k$. In correspondence with this, cycles represented in Fig. 7b, c on the left are split into the sum of two summands. In the first of them, the original cycle is transported along the path, in the second, a multiple of $\delta_k$. The first summand is homologous to the original cycle: the difference of the corresponding cycles on the z-plane (the solid line minus the dashed one) is the boundary of the shaded domain. The second summand in Fig. 7b is simply $\hat{e}_{(k)}$ (with the corresponding numerical coefficient), and in Fig. 7c it differs by the factor $q^{-1}$. The latter is clear from the left part of Fig. 7c; the path issuing from $z^{(j)}$, before hitting $z^{(k)}$, intersects the slit.

5°. We calculate the monodromy and the intersection form independently of one another in terms of the intersection index of the vanishing cycles $\{\delta_k\}$. One gets the twisted Picard–Lefschetz formula by comparing the results of these calculations.

## 4. Examples and Generalizations

1. Burau Representation. Application of the preceding constructions to the family of polynomials in one variable $F = x^{\mu+1} + \lambda_1 x^{\mu-1} + \ldots + \lambda_\mu$ leads to a representation $\rho$ of the Artin braid group $\pi_1(\mathbf{C}^\mu \setminus \Delta)$ on the $\mu$-dimensional space of twisted cycles on $\mathbf{C} \setminus \{\mu + 1 \text{ points}\}$, known in the theory of braids as the Burau representation (cf. [19]). This representation plays an important role in the theory of links. Namely, $P_\sigma(q, q^{-1}) = \det(\rho(\sigma) - 1)$ is the Alexander polynomial, an invariant of a link of the braid $\sigma$ represented. There is an old conjecture about the faithfulness of the Burau representation, easily proved for groups of braids of 2 and 3 threads, but open for $\mu \geqslant 3$. The author feels an irresistible temptation to formulate a more general conjecture about complements of discriminants of isolated singularities of hypersurfaces.

2. Singularities of Powers of Volume Forms and de Rham's Theorem. The twisted cohomology originates in the problem of singularities of differential forms $f(x_1, \ldots, x_n) \times (dx_1 \wedge \ldots \wedge dx_n)^\beta$ of degree $\beta$. The question is, if you like, of the classification of singularities of functions $f\colon \mathbf{C}^n, 0 \to \mathbf{C}, 0$ with respect to the pseudogroup of diffeomorphisms $\mathbf{C}^n, 0 \to \mathbf{C}^n, 0$, acting on f by change of variables and by simultaneous multiplication by the $\beta$-th power of the Jacobian, $\beta \in \mathbf{C}$. A similar problem arose independently for $n = 1$, $\beta = 1/2$ in the author's paper [10], and was posed in the formulation cited by V. I. Arnol'd. A comparative classification of singularities of $\beta$-forms and functions ($\beta = 0$) was found by Varchenko [7] and Lando [15]. The versality theorem for $\beta$-forms was proved by Kostov [14] for $n = 1$, and in the dissertation of Lando for any n. Lando's theorem asserts that $F(x, \lambda)(dx)^\beta$ is a versal deformation of the form $f(x)(dx)^\beta$ (i.e., transverse to the orbit, for a precise definition of versal deformations, cf., e.g., [3]), if F is a versal deformation of a germ of a function f, and the set $-1/\beta + \mathbf{N}$ does not contain roots of the Bernshtein polynomial (cf. [5]) of this germ.

The condition of transversality of $F(dx)^\beta$ to an orbit of the pseudogroup is the condition of solvability of the equation

$$\varphi = \sum \left(\omega_i \frac{\partial F}{\partial x_i} + \beta \frac{\partial \omega_i}{\partial x_i} F\right) + \sum v_j \frac{\partial F}{\partial \lambda_j} \tag{1}$$

with respect to the functions $\omega_i(x, \lambda)$, $v_j(\lambda)$ for any $\varphi(x, \lambda)$. This equation has the following interpretation. Let $(\Omega^*_\lambda, d_\alpha)$ be the complex of holomorphic differential forms in $\mathbf{C}^n \setminus V_\lambda$, $d_\lambda = d - \alpha \frac{dF}{F} \wedge$. It follows from (1) that for all $\lambda$ the n-form $\varphi \frac{dx}{F_\lambda}$ is cohomologous to a linear combination of the forms $(\partial F_\lambda/\partial\lambda_j)\, dx/F_\lambda$ in this complex with $\alpha = -1/\beta$. The following "de Rham theorem" gives the connection with twisted cohomology.

Proposition. The twisted cohomology $H^*(\mathbf{C}^n \setminus V; \mathbf{C}(q))$ of the complement to a nonsingular hypersurface is isomorphic to the cohomology of the complex $(\Omega^*, d_\alpha)$ of holomorphic forms on $\mathbf{C}^n \setminus V$, if $q = e^{2\pi i\alpha}$.

Proof. Let $(\Omega^*, d_\alpha)$ be the complex of sheaves of holomorphic forms in $\mathbf{C}^n \setminus V$. With it there is connected a double complex $\mathscr{C}^r\Omega^s$ of Cech cochains with differentials $\delta\colon \mathscr{C}^r\Omega^s \to \mathscr{C}^{r+1}\Omega^s$ (Cech coboundary) and $d_\alpha\colon \mathscr{C}^r\Omega^s \to \mathscr{C}^r\Omega^{s+1}$. The hypercohomology $\mathbf{H}^*(\Omega^*, d_\alpha)$ of the complex of sheaves is defined as the cohomology of the complex $(\bigoplus_{r+s=*} \mathscr{C}^r\Omega^s, d_\alpha + \delta)$. In the present case it is unimportant what the hypercohomology is, but it is important that two spectral sequences $E^{r,s}$ and $\tilde{E}^{s,r}$ corresponding to two filtrations of the spaces $\bigoplus_{r+s=*} \mathscr{C}^r\Omega^s$, by r and by s (cf. [13]) converge to it.

The term $E_1$ of the first spectral sequence consists of Cech cochains of the cohomology sheaves of the complex $(\Omega^*, d_\alpha)$. The cohomology sheaves $\mathscr{H}^s(\Omega^*, d_\alpha)$ assign to the open set $U \subset \mathbf{C}^n$ the cohomology spaces of the complex $(\Omega^*|_U, d_\alpha)$. Local calculation shows that the "Poincaré lemma" holds: the sheaf $\mathscr{H}^s$ is zero for $s > 0$ and $\mathscr{H}^0 \simeq \mathbf{C}(q)$. Thus, in the first row of the array $E_1^{r,s}$ there is the complex of Cech chains of the sheaf $\mathbf{C}(q)$, and the other rows ($s > 0$) are zero.

The spaces $\tilde{E}_1^{r,s}$ of the second spectral sequence are the Cech cohomology spaces $H^r(\mathbf{C}^n; \Omega^s)$. Sheaves of holomorphic forms on Stein manifolds are acyclic. Hence $\tilde{E}_1^{s,r} = 0$ for r > 0, and $\tilde{E}_1^{*,0} = (\Omega^*, d_\alpha)$ is the complex of holomorphic forms.

Thus, $E_2 = E_\infty$, $\tilde{E}_2 = \tilde{E}_\infty$, and the equation $E_2 \simeq \mathbf{H} \simeq \tilde{E}_2$ is the assertion being proved.

3. Theory of Singularities of Divisors and Hypergeometric Functions. A direct generalization of the operator $d_\alpha$ is the differential $d - \alpha\, df/f - \beta\, dg/g - \ldots - \gamma\, dh/h$ in the complex of holomorphic forms in $\mathbf{C}^n$ with poles on the divisor $f^{-1}(0) \cup g^{-1}(0) \cup \ldots \cup h^{-1}(0)$.* The corresponding local theory is the theory of singularities of divisors (a nonsingular is a divisor with normal intersections). The twisted Picard–Lefschetz theory for divisors is needed to describe the monodromy of cycles on the complement of the divisor, endowed with the sheaf of coefficients "$f^\alpha g^\beta \ldots h^\gamma$." We shall return to this theory in another paper, and here we only note the numerous connections of this subject with others.

a) Simple singularities of divisors in $\mathbf{C}^n$, $n > 1$ are classified by Dynkin diagrams $A_\mu$, $B_\mu$, $C_\mu$, $D_\mu$, $E_6$, $E_7$, $E_8$, $F_4$.

b) The theory of singularities of functions of the form $f^\alpha g^\beta$ with nonsingular g contains the theory of Arnol'd [2] of critical points of functions on manifolds with boundary ($\alpha = 1$, $\beta = 0$) and the theory of Goryunov [12] of projections to a line ($\alpha = 0$, $\beta = 1$).

c) The case of linear functions f, g,...,h was considered in [18, 8, 9] in connection with a single theory of multidimensional hypergeometric functions.

Thus, the Picard–Lefschetz theory for a divisor of three merging points on a line describes the branching of ordinary hypergeometric integrals $I_\lambda(z) = \int_0^\lambda x^\alpha (x-\lambda)^\beta (x - z\lambda)^\gamma\, dx$.

4. Classical Monodromy Operator and Signature of Intersection Form. For a germ $f\colon \mathbf{C}^n, 0 \to \mathbf{C}, 0$ of a holomorphic function at a $\mu$-fold critical point we define the operator

$$M\colon H_n(E_\lambda; \mathbf{Z}(q)) \to H_n(E_\lambda; \mathbf{Z}(q))$$

of classical monodromy by transporting twisted cycles along the path $\lambda = \varepsilon e^{it}$, $t \in [0, 2\pi]$; $M = M_\mu \ldots M_1$ (in the notation of Sec. 3).

For (odd) even n and $q = e^{2\pi i\alpha}$ ($\alpha \in \mathbf{R}$) the intersection form $(\cdot, \cdot)$ in the complexified space of twisted cycles $H_n(E_\lambda; \mathbf{C}(q))$ is (skew)-Hermitian with respect to complex conjugation. The form $"\cdot, \cdot" = (\cdot, \cdot)/(q^{1/2} - q^{-1/2})$ is, on the contrary, Hermitian for odd n. In a distinguished basis for the space $H_n(E_\lambda; \mathbf{C}(q))$ it has the matrix $e^{\pi i\alpha}V + e^{-\pi i\alpha}V^t$. We denote the signature of this form by $(\mathrm{ind}_+, \mathrm{ind}_0, \mathrm{ind}_-)$.

In the theory of singularities one associates with a germ f its spectrum Sp, a collection of $\mu$ rational numbers $\alpha_1, \ldots, \alpha_\mu \in (0, n)$, symmetric with respect to n/2.

Proposition. 1) The classical monodromy operator has, with respect to a distinguished basis, the matrix $M = (-1)^n q (V^t)^{-1} V$. The numbers $qe^{2\pi i\alpha_k}$ form its spectrum.

2) For odd n and $q = e^{2\pi i\alpha}$, $\alpha \in \mathbf{R}$,

$$\mathrm{ind}_0 = \#\,\mathrm{Sp} \cap (\mathbf{Z} - \alpha),$$
$$\mathrm{ind}_+ = \#\,\mathrm{Sp} \cap [\bigcup_{k\in\mathbf{Z}} (2k - \alpha, 2k + 1 - \alpha)],$$
$$\mathrm{ind}_- = \#\,\mathrm{Sp} \cap [\bigcup_{k\in\mathbf{Z}} (2k - 1 - \alpha, 2k - \alpha)],$$

if the monodromy operator $M_0 = -(V^t)^{-1}V$ is diagonalizable.†

Remark. Up to the sign $(-1)^n$, the matrix V coincides with the matrix of the inverse variation operator

$$\mathrm{Var}^{-1}\colon H_{n-1}(V_\lambda) \to H_{n-1}(V_\lambda, \partial V_\mu)$$

*$d_\alpha$ also acts on the complex of meromorphic forms with pole on the divisor and on the logarithmic complex.

†This is so, for example, for quasihomogeneous singularities.

in a distinguished basis (cf. [4]), $V-(-1)^n V^t$ is the matrix of the intersection form in $H_{n-1}(V_\lambda)$, $(-1)^n (V^t)^{-1} V$ is the matrix of the classical monodromy operator $M_0$: $H_{n-1}(V_\lambda) \to H_{n-1}(V_\lambda)$ in this basis. The proposition is derived from this purely algebraically taking into account the description in terms of the spectrum for the signature of the intersection form in $H_{n-1} \times (V_\lambda)$, given by Steenbrink [20]. One can define the spectrum of a singularity with the help of the asymptotics as $\lambda \to 0$ of the integrals of holomorphic forms in $\mathbf{C}^n$ over cycles on $V_\lambda$ (cf. [4]). If analogously one defines the spectrum with the help of integrals $\int f^\alpha \omega$ over twisted cycles, then it turns out that one gets such a spectrum from Sp by translation by $\alpha$.

5. Twisting and Suspension. Let $V_\lambda \subset \mathbf{C}^n$ be a nonsingular local level manifold of a germ f of finite multiplicity (cf. Sec. 1), $E_\lambda = \mathbf{C}^n \setminus V_\lambda$ be its complement, $\tilde{E}_\lambda$ be a $\mathbf{Z}_p$ covering of $\mathbf{C}^n$, branched along $V_\lambda$.

Proposition. There is a canonical isomorphism

$$H_n(\tilde{E}_\lambda) = \oplus_{k=1}^{p-1} H_n\left(E_\lambda, \partial E_\lambda; \mathbf{Z}\left(\exp\left(\frac{2\pi i k}{p}\right)\right)\right).$$

One gets the proof easily from the exact sequence of the pair $(\tilde{E}_\lambda, V_\lambda)$. It contains the fragment

$$0 \to H_n(\tilde{E}_\lambda) \to H_n(\tilde{E}_\lambda, V_\lambda) \to \tilde{H}_{n-1}(V_\lambda) \to 0.$$

It is clear that $H_n(\tilde{E}_\lambda, V_\lambda) = \bigoplus_{\{q: q^p=1\}} H_n(E_\lambda, \partial E_\lambda; \mathbf{Z}(q))$. The decomposition into a direct sum here coincides with the decomposition of $H_n(\tilde{E}_\lambda, V_\lambda)$ by characters of the group of covering transformations $\mathbf{Z}_p$, and on $\tilde{H}_{n-1}(V_\lambda)$ this group acts trivially.

This proposition explains why the twisted Picard–Lefschetz formula interpolates between the even and odd versions of the classical theory. Under suspension $f \rightsquigarrow f - y^2$ the manifold $V_\lambda$ is replaced by the two-sheeted covering $\tilde{E}_\lambda$, branched along $V_\lambda$, and we get canonical isomorphisms

$$H_n(\tilde{E}_\lambda) = H_n(E_\lambda, \partial E_\lambda; \mathbf{Z}(-1)), \quad H_n(V_\lambda) = H_n(E_\lambda, \partial E_\lambda; \mathbf{Z}(1)).$$

6. Analytic Theory. Let $\Omega^*_{\log}$ be the space of meromorphic forms in $\mathbf{C}^n$ with a logarithmic pole on $V_\lambda$. The differential $d_\alpha = d - \alpha \frac{df_\lambda}{f_\lambda} \wedge$ turns $\Omega^*_{\log}$ into a complex. By the analytic theory of twisted cohomology we mean the study of the properties of the logarithmic complex. These properties depend on $\alpha$, and not only on $q = e^{2\pi i\alpha}$, as was the case in the topological theory considered above. Studying the asymptotics of the integrals $\int f^\alpha \omega$ of such forms over twisted cycles, one can endow the twisted cohomology spaces with a filtration which apparently has good "Hodge" properties, define the spectrum of a singularity, etc. It seems to me that the analytic theory of twisted cohomology happens to be the natural language for proving the theorem on the semicontinuity of the spectrum of a singularity. It suffices to recall the role which the behavior of the Hodge structure plays in the cohomology of branched covering under decomposition by the characters of a cyclic group in the proof of this theorem given by A. N. Varchenko and J. Steenbrink [6, 21].

LITERATURE CITED

1. V. I. Arnol'd, "Normal forms of functions near degenerate critical points, the Weyl groups $A_k$, $D_k$, $E_k$, and Lagrangian singularities," Funkts. Anal. Prilozhen., 5, No. 4, 3-25 (1972).
2. V. I. Arnol'd, "Critical points of functions on manifolds with boundary, simple Lie groups $B_k$, $C_k$, $F_4$, and singularities of evolutes," Usp. Mat. Fiz., 33, No. 5, 91-105 (1978).
3. V. I. Arnol'd, A. N. Varchenko, and S. M. Gusein-Zade, Singularities of Differentiable Maps [in Russian], Vol. 1, Nauka, Moscow (1982).
4. V. I. Arnol'd, A. N. Varchenko, and S. M. Gusein-Zade, Singularities of Differentiable Maps [in Russian], Vol. 2, Nauka, Moscow (1984).
5. I. N. Bernshtein, "Analytic continuation of generalized functions with respect to a parameter," Funkts. Anal. Prilozhen., 6, No. 4, 26-40 (1972).

6. A. N. Varchenko, "Semicontinuity of the spectrum and an upper bound for the number of singular points of a projective hypersurface," Dokl. Akad. Nauk SSSR, 270, No. 6, 1294-1297 (1983).
7. A. N. Varchenko, "Local classification of volume forms in the presence of a hypersurface," Funkts. Anal. Prilozhen., 19, No. 4, 23-31 (1985).
8. A. N. Varchenko, "Combinatorics and topology of the disposition of affine hyperplanes in real space," Funkts. Anal. Prilozhen., 21, No. 1, 11-22 (1987).
9. V. A. Vasil'ev, I. M. Gel'fand, and A. V. Zelevinskii, "General hypergeometric functions on complex Grassmanians," Funkts. Anal. Prilozhen., 21, No. 1, 23-38 (1987).
10. A. B. Givental', "Lagrangian manifolds with singularities and irreducible $s\ell_2$-modules," Usp. Mat. Nauk, 38, No. 6, 109-110 (1983).
11. A. B. Givental' and V. V. Shekhtman, "Monodromy groups and Hecke algebras," Usp. Mat. Nauk, 42, No. 4, 138 (1987).
12. V. V. Goryunov, "Geometry of bifurcation diagrams of simple projections to a line," Funkts. Anal. Prilozhen., 15, No. 2, 1-8 (1981).
13. P. Griffiths and J. Harris, Principles of Algebraic Geometry [Russian translation], Mir, Moscow (1982).
14. V. P. Kostov, "Versal deformations of differential forms of degree $\alpha$ on the line," Funkts. Anal. Prilozhen., 18, No. 4, 81-82 (1984).
15. S. K. Lando, "Normal forms of powers of the volume form," Funkts. Anal. Prilozhen., 19, No. 2, 78-79 (1985).
16. J. Milnor, Singular Points of Complex Hypersurfaces [Russian translation], Mir, Moscow (1971).
17. F. Fam, "Generalized Picard—Lefschetz formulas and branched integrals," Matematika, 13, No. 4, 61-93 (1969).
18. K. Aomoto, "On the structure of integrals of power product of linear functions," Sc. Papers Coll. General Educ. Univ. Tokyo, 27, 49-61 (1977).
19. T. Kohno, "Linear representations of braid groups and classical Yang—Baxter equation," in: Artin's Braid Groups, Santa Cruz (1987).
20. J. Steenbrink, "Mixed Hodge structure on the vanishing cohomology," in: Real and Complex Singularities, Nordic Summer School, Oslo (1976).
21. J. Steenbrink, "Semicontinuity of the singularity spectrum," Invent. Math., 79, No. 3, 557-565 (1985).
22. W. Ebeling, "On the monodromy groups of singularities," in: Proc. Symp. Pure Math., Vol. 40, Part 1 (1983), pp. 327-336.
23. S. V. Chmutov, "Monodromy groups of singularities of functions of two variables," Funkts. Anal. Prilozhen., 15, No. 1, 61-66 (1981).
24. G. G. Il'yuta, "Monodromy and vanishing cycles of boundary singularities," Funkts. Anal. Prilozhen., 19, No. 3, 11-21 (1985).