NORMAL FORMS FOR FUNCTIONS NEAR DEGENERATE CRITICAL POINTS, THE WEYL GROUPS OF A_k, D_k, E_k AND LAGRANGIAN SINGULARITIES

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There exist interesting connections between the simplest degenerate critical points of functions and the simple Lie algebras A_k, D_k, E_k (or at least their Weyl groups). In the present paper it is shown that critical points which are simple (without moduli) are classified by the series A_k, D_k, E_k. All degeneracies of codimension not greater than 5 turn out to be simple. Hence this paper also contains the classification of all degenerate critical points of codimension not greater than 5.

Critical points of functions are closely connected with singularities of projections of lagrangian manifolds. At the end of the paper, the classification of simple singularities of lagrangian maps is given. In particular, a table is computed of normal forms of projections of lagrangian manifolds in general position up to dimension 6, where moduli first appear.

§1. Introduction

The behavior of a function near a nondegenerate critical point is determined by Morse's lemma: an appropriate choice of coordinate functions will lead to the morsian normal form $-x_1^2 - \ldots - x_\nu^2 + x_{\nu+1}^2 + \ldots + x_n^2 + C$. It is also known that near an isolated critical point, even if it is degenerate, an analytic function is transformed into its Taylor polynomial by an appropriate analytic change of variables (see, e.g., [1], [2]).

In this sense, a function near an isolated critical point can always be reduced to polynomial normal form. However, there is an essential difference between this polynomial normal form and the morsian one: the coefficients of the Taylor polynomial are continuous parameters for the polynomial normal forms, while the morsian normal forms are determined by a discrete parameter (the index of inertia ν).

It turns out, that for the simplest degeneracies one can give normal forms similar to the morsian ones, i.e., which do not involve continuous parameters. In the present paper the classification of degeneracies of codimension less than 6 is given. We shall give a finite list of normal forms such that in typical $l < 6$-parameter families of functions no singularities occur other than those enumerated.

Carrying out an analogous classification of all critical points of codimension 6 is impossible, since here the normal forms must inevitably contain parameters ("moduli"). However, there are some special degeneracies (we shall call them simple) of any codimension, near which there are no moduli (for precise definitions see § 2). The basic result of this paper is the classification of all simple critical points.

A complete list of normal forms of a function in the neighborhood of a simple critical point appears as follows (we assume that the critical value is equal to zero):

$$\begin{array}{lll} A_k: f = \pm x_1^{k+1} \pm x_2^2 + Q, & k \geqslant 1, & \operatorname{codim} A_k = k-1, \\ D_k: f = x_1^2 x_2 \pm x_2^{k-1} + Q, & k \geqslant 4, & \operatorname{codim} D_k = k-1, \\ E_6: f = x_1^3 \pm x_2^4 + Q, & & \operatorname{codim} E_6 = 5, \\ E_7: f = x_1^3 + x_1 x_2^3 + Q, & & \operatorname{codim} E_7 = 6, \\ E_8: f = x_1^3 + x_2^5 + Q, & & \operatorname{codim} E_8 = 7, \end{array} \tag{1.1}$$

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where Q denotes the standard quadratic form

$$Q = -x_3^2 - \ldots - x_s^2 + x_{s+1}^2 + \ldots + x_n^2.$$

The connection between these series and the series of simple Lie groups A_k (linear), D_k (orthogonal) and E_k is discussed in §9.

All singularities of codimension $l < 6$ are simple. They are indicated exhaustively in the following table:

l	1	2	3	4	5
Singularity	A_2	A_3	A_4, D_4	A_5, D_5	A_6, D_6, E_6

(1.2)

In codimension $l \geq 6$ with $n \geq 3$ variables, along with the singularities A_k, D_k, E_k one meets singularities of other types: their normal forms inevitably contain parameters. For $n = 2$, the parameters show up beginning with codimension 7.

§2. Formulation of Results

We shall consider smooth real functions with critical point $0 \in \mathbf{R}^n$ and critical value 0 (the complex, **R**-analytic and formal cases differ with slight simplifications). The group of germs at 0 of local diffeomorphisms of the space $\mathbf{R}^n$ which leave 0 fixed, acts on the space of germs of our functions at 0. We are interested in the orbits of this action. Orbits "in general position" have codimension 0: they consist of functions with nondegenerate critical points. The remaining orbits have positive codimension.

Orbits of positive codimension can form discrete "stratifications" or continuous families. In the first case we shall call the orbits simple. In order to give the exact definition, we shall first consider the action $G \times M \to M$ of a finite-dimensional Lie group G on a finite-dimensional manifold M.

Definition 2.1. An orbit W of the action of G on M abuts an orbit V at some (and then any) point $v \in V$, if any neighborhood of the point v in M meets W.

Definition 2.2. An orbit V is called simple, if a sufficiently small neighborhood of one (and then any) of its points v meets only a finite number of orbits.

Remark 2.3. If the action of G on M is algebraic, then a neighborhood of a point v meets either a finite number of orbits, or a continuous family.

Remark 2.4. If in definitions 2.1 and 2.2 one replaces neighborhoods by local transversals, then these definitions will apply to orbits of finite codimension even in the infinite-dimensional case.

We consider as M the manifold of r-jets at the point 0 of functions with critical point $0 \in \mathbf{R}^n$ and critical value 0. On this manifold the group of r-jets of diffeomorphisms $\mathbf{R}^n \to \mathbf{R}^n$ which leave 0 fixed, acts algebraically.

Definition 2.5. A germ of a function f with critical point $0 \in \mathbf{R}^n$ and critical value 0 will be called simple if for sufficiently large r, 1) its orbit in the space of r-jets is simple and 2) the number of abutting orbits in the space of r-jets remains bounded as $r \to \infty$.

Example 2.6. The germ of the function of one variable $v(x) = x^k$ is simple, since its orbit in the space of r-jets for $r \geq k$ abuts only the orbits of the germs $\pm x^m$, where $m < k$.

Example 2.7. The germ of the function $v(x_1, x_2) = x_1x_2(x_1^2 - x_2^2)$ is not simple, since among the nearby functions $x_1(x_1^2 - x_2^2)(x_2 + tx_1)$ (where t is small) an infinite number belong to different orbits. Thus the cross-ratio of the four tangents to the branches of the zero level lines is an invariant diffeomorphism.

Example 2.8. The germ of the function of one variable $v(x) \equiv 0$ is not simple, although its r-jet for any r is simple.

The critical point 0 in this example is not isolated. A phenomenon, similar to what has been observed in this example is possible only for critical points of infinite multiplicity (in the analytic case not isolated). In fact, if a critical point is of finite multiplicity (in the analytic case isolated), sufficiently long r-jets define the germ up to diffeomorphisms, hence under passage to longer jets the number of abutting orbits is unchanged.

Definition 2.9. An r-jet of a function at 0 is called sufficient, if any 2 germs f_1, f_2 with this r-jet are carried into one another by a local diffeomorphism g which leaves 0 fixed: $f_1(x) = f_2(g(x))$.

The set of germs which do not have a sufficient jet (i.e., the set of germs with singularities of infinite multiplicity), has infinite codimension in the space of all germs.

Disregarding these completely unusual cases allows us to reduce the analytic problem of the classification of simple germs to an algebraic one (on orbits in the space of jets) and has no influence on the decisive results which are the following.

THEOREM 2.10. Each simple germ can be described in an appropriate coordinate system as one of the normal forms $A_k (k \geq 1)$, $D_k (k \geq 4)$, $E_k (6 \leq k \leq 8)$ of the list (1.1).

THEOREM 2.11. The set of all nonsimple germs has codimension 6 in the space of germs of functions of $n > 2$ variables with critical point 0 and critical value 0 (if it is desired, in the space of r-jets where $r \geq 3$). For $n = 2$ the codimension of the set of all nonsimple germs (or r-jets, where $r \geq 4$) is equal to 7, and for $n = 1$ it is infinite.

COROLLARY 2.12. The classification of degeneracies of codimension $l < 6$ is given by table (1.2). For example, in a typical five-parameter family of functions with separated values of the parameters, critical points of types A_6, D_6, E_6 occur, on some union of their curves types A_5, D_5, on some union of these curved surfaces types A_4 and D_4; the singularity A_3 occurs on a three-dimensional and A_2 on a four-dimensional submanifold of the parameter space. For the remaining values of the parameters, critical points are morsian (nondegenerate).

Remark 2.13. The results formulated are equally valid in C^∞ and **R**- or **C**-analytic situations and also in the framework of formal power series. In the complex case the number of normal forms is naturally lower, replacing all minus signs in (1.1) by plus signs. For even k the singularities A_k with x_1^{k+1} are equivalent even in the real domain (i.e., they belong to the same orbit). With this exception the remaining singularities in the list (1.1) are pairwise nonequivalent.

Remark 2.14. Further consequences of theorems 2.10 and 2.11 are mentioned in §§8-11. Among other things, from the results cited there it follows that in a typical family of functions of the special form $F(x, y) = S(x) - \langle x, y \rangle$ $(x \in \mathbf{R}^n, y \in \mathbf{R}^n)$, depending on n variables x and n parameters y, there occur exactly the same degenerate critical points as in a typical family F of general form which depends arbitrarily on n parameters.

The proof of Theorems 2.10 and 2.11 is given in §§ 3-7.

§ 3. Sufficient Jets

The use of the simple lemmas given below greatly shortens the calculations in the proof of the classification theorem.

We consider the ring of germs of smooth functions at the point $0 \in \mathbf{R}^n$. We denote by $\mathfrak{m}$ the maximal ideal of this ring, i.e., the collection of germs of functions equal to zero at the point 0. Thus, the notation $f \in \mathfrak{m}^s$ denotes that f has a zero of order s at 0.

LEMMA 3.1. Let f be a germ of a function at the point $0 \in \mathbf{R}^n$. We fix a natural number r and assume that f satisfies the following condition:

$$\forall \alpha \in \mathfrak{m}^{r+1}\ \exists h_i \in \mathfrak{m}^2:\ \alpha = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \bmod \mathfrak{m}^{r+2}. \qquad (3.1)_r$$

Then any function f' for which $f' = f \bmod \mathfrak{m}^{r+1}$ also satisfies $(3.1)_r$.

Proof. $\sum_{i=1}^{n} \frac{\partial (f'-f)}{\partial x_i} h_i \in \mathfrak{m}^{r+2}$, hence it suffices to take $h_i' = h_i$.

Remark. Thus, $(3.1)_r$ is a restriction on r-jets, not on germs.

LEMMA 3.2. Let the r-jet of the function f at 0 satisfy $(3.1)_r$. Then the r-jet of the function f at 0 is sufficient (see definition 2.9).

Proof. 1°. Let J^s be the space of s-jets of functions at the point 0 in the space $\mathbf{R}^n$. Let $m \geq 1$. We consider in the jet space J^{r+m} the submanifold consisting of all $r + m$-jets, whose r-jet coincides with the r-jet of the function f. We denote this submanifold by F_m; thus, $F_m \subset J^{r+m}$. To prove Lemma 3.2, we shall show first of all the following

Assertion A_m. Under the condition $(3.1)_r$ the orbit of the jet of the function f in J^{r+m} contains F_m for any $m \geq 1$.

2°. We shall prove assertion A_1. The action $G \times J \to J$ of the Lie group of s-jets of diffeomorphisms leaving 0 fixed, on the manifold of jets $J = J^s$ gives a map of tangent spaces (derived from the action on the first argument) $T_eG \to T_jJ$ (e is the unit of the group, j is the jet of the function f). In coordinate notation, this map associates with the s-jet of the vector field $h = \sum h_i \partial/\partial x_i$ at zero the s-jet of the function $\sum h_i \partial f/\partial x_i$ at zero. Hence, from condition $(3.1)_r$ it follows that the tangent space to the orbit of the $r + 1$-jet of the function f contains the tangent space to F_1 at the point which is the $r + 1$-jet of the function f.

But by Lemma 3.1, condition $(3.1)_r$ is also satisfied for all $f \in F_1$. Consequently, the tangent space to the orbit of each point of F_1 contains the entire tangent space to F_1 at this point.

Whence it follows that each curve in F_1 is tangent to the orbit of each of its points, and so belongs entirely to an orbit. Since the space F_1 is arcwise connected (it is diffeomorphic to euclidean space), F_1 belongs entirely to an orbit. A_1 is proved.

3°. We shall show that $(3.1)_r \Rightarrow (3.1)_{r+1}$. Let $\alpha \in \mathfrak{m}^{r+2}$. Then $\alpha = \sum x_i \alpha_i$ $(\alpha_i \in \mathfrak{m}^{r+1})$ (Hadamard's lemma). We represent α_i in the form $\alpha_i = \sum_j h_{ij} \cdot \partial f/\partial x_j \bmod \mathfrak{m}^{r+2}, h_{ij} \in \mathfrak{m}^2$, by virtue of $(3.1)_r$. We set $h_j = \sum_i x_i h_{ij}$. Then $\alpha = \sum h_j \cdot \partial f/\partial x_j \bmod \mathfrak{m}^{r+3}$ $(h_j \in \mathfrak{m}^3 \subset \mathfrak{m}^2)$, which is what was needed.

Applying what was proved $s - r$ times, we get $(3.1)_r \Rightarrow (3.1)_s$ for any $s \geq r$.

4°. Let f_1 and f_2 be two functions whose r-jets coincide with the r-jet of the function f. We shall show that their s-jets belong to one orbit in J^s for $s \geq r$. This was already proved in Par. 2° for $s = r + 1$. Let the k-jets of the functions f_1 and f_2 belong to one orbit in J^k, we shall show that their $k + 1$-jets belong to one orbit in J^{k+1}. We make a diffeomorphism which carries the k-jet of the function f_2 into the k-jet of the function f_1. The function f_2 is carried into a function f_3, whose $k + 1$-jet lies in the same orbit in J^{k+1} as the $k + 1$-jet of the function f_2. But the k-jets of the functions f_1 and f_3 coincide and satisfy $(3.1)_k$ by virtue of 3°. From A_1 (proved in 2°) it follows that the $k + 1$-jet of the function f_3 belongs to the orbit of the $k + 1$-jet of the function f_1 in J^{k+1}. This means that the $k + 1$-jets of f_1 and f_2 lie in one orbit in J^{k+1}, and A_m is proved for any $m \geq 1$.

5°. For sufficiently large s the s-jet of the function f is sufficient (see [2] or [1] and [3]). Whence and from A_m which was proved in 4° it follows that the r-jet is already sufficient.

In fact, let f_4 be another function with the same r-jet as f. There exists a diffeomorphism which carries f_4 into a function f_5 with the same s-jet as f (A_{s-r}). Since the s-jet of the function f is sufficient, there exists a diffeomorphism carrying f_5 into f. Thus, there exists a diffeomorphism carrying f_4 into f, and Lemma 3.2 is proved.

Example 3.3. We consider the function $x^3 + y^4$ of the two variables x, y. Its 4-jet at 0 is sufficient by Lemma 3.2. In fact, any monomial of degree 5 in x, y can be represented either in the form $3x^2h_1$ or in the form $4y^3h_2$, where h_1 is a monomial of degree 3 and h_2 is a monomial of degree 2.

Remark 3.4. To replace $h \in \mathfrak{m}^2$ by $h \in \mathfrak{m}$ in lemmas 3.1 and 3.2 is impossible. In fact, let $f(x) = x^2$, $r = 1$. Then $\forall \alpha \in \mathfrak{m}^2 \, \exists h \in \mathfrak{m}$: $\alpha = 2xh \bmod \mathfrak{m}^3$, however, the 1-jet of the function f at zero is not sufficient. In the same example the function $f' \equiv 0$ satisfies the condition $f = f' \bmod \mathfrak{m}^2$, but does not admit a decomposition $\alpha = h\partial f/\partial x$, although f admits such a decomposition.

Remark 3.5. The assertions and proofs of lemmas 3.1 and 3.2 are still valid in the R- or C-analytic and R- or C- formal cases.

§4. Classification of Simple Germs: the Series A_k

Let f be a germ of a smooth function at the point $0 \in \mathbf{R}^n$; we shall assume that 0 is a critical point $(df|_0 = 0)$ and that $f(0) = 0$. Let ρ be the rank of the second differential $d^2f|_0$, $\mathfrak{m}$ the maximal ideal of the ring of germs so that germs in $\mathfrak{m}^k$ have zeroes of order k at 0.

LEMMA 4.1. In a neighborhood of the point 0 there exists a smooth system of coordinates x, y in which f can be written in the form

$$f = \varphi(x) + Q(y) \qquad (\varphi \in \mathfrak{m}^3),$$

where Q is a nondegenerate quadratic form (dim $\{y\} = \rho$, dim $\{x\} = n - \rho$).

Proof. First we reduce 2-jets to canonical form. By the theorem of Jacobi on quadratic forms, there exists a system of coordinates x, z(dim $\{z\} = \rho$), in which the 2-jet of the function f takes the form

$$(j_2f)(x, z) = \sum \varepsilon_i z_i^2 \quad (i = 1, \ldots, \rho, \ \varepsilon_i = \pm 1). \tag{4.1}$$

We consider the restriction f_t of the function f to the plane x = t. For small $|t|$ the function f_t has a unique critical point near z = 0; this critical point depends smoothly on t, and at it $|z| = o(|t|)$ all of this follows from the nondegeneracy of the critical point z = 0 of the function f_0 and the implicit function theorem.

We denote by $\varphi(t)$ the value of the function f at this critical point; from formula (4.1), since $|z| = o(|t|)$ it follows that $|\varphi(t)| = O(|t|^3)$.

The difference $g(x, z) = f(x, z) - \varphi(x)$ can be considered as a family of functions of z depending smoothly on the parameter x, with nondegenerate critical point which depends smoothly on x and with critical value zero.

The generalized lemma of Morse asserts that such a family g can be reduced to the form $g(x, z) = \pm y_1^2 \pm \ldots \pm y_e^2$ by a smooth change of variables $y_i = y_i(x, z)$ (this lemma is proved in the same way as the ordinary lemma of Morse in which there is no parameter x; see, e.g., [4]). Now

$$f = \varphi(x) \pm y_1^2 \pm \cdots \pm y_\rho^2 \qquad (\varphi \in \mathfrak{m}^3),$$

which is what had to be proved.

The number of deficient squares $n - \rho = \dim\{x\}$ will be called the corank of the function f at zero. Thus, the corank of a function at a nondegenerate critical point is equal to zero.

LEMMA 4.2. The corank of a simple germ does not exceed two.

Proof. We consider the cubical Taylor polynomial of the function f at a critical point of corank k. The restriction to the null space of the second differential defines on this k-dimensional linear subspace of the tangent space a cubical form which is independent of the coordinate system.

The action of the group of diffeomorphisms on the space J^3 (3-jets of functions) induces an action of the linear group $GL(R^k)$ in the form of a linear substitution in the cubical forms on R^k. If the cubical forms of the functions f and g lie in different orbits of the action of $GL(R^k)$ on 3-jets, these functions lie in different orbits of the action of the group of diffeomorphisms on the space of jets J^3.

But the dimension of the space E of cubical forms in R^k is equal to C_{k+2}^3, and the dimension of the group $GL(R^k)$ is equal to k^2. For $k \geq 3$ we have $C_{k+2}^3 > k^2$. Consequently the dimension of all orbits of $GL(R^k)$ in E is less than the dimension of E. Hence the orbits form a continuous family in E for $k \geq 3$. This means that the orbits of the group of diffeomorphisms in J^3 also form a continuous family near the 3-jet of the function f.

Thus, a germ of corank $k \geq 3$ cannot be simple, and Lemma 4.2 is proved.

It remains for us to classify simple germs of coranks 1 and 2.

LEMMA 4.3. A simple germ of corank 1 can be reduced to one of the normal forms of type A_k with some $k \geq 2$,

$$f = \pm x^{k+1} + Q, \quad Q = \sum_{i=1}^{n-1} \varepsilon_i y_i^2 \qquad (\varepsilon_i = \pm 1). \tag{4.2}$$

Proof. By virtue of Lemma 4.1, in some coordinate system x, y we have $f = \varphi(x) + Q(y)$, where $\varphi \in \mathfrak{m}^3$. We investigate the function φ of one variable x. If all derivatives of φ at zero are equal to 0, then the germ f is not simple (infinite number of abutting orbits $-x^k + Q$). Let, for some $k \geq 2$, all derivatives of φ at 0 up to order k inclusive be equal to zero, and the derivative of order k + 1 be different from zero.

Then f reduces to normal form (4.2). In fact, by virtue of Lemma 3.2 the k + 1-jet

$$c^{k+1} \pm y_1^2 \pm \dots \pm y_{n-1}^2 \qquad (c \neq 0) \tag{4.3}$$

is sufficient: this follows from the factorization of all monomials in x, y of degree k + 2:

$$x^{k+2} = (k+1)\, c x^k h_0, \quad y_i g_\alpha(x, y) = \pm 2 y_i h_{i\alpha} \qquad (h_0, h_{i\alpha} \in \mathfrak{m}^2).$$

Thus, we can write f in the form (4.3), and hence, (after substituting $x' = |c|^{1/k+1} x$), also in the form (4.2). Lemma 4.3 is proved.

It is not difficult to verify that all germs (4.2) are simple (see below §8).

Remark 4.4. Actually, we have proved more than was formulated in Lemma 4.3. Namely, we have proved that every germ of corank 1 either reduces to one of the normal forms (4.2) or has a critical point of infinite multiplicity, and hence, belongs to a set of infinite codimension in the space of germs.

§5. Classification of Simple Germs of Corank 2: the Series D_k

Lemma 4.1 reduces the study of germs of corank k to the case of functions of k variables: the reduction is brought about in the general case just as it was done for the case k = 1 in the proof of lemma 4.3. Hence for the classification of simple germs of corank 2 it suffices to deal with functions of two variables. (However, the calculation carried out below could also be done for a larger number of variables, without using the above-cited reduction.)

A germ of a function of two variables has corank 2 if its 2-jet is zero. In this case the 3-jet determines a cubical form on the tangent plane. It is easy to prove

LEMMA 5.1. A cubical form on the plane R^2 is reduced by a linear change of variables to one of the following types: $x^2y \pm y^3$, x^2y, x^3, 0.

The proof depends on the fact that three different points in a projective line can be transformed into any other three by a projective transformation.

LEMMA 5.2. If the germ of a function of two variables of corank 2 is simple, then its cubical form is different from 0. Moreover, if a germ of corank 2 of a function of any number of variables has zero cubical form on the null plane of the second differential, then it is not simple.

Proof. Under the given hypotheses the 4-jet can be reduced to the form $\varphi(x_1, x_2) \pm x_3^2 \pm \dots \pm x_n^2$ mod $\mathfrak{m}^5$, where φ is a homogeneous polynomial of degree 4. This polynomial gives a form of degree 4 on the null plane of the second differential, which does not depend on the coordinate system. The cross-ratio, defined on zeros of φ, is an invariant action of the diffeomorphisms on the 4-jets. The existence of a continuous family of orbits distinguishing values of this invariant is an obstruction to the simplicity of the germ.

LEMMA 5.3. The 3-jet of the function $x^2y \pm y^3$ at 0 is sufficient.

Proof. For calculation it is more convenient to take the (obviously equivalent) form $f = x^2y \pm (y^3/3)$. Then $\partial f/\partial y = x^2 \pm y^2$, $\partial f/\partial x = 2xy$, and Lemma 5.3 follows from Lemma 3.2 and the decompositions

$$y^4 = (x^2 \pm y^2)(\pm y^2) \mp 2xy\,(xy/2), \quad y^3x = 2xy\,(y^2/2),$$
$$x^2y^2 = 2xy\,(xy/2), \quad yx^3 = 2xy\,(x^2/2), \quad x^4 = (x^2 \pm y^2)x^2 \mp 2xy\,(xy/2).$$

From Lemma 5.3 and 5.1 it follows that if the cubical form is not degenerate, then the function can be reduced to the normal form

$$D_4 : f = x^2y \pm y^3.$$

It is not hard to verify that both germs of type D_4 are simple (see §8).

We now consider the case of a cubical form of type x^2y.

LEMMA 5.4. A simple germ of a function of two variables of corank 2 with cubical form x^2y can be reduced to one of the normal forms

$$D_k : f = x^2y \pm y^{k-1} \ (k \geqslant 5).$$

Proof. We assume that the s-jet of the function f in some coordinate system has the form $x^2y + \varphi_s(x, y)$, where φ_s is a homogeneous polynomial of degree s. (For s = 4 there is automatically such a coordinate system). We represent φ_s in the form

$$\varphi_s = ay^s + 2bxy^{s-1} + x^2\psi(x, y), \text{ where } \psi \in \mathfrak{m}^{s-2}.$$

Substituting $x - by^{s-2} = x_1$, $y - \psi(x, y) = y_1$ reduces f to the form $f = x_1^2y_1 + ay_1^s \mod \mathfrak{m}^{s+1}$ (because $\mathfrak{m}^{2s-3} \subset \mathfrak{m}^{s+1}$ for s > 3). Next, two cases are possible: $a = 0$ and $a \neq 0$. If $a = 0$, then the s-jet of the function f is reduced to the form x^2y. Hence the s + 1-jet has the form $x_1^2y_1 + \varphi_{s+1}(x_1, y_1)$, and we can repeat the previous argument. Going on, we will either get $a = 0$ for all s = 4, 5, . . ., or at some step we get $a \neq 0$.

If $a \neq 0$ never occurs, then the singularity has infinite codimension and the germ of f at 0 is not simple (for example, its orbit in J^s abuts all orbits of $x^2y + x^k$ for $k \leq s$).

Now if for some s one has $a \neq 0$, then the germ of the function f at zero will be equivalent to the germ of $x^2y + ay^s$. For the proof, we verify that the s-jet $x^2y + (ay^s/s)$ ($s \geq 4$, $a \neq 0$) is sufficient. By Lemma 3.2 this follows from the following decompositions:

$$x^\alpha y^\beta = 2xyh(x, y), \quad h \in \mathfrak{m}^2 \text{ for } \alpha + \beta = s + 1 > 3 \quad (\alpha > 0, \ \beta > 0),$$
$$x^{s+1} = (x^2 + ay^{s-1})x^{s-1} + 2xy(-x^{s-2}y^{s-2}/2a) \quad (x^{s-1} \in \mathfrak{m}^2, \ (xy)^{s-2} \in \mathfrak{m}^2),$$
$$y^{s+1} = (x^2 + ay^{s-1})(y^2/a) + 2xy(-xy/2) \quad (y^2 \in \mathfrak{m}^2, \ (-xy/2) \in \mathfrak{m}^2).$$

In order to convert $x^2y + (ay^s/s)$ into $x^2y \pm y^s$, it suffices to substitute $x_1 = px$, $y_1 = qy$. Lemma 5.4 is proved.

Remark 5.5. Actually, more was proved than Lemma 5.4. Namely, we have proved that every germ of corank 2 with cubical form x^2y either reduces to one of the normal forms $D_k(f = x_1^2x_2 \pm x_2^{k-1} \pm x_3^2 \pm \ldots \pm x_n^2)$, or has a critical point of infinite multiplicity, and hence belongs to a set of infinite codimension in the space of germs.

It is not hard to verify that the germs D_k, $k \geq 4$, are simple (see § 8).

§6. The Germs E_6, E_7, E_8

We shall look for simple germs of functions of two variables of corank 2 with cubical form x^3.

LEMMA 6.1. A simple germ of a function of two variables of corank 2 with cubical form x^3 reduces to one of the following normal forms:

$$E_6: f = x^3 \pm y^4, \ E_7: f = x^3 + xy^3, \ E_8: f = x^3 + y^5.$$

Proof. 1°. We write the 4-jet of the function considered in the form

$$x^3 + ay^4 + bxy^3 + 3x^2\varphi(x, y) \mod \mathfrak{m}^5, \text{where } \varphi \in \mathfrak{m}^2.$$

Substituting $x + \varphi(x, y) = x_1$ turns the 4-jet into $x_1^3 + ay^4 + bx_1y^3 \mod \mathfrak{m}^5$.

2°. We assume that $a \neq 0$. Then one can reduce the 4-jet to the form $x_1^3 \pm y_1^4 + 4cx_1y_1^3 \mod \mathfrak{m}^5$ by substituting $y = py_1$. After this we set $y_2 = y_1 \pm cx_1$. Then the 4-jet assumes the form $x_1^3 \pm y_2^4 + 3x_1^2\psi \mod \mathfrak{m}^5$, where $\varphi \in \mathfrak{m}^2$. Now, substituting $x_1 + \psi = x_2$ reduces the 4-jet to the form $x_2^3 \pm y_2^4 \mod \mathfrak{m}^5$. By Lemma 3.2 this jet is sufficient (see example 3.3). Thus, in the case $a \neq 0$ the germ is reduced to the normal form E_6.

3°. We assume that $a = 0$, $b \neq 0$. Then the 4-jet has the form $x_1^3 + bx_1y^3$, and it can be reduced to the form $x_1^3 + x_1y_1^3 \mod \mathfrak{m}^5$ by substituting $y = py_1$.

We shall show that the 4-jet $x^3 + xy^3$ is sufficient. First, we shall show that the orbit of any 5-jet $x^3 + xy^3 + \varphi \mod \mathfrak{m}^6$ ($\varphi \in \mathfrak{m}^5$) is open in the space of 5-jets with 4-jet $x^3 + xy^3$. For this it suffices to represent any monomial 5-jet $x^\alpha y^\beta (\alpha + \beta = 5)$ in the form

$$x^\alpha y^\beta = (3x^2 + y^3 + A)h_1 + (3xy^2 + B)h_2 \mod \mathfrak{m}^6, \tag{6.1}$$

where $A = \partial\varphi/\partial x$, $B = \partial\varphi/\partial y \in \mathfrak{m}^4$, $h \in \mathfrak{m}^2$. For the decompositions of xy^4, x^2y^3 and x^3y^2 we take $h_1 = 0$, $h_2 = x^{\alpha-1}y^{\beta-2}/3$. Further,

$$x^5 = (3x^2 + y^3 + A)(x^3/3) + (3xy^2 + B)(-x^2y/3) \mod \mathfrak{m}^6.$$

Thus, for all monomials divisible by x, the decomposition (6.1) is achieved. Finally,

$$y^5 = (3x^2 + y^2 + A)y^2 + (3xy^2 + B)(-x) + Bx \bmod \mathfrak{m}^6;$$

Here $Bx \bmod \mathfrak{m}^6$ is a monomial of degree 5, which is divisible by x. The decomposition (6.1) with left side Bx exists by what was proved. This means it also exists for y^5.

Thus, the orbit of the 5-jet $x^3 + xy^3 + \varphi(\varphi \in \mathfrak{m}^5)$ is open in the space of 5-jets with 4-jet $x^3 + xy^3$ and hence contains this entire space. Consequently, any 5-jet with 4-jet $x^3 + xy^3 \bmod \mathfrak{m}^5$ can be reduced to the form $x^3 + xy^3 \bmod \mathfrak{m}^6$.

4°. We shall show that the 4-jet of $x^3 + xy^3 \bmod \mathfrak{m}^6$ is sufficient. Our computation in Par. 3° shows for any $\gamma \in \mathfrak{m}^5$ there exists a decomposition of the form (6.1)

$$\gamma = (3x^2 + y^3)h_1 + (3xy^2)h_2 \bmod \mathfrak{m}^6, \text{where } h_i \in \mathfrak{m}.$$

Consequently, for $\delta \in \mathfrak{m}^6$ there exists a decomposition

$$\delta = (3x^2 + y^3)h_1 + (3xy^2)h_2 \bmod \mathfrak{m}^7, \text{ where } h_i \in \mathfrak{m}^2.$$

Thus, the 5-jet of $x^3 + xy^3$ is sufficient by Lemma 3.2.

But by virtue of 3°, every 5-jet with 4-jet $x^3 + xy^3 \bmod \mathfrak{m}^5$ can be transformed into the form $x^3 + xy^3 \bmod \mathfrak{m}^6$. Thus, the 4-jet of $x^3 + xy^3 \bmod \mathfrak{m}^5$ is already sufficient.

We have thus proved that if in 1°, $a = 0$ and $b \neq 0$, then the function can be transformed to the normal form E_7.

5°. We assume that $a = 0$, $b = 0$. Then the 4-jet has the form $x^3 \bmod \mathfrak{m}^5$. We write the 5-jet of our function in the form

$$x^3 + a'y^5 + b'xy^4 + 3x^2\varphi(x, y) \bmod \mathfrak{m}^6, \text{where } \varphi \in \mathfrak{m}^3.$$

Substituting $x + \varphi(x, y) = x_1$ transforms the 5-jet into $x_1^3 + a'y^5 + b'x_1y^4 \bmod \mathfrak{m}^6$.

6°. We assume that $a' \neq 0$. Then one can reduce the 5-jet to the form $x_1^3 + y_1^5 + 5c'x_1y_1^4 \bmod \mathfrak{m}^6$ by substituting $y = py_1$. After this we set $y_2 = y_1 \pm c'x_1$. Then the 5-jet assumes the form $x_1^3 + y_2^5 + 3x_1^2\psi \bmod \mathfrak{m}^6$, where $\psi \in \mathfrak{m}^3$. Now substituting $x_1 + \psi = x_2$ transforms the 5-jet to the form $x_2^3 + y_2^5 \bmod \mathfrak{m}^6$.

By Lemma 3.2, the 5-jet $x^3 + y^5 \bmod \mathfrak{m}^6$ is sufficient (the monomial $x^\alpha y^\beta$ for $\alpha + \beta = 6$ is divisible either by x^2 or by y^4). Thus, in the case $a' \neq 0$ the function reduces to the normal form E_8.

7°. We assume $a' = 0$. Then the 5-jet has the form $x^3 + b'xy^4 \bmod \mathfrak{m}^6$. We shall show that this jet is not sufficient. Moreover, the partition of the space of 6-jets with this 5-jet into orbits is continuous (at least for $b' \neq 0$) in the neighborhood of any such 6-jet. Thus we will show that for simple jets $a' \neq 0$.

8°. We shall show that our 6-jet for $b' \neq 0$ can be reduced to the form $x^3 \pm xy^4 + \lambda y^6 \bmod \mathfrak{m}^7$. In fact, we shall show that the coefficient b' in the 5-jet can be made equal to ± 1 by an axial dilatation. After this we shall show that the tangent plane to the orbit of any 6-jet $x^3 \pm xy^4 + \varphi \bmod \mathfrak{m}^7$ (where $\varphi \in \mathfrak{m}^6$) contains the entire space of monomials of degree 6 which are divisible by x ($\bmod \mathfrak{m}^7$). For this, we shall give a decomposition of the monomial $x^\alpha y^\beta$ ($\alpha + \beta = 6$, $\alpha \geq 1$):

$$x^\alpha y^\beta = (3x^2 \pm y^4 + A)h_1 + (\pm 4y^3 + B)h_2 \bmod \mathfrak{m}^7,$$

where $A = \partial\varphi/\partial x$, $B = \partial\varphi/\partial y$, A and $B \in \mathfrak{m}^5$. Namely, for $1 \leq \alpha \leq 3$ we set $h_1 = 0$, $h_2 = \pm x^{\alpha-1}y^{\beta-3}/4 \in \mathfrak{m}^2$, and for $4 \leq \alpha \leq 6$ we set $h_1 = x^{\alpha-2}y^\beta/3 \in \mathfrak{m}^4$, $h_2 = 0$.

Consequently, the orbit of the 6-jet $x^3 \pm xy^4 + \varphi \bmod \mathfrak{m}^7$ contains a representative with $\varphi = \lambda y^6$, which is what was asserted.

9°. We consider the family of functions depending on a parameter λ, $f_\lambda = x^3 \pm xy^4 + \lambda y^6$. We shall show that the orbit of the germ of the function f_λ at zero (and even the orbit of its 6-jet) varies continuously with λ.

With this aim, we note that the zeros of f_λ form 3 parabolas $x = t_i y^2$, where t_i are the roots of the equation $t^3 \pm t + \lambda = 0$. We shall show that the ratio $(t_3 - t_1)/(t_2 - t_1)$ (definite to within the order of the roots) is an invariant of the system of three tangent parabolas with respect to diffeomorphisms (it is sufficient to consider the 2-jets of the parabolas and the 1-jets of the diffeomorphisms).

We shall verify that the ordered triple of parabolas $x = 0$, $x = y^2$, $x = ky^2$ $(k \neq 0, 1)$ is never transformed by a diffeomorphism of the plane (near $x = y = 0$) into another triple of the same form with different k (one parabola can be transformed by the diffeomorphism into $x = 0$, and the second into $x = y^2$).

A diffeomorphism which carries the first triple into the second leaves the y axis fixed and hence has the form

$$x' = x(a_{11} + u(x, y)), \quad y' = a_{21}x + a_{22}y + v(x, y), \text{where } u \in \mathfrak{m}, \ v \in \mathfrak{m}^2.$$

In order that the parabola $x = y^2$ remain in place, one must have $a_{11} = a_{22}^2$. But then the image of the parabola $x = ky^2$ will also be $x' = ky'^2 \bmod y'^3$, which cannot be a parabola $x' = k'y'^2$ for $k' \neq k$.

10°. We return to the 6-jet of 8°. The number $(t_3 - t_1)/(t_2 - t_1)$, constructed from this jet, varies continuously with λ. Whence it follows that the 6-jets of the functions f_λ belong to different orbits (which vary continuously with λ). This means that not one of the germs with $a' = 0$ is simple, and Lemma 6.1 is proved.

One can verify (see §8), that the germs E_6, E_7, E_8 are simple.

Remark 6.2. Actually, we have proved more than Lemma 6.1. Namely, we have proved that every germ of corank 2 with cubical form x^3 either reduces to one of the forms

$$E_6\colon f = x_1^3 \pm x_2^4 \pm x_3^2 \pm \ldots \pm x_n^2, \quad E_7\colon f = x_1^3 + x_1x_2^3 \pm x_3^2 \pm \ldots \pm x_n^2,$$

$$E_8\colon f = x_1^3 + x_2^5 \pm x_3^2 \pm \ldots \pm x_n^2,$$

or belongs to the set of codimension 8, which consists of germs whose 5-jets reduce to the form

$$x_1^3 \pm x_1x_2^4 \pm x_3^2 \pm \ldots \pm x_n^2 \bmod \mathfrak{m}^6.$$

All germs of this last type are not simple.

The last part of the assertion (that the germs with 5-jet $x_1^3 \pm x_1x_2^4 \pm x_3^2 \pm \ldots \pm x_n^2$ are not simple) for $n > 2$ does not follow formally from Lemma 6.1, but was also proved.

We shall show, for example, that the tangent plane to the orbit of the 6-jet $x_1^3 \pm x_1x_2^4 + \lambda x_2^6 \pm x_3^2 \pm \ldots \pm x_n^2$ (and every 6-jet with the given 5-jet reduces to this form) does not contain the direction x_2^6, i.e., that the equation relative to $h_i \in \mathfrak{m}$

$$(3x_1^2 \pm x_2^4)h_1 + (6\lambda x_2^5 \pm 4x_1x_2^3)h_2 \pm 2x_3h_3 \pm \ldots \pm 2x_nh_n = x_2^6 \bmod \mathfrak{m}^7$$

is unsolvable. In fact, we set $\varepsilon = \pm 1$, $x_3 = \ldots = x_n = 0$, and we introduce the notation $h_\varepsilon(x_2) = h_2(\varepsilon\sqrt{\mp 1/3}\,x_2^2, x_2, 0, \ldots, 0)$. Then we get

$$(6\lambda \pm 4\varepsilon\sqrt{\mp 1/3})h^\varepsilon x_2^5 = x_2^6 \bmod x_2^7.$$

Consequently,

$$h^\varepsilon(x_2) = cx_2 \bmod x_2^2, \quad c = 6\lambda + 4\sqrt{\mp 1/3} = 6\lambda - 4\sqrt{\mp 1/3},$$

which is impossible for such a λ.

Thus, the orbits of the 6-jets $x_1^3 \pm x_1, x_2^4 + \lambda x_2^6 \pm x_3^2 \pm \ldots \pm x_n^2 \bmod \mathfrak{m}^7$ vary continuously with λ, so no germ with such a 6-jet is simple.

Remark 6.3. The passage from the classification of germs of functions of k variables to germs of corank k of functions of any number of variables can also be justified with the help of any of the following assertions.

Proposition 6.4. The map which associates with the germ of φ at zero the germ of the function

$$f(x, y) = \varphi(x) + Q(y), \text{where } Q = \pm y_1^2 \pm \ldots \pm y_n^2 \quad (x \in \mathbf{R}^k, y \in \mathbf{R}^n, \varphi \in \mathfrak{m}^3),$$

at the point (0, 0) is transversal to the orbit of the germ of f at this point.

Proposition 6.5. If the germ of the functions $f_1(x, y) = \varphi_1(x) + Q(y)$ and $f_2(x, y) = \varphi_2(x) + Q(y)$ and (where $\varphi_1 \in \mathfrak{m}^3$, $\varphi_2 \in \mathfrak{m}^3$) at the point (0, 0) are smoothly equivalent and of finite multiplicity, then the germs of the functions φ_1 and φ_2 are also equivalent.

The proofs are not given since they are long and it is not clear if the requirement of finite multiplicity is essential. We shall not use propositions 6.4 and 6.5 in what follows.

§ 7. Classification of Singularities up to Codimension 6

Theorem 2.10 follows immediately from the lemmas proved above in §§ 4-6. As a matter of fact, these lemmas prove somewhat more, namely, from remarks 4.4, 5.5, and 6.2 follow the

THEOREM 7.1. Any germ of a function at a critical point* can either

1) be described in one of the forms A_k, D_k, E_6, E_7, E_8, or

2) belongs to the set of codimension 6, formed by all germs of corank greater than 2, or

3) belongs to the set of codimension 7, formed by those germs of corank 2, whose 3-jet reduces to the form $\pm x_3^2 \pm \ldots \pm x_n^2$, or

4) belongs to the set of codimension 8, formed by all germs of corank 2 with 5-jet which reduces to the form $x_1^3 \pm x_1 x_2^4 \pm x_3^2 \pm \ldots \pm x_n^2$, or

5) belongs to the set of infinite codimension, formed by all germs of corank 1 with critical point of infinite multiplicity.

In case 1) the germ is simple, and in the remaining cases it is not.

Each of the sets described in 2)-5), has, for $n \geq 3$, codimension not less than 6 (for $n = 1$, not less than 7, for $n = 2$, infinite codimension). Hence from Theorem 7.1 follows Theorem 2.10 as well as Theorem 2.11 and Corollary 2.12.

§8. Versal Deformations of Functions

We recall the definition of versal deformations of functions (for more details on versal deformations (see [5], [6]).

Let f be the germ of a smooth† function f: $R^n \to R$ at the point 0. A deformation F of the function f is a germ of a smooth function F: $R^n \times R^l \to R$ at the point (0, 0), for which $F(x, 0) = f(x)$. The space R^l of the second argument of F is called the base of the deformation, and its elements are called the parameters of the deformation. The dimensions of the bases of different deformations of one function f can be distinct.

Definition 8.1. A deformation F of the germ of a function f is called versal, if every other deformation of the function f is equivalent to one induced from F.

This means that for any function G: $R^n \times R^n \to R$ which is smooth near the point (0, 0), such that $G(y, 0) = f(y)$, there exist 1) a map ψ: $R^m \to R^l$ which is smooth near 0 and $\psi(0) = 0$ (change of parameter), 2) a diffeomorphism ($x = X(y, \mu)$, where X: $R^n \times R^m \to R^n$, $X(0, 0) = 0$, $\det (D_1 X)(0, 0) \neq 0$) which depends smoothly on the parameter $\mu \in R^l$, which turns into the identity transformation for $\mu = 0$ ($X(y, 0) = y$), such that

$$G(y, \mu) \equiv F(X(y, \mu), \psi(\mu)).$$

Differentiating this relation, we arrive at the following definition.

Definition 8.2. A deformation F: $R^n \times R^l \to R$ of the germ of a function f: $R^n \to R$ at zero is called infinitesimally versal if every germ at 0 of a function α: $R^n \to R$ can be represented in the form

$$\alpha(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i(x) + \sum_{j=1}^{l} c_j \varphi_j(x),$$

where $\varphi_j = \left.\frac{\partial F}{\partial \lambda_j}\right|_{\lambda=0}$ $(\lambda = (\lambda_1, \ldots, \lambda_l) \in R^l)$, h_i are smooth functions, c_i are numbers.

In other words, a deformation of the germ of a function at a critical point is infinitesimally versal if the germ of its derivative with respect to the parameter generates the local ring of gradient maps at the critical point.

*With critical value zero; the codimension is also in the space of germs with critical value zero.

†There exist C^∞, R-, and C-analytic and formal variants of the following definitions; the results carry over to all these cases.

THEOREM 8.3. Every infinitesimally versal deformation is versal.

The proof is too long to give it here; it can be carried out by any of a series of standard schemes (see, e.g., [1], [7]).

Choosing generators of the local ring for germs of the functions A_k, D_k, E_k, we get the basic result of the present paragraph.

COROLLARY 8.4. As versal deformations for the germs of functions A_k, D_k, and E_k [see formulas (1.1)] one can take the following k-parameter deformations:

$$A_k:\ F(x,\lambda) = \pm x_1^{k+1} \pm x_2^2 + Q + \lambda_{k-1}x_1^{k-1} + \lambda_{k-2}x_1^{k-2} + \ldots + \lambda_1 x_1 + \lambda_0,$$
$$D_k:\ F(x,\lambda) = x_1^2 x_2 \pm x_2^{k-1} + Q + \lambda_{k-1}x_1 + \lambda_{k-2}x_2^{k-2} + \ldots + \lambda_1 x_2 + \lambda_0,$$
$$E_6:\ F(x,\lambda) = x_1^3 \pm x_2^4 + Q + \lambda_5 x_1 x_2^2 + \lambda_4 x_1 x_2 + \lambda_3 x_2^2 + \lambda_2 x_2 + \lambda_1 x_1 + \lambda_0,$$
$$E_7:\ F(x,\lambda) = x_1^3 + x_1 x_2^3 + Q + \lambda_6 x_1 x_2 + \lambda_5 x_2^4 + \lambda_4 x_2^3 + \lambda_3 x_2^2 + \lambda_2 x_2 + \lambda_1 x_1 + \lambda_0,$$
$$E_8:\ F(x,\lambda) = x_1^3 + x_2^5 + Q + \lambda_7 x_1 x_2^3 + \lambda_6 x_1 x_2^2 + \lambda_5 x_1 x_2 + \lambda_4 x_2^3 + \lambda_3 x_2^2 + \lambda_2 x_2 + \lambda_1 x_1 + \lambda_0.$$

Here Q is the standard quadratic form

$$-x_3^2 - \ldots - x_s^2 + x_{s+1}^2 + \ldots + x_n^2.$$

COROLLARY 8.5. As transversals to the orbits of the simple germs f in the space of r-jets of functions with critical point 0 and critical value zero, one can take the linear family with k − 1 parameters $\varepsilon_2, \ldots, \varepsilon_k$ of the form

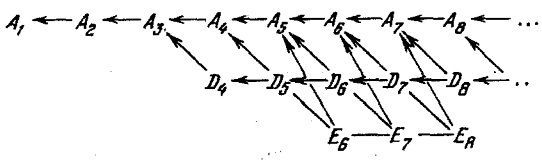
$$A_k:\ \pm x_1^{k+1} \pm x_2^2 + Q + \varepsilon_2 x_1^2 + \varepsilon_3 x_1^3 + \ldots + \varepsilon_k x_1^k \qquad (r \geqslant k+1),$$
$$D_k:\ x_1^2 x_2 \pm x_2^{k-1} + Q + \varepsilon_2 x_2^2 + \ldots + \varepsilon_{k-2} x_2^{k-2} + \varepsilon_{k-1} x_1 x_2 + \varepsilon_k x_1^2 \qquad (r \geqslant k-1),$$
$$E_6:\ x_1^3 \pm x_2^4 + Q + \varepsilon_2 x_2^2 + \varepsilon_3 x_2^3 + \varepsilon_4 x_1^2 + \varepsilon_5 x_1 x_2 + \varepsilon_6 x_1 x_2^2 \qquad (r \geqslant 4),$$
$$E_7:\ x_1^3 + x_1 x_2^3 + Q + \varepsilon_2 x_2^2 + \varepsilon_3 x_2^3 + \varepsilon_4 x_2^4 + \varepsilon_5 x_1^2 + \varepsilon_6 x_1 x_2 + \varepsilon_7 x_1 x_2^2 \qquad (r \geqslant 4),$$
$$E_8:\ x_1^3 + x_2^5 + Q + \varepsilon_2 x_2^2 + \varepsilon_3 x_2^3 + \varepsilon_4 x_2^4 + \varepsilon_5 x_1^2 + \varepsilon_6 x_1 x_2 + \varepsilon_7 x_1 x_2^2 + \varepsilon_8 x_1 x_2^3 \qquad (r \geqslant 5).$$

From these formulas, in particular, follows

COROLLARY 8.6. The germs of types A_k, D_k, E_6, E_7, E_8 are simple.

In fact, by direct calculation it is easy to see that for all ε the germs indicated in Corollary 8.5 belong to the orbits of types A_l, D_l, E_l ($l \leq k$), which (for fixed k and n) are finite in number. The formulas of Corollary 8.5 also clearly permit one to enumerate all orbits which abut orbits of the given simple germ.

COROLLARY 8.7. The diagram of abutting of simple germs has the form (in the complex case)



Here P ← S denotes that the orbit P abuts the orbit S; the complete set of orbits abutting S is obtained from S by moving along arrows in the diagram.

It is easy to foresee the result of Corollary 8.7 heuristically, by considering the inclusions of the Dynkin diagrams of A_k, D_k, E_k in one another (see §9). However, the proof requires some calculation.

First of all, from dimensional considerations it is clear that S_k can abut only P_l with $l < k$ (P and S run through the values A, D, E). Further, from the semicontinuity of the rank and multiplicity, it follows that A_k can abut only A_l, and that D_k can abut only D_l and A_l. Thus, only the abutments indicated in the diagram are possible.

In order to prove that the abutment $P_{k-1} \leftarrow S_k$ is realized, it is sufficient to indicate a curve in the space of jets ($t \to f_t$), for which f_0 has type S_k, and f_t for $t \neq 0$ has type P_{k-1}.

As such a curve, one can take, for example, the intersection of the transversals to S_k, indicated in Corollary 8.5, with orbit P_{k-1}.

The calculations (rather tedious, especially for $A_7 \leftarrow E_8$) give the following curves:

$$
\begin{aligned}
&A_{k-1} \leftarrow A_k\colon\ f_t = x_1^{k+1} + tx_1^k + x_2^2 + Q,\\
&D_{k-1} \leftarrow D_k\colon\ f_t = x_1^2x_2 + x_2^{k-1} + tx_2^{k-2} + Q,\\
&A_{k-1} \leftarrow D_k\colon\ f_t = x_1^2x_2 - x_2^{k-1} - [tx_2^{k-2} + t^2x_2^{k-3} + \ldots + t^{k-3}x_2^2] + 2\sqrt{t^{k-2}}x_1x_2 - tx_1^2 + Q,\\
&D_5 \leftarrow E_6\colon\ f_t = (x_1 - tx_2)^2(x_1 + 2tx_2) + x_2^4 + Q,\\
&A_5 \leftarrow E_6\colon\ f_t = x_1^3 + (x_2^2 + tx_1)^2 + Q,\\
&D_6 \leftarrow E_7\colon\ f_t = (x_1 - tx_2)^2(x_1 + 2tx_2) + x_2^3(x_1 - tx_2) + Q,\\
&A_6 \leftarrow E_7\colon\ f_t = 16t^3x_1^2 + x_1^3 - 4tx_1^2x_2 - 8t^2x_1x_2^2 + x_2^3(x_1 + tx_2) + Q,\\
&D_7 \leftarrow E_8\colon\ f_t = (x_1 - 3tx_2)^2(x_1 + 6tx_2) - 18t^3x_2^4 + 6tx_2^3x_1 + x_2^5 + Q,\\
&A_7 \leftarrow E_8\colon\ f_t = t^5(x_1 - t^2x_2)^2 - x_1(5t^4x_2^2 + 4tx_2^3) + x_1^3 + 4t^6x_2^3 + 5t^3x_2^4 + x_2^5 + Q,\\
&E_6 \leftarrow E_7\colon f_t = x_1^3 + x_1x_2^3 + tx_2^4 + Q,\\
&E_7 \leftarrow E_8\colon f_t = x_1^3 + x_2^5 + tx_1x_2^3 + Q.
\end{aligned}
$$

We shall show, for example, that the function f_t indicated in the line $A_5 \leftarrow E_6$, for $t \neq 0$, belongs to A_5. In fact, for $t \neq 0$ the corank of the germ of f_t at 0 is equal to 1, so f_t has type A_k with some $k \leq \infty$. In order to find k it suffices to consider the function $\varphi_t(x_2) = f_t(x_1(x_2), x_2)$, where the function $x_1(x_2)$ is defined by the condition $\left.\frac{\partial f_t}{\partial x_1}\right|_{x_1(x_2),\, x_2} = 0$; the order of zero of the function φ_t is then equal to $k + 1$. We find successively

$$
\begin{gathered}
3x_1^2 + 2t(x_2^2 + tx_1) = 0, \quad x_1(x_2) = -(x_2^2/t) + o(x_2^2),\\
(x_2^2 + tx_1(x_2))^2 = 9x_1^4/4t^2 = o(x_2^6), \quad \varphi_t = -x_2^6/t^3 + o(x_2^6).
\end{gathered}
$$

Thus, $k + 1 = 6$, i.e., f_t has type A_5.

§9. Bifurcation Diagrams, Braids, and the Weyl Group

We consider the bases of the versal deformations indicated in Corollary 8.4. These bases are naturally stratified according to the singularities of the zeros of the level surfaces of the function $F(\cdot, \lambda)$. The set of all values of the parameter $\lambda \in C^k$ for which the function $F(\cdot, \lambda)$ has zero as a critical value, forms a hypersurface Σ in C^k (the bifurcation diagram), which can be called the generalized swallow's tail of the corresponding singularity (A_k, D_k or E_k); the ordinary swallow's tail in C^3 is obtained for the singularity A_3.

In this paragraph we consider (without proofs) more or less new propositions about bifurcation diagrams of germs of type A_k, D_k, E_k.

PROPOSITION 9.1. The complement in C^k to the bifurcation diagram Σ of a germ of type A_k, D_k or E_k is an Eilenberg-MacLane space $K(\pi, 1)$: $\pi_i(C^k - \Sigma) = 0$ for $i \geq 2$.

In order to describe the fundamental group of this complement, we recall the construction of E. Brieskorn of the group of braids of the group generated by a map (see [8], [9], [10]).

Let Γ be a finite group generated by a map of R^m into some hyperplane.

The complexification of the action of Γ on R^m gives a finite group in C^m. Let X be the domain in C^m formed by points of "general type" (with orbits of the smallest number of points). The domain X is obtained from C^k by throwing out a certain number of hyperplanes ("diagonals").

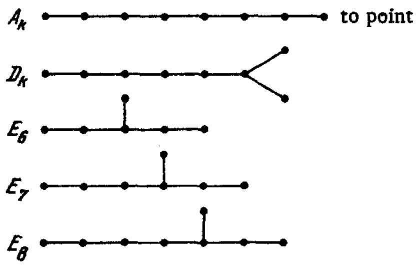
Definition 9.2. The braid group of the group Γ is the fundamental group of the quotient space X/Γ : $B(\Gamma) = \pi_1(X/\Gamma)$.

PROPOSITION 9.3. The fundamental group of the complement of the bifurcation diagram of a germ of type A_k, D_k, or E_k is the braid group of the corresponding Weyl group: $\pi_1(C^k - \Sigma) = B(\Gamma)$; $C^k - \Sigma$ is homeomorphic to X/Γ.

The complement $C^k - \Sigma$ of the bifurcation surface serves as the base of a series of important fibrations. One of them has fiber over the point $\lambda \in C^k - \Sigma$ the hypersurface in C^n with equation $F(\cdot, \lambda) = 0$. Since λ does not belong to the bifurcation diagram Σ, the fiber is a nonsingular hypersurface. We denote it by $V = V_\lambda$.

PROPOSITION 9.4. The nonsingular fiber V_λ of a versal deformation of a singularity A_k, D_k, or E_k is homotopically equivalent to a bouquet of spheres of dimension $n - 1$.

As these spheres one can take the vanishing cycles of Picard-Lefschetz. Moreover, if n is odd (so that the intersection index in $H_{n-1}(V_\lambda)$ is symmetric), then these k cycles can be chosen so that their intersection indices will give the Dynkin diagram:



Here each point represents a sphere (a generator in $H_{n-1}(V_\lambda)$). The self-intersection indices of the generators are equal to $(-1)^{n-1/2}2$, if the points α and β are joined by a segment, and $(\alpha, \beta) = (-1)^{n+1/2}$, if the points α and β are joined by a segment, and $(\alpha, \beta) = 0$, if there is no segment.

PROPOSITION 9.5. The action of the fundamental group $\pi_1(C^k - \Sigma)$ of the base of the fibration considered on the homology of the fiber V_λ (for odd n) is the standard representation of the generalized braid group as the Weyl group; the Picard-Lefschetz transformation is realized as a map into $H_{n-1}(V_\lambda)$ (for A_k, see the work of A. N. Varchenko [11]).

Remark 9.6. An interesting special case is the case $n = 3$. In this case the hypersurface V_λ is an ordinary surface. The singularities of surfaces of types A_k, D_k, E_k have been thoroughly studied under the name of "double rational points" (see [12], [13], [14]). In the theory of double rational points, in particular, it is proved that a singular point of a surface which satisfies some rigidity condition, can be reduced to one of the types A_k, D_k, E_6, E_7, E_8. Moreover, it turns out that the set of lines joined in minimal resolutions of double rational points is described by a Dynkin diagram (points – lines, segments – intersections).

Surfaces with double rational points, their versal deformations, and their resolutions can be obtained from the corresponding Lie groups (A_k, D_k, E_k): they appear in the description of the singularities of a map, associating with a matrix (an element of the Lie algebra) its characteristic polynomial. In this connection, see [15], [16], [17]. With the indicated map is also connected a family of algebraic manifolds of other (but always even) dimensions – it would not be surprising to meet among them $\{V_\lambda\}$ not only for $n - 1 = 2$ but also for $n - 1 = 2l > 2$.

Despite the facts mentioned above, the connection of singularities with Weyl groups does not seem to be well understood. For example, it is not clear whether connected with singularities are only coxeter groups generated by reflections, or real Lie groups with their Weyl groups – in the latter case one could hope to obtain some information about nonsimple germs.

§10. Lagrangian Singularities

The investigation above of the classification of simple germs of functions has applications in the theory of singularities of projections of so-called lagrangian manifolds.

Actually, this classification was found for the solution of the problems of asymptotic integrals of rapidly oscillating functions [18], which is closely connected with lagrangian singularities. In the present paragraph is communicated (without detailed proof) preliminary information about lagrangian singularities; the classification results obtained with its help is contained in §11.

Definition 10.1. A symplectic manifold is a pair (M^{2n}, ω^2), where M^{2n} is a smooth even-dimensional manifold, and ω^2 is a closed nondegenerate differential 2-form on it. A diffeomorphism of symplectic manfolds $f: M_1^{2n} \to M_2^{2n}$ is called symplectic if $f^*\omega_2^2 = \omega_1^2$.

Definition 10.2. A submanifold L^n of a symplectic manifold (M^{2n}, ω^2) is called a lagrangian manifold if the form $i^*\omega^2$ induced by the inclusion $i: L \to M$ is equal to zero.

Definition 10.3. A fibration $p: M \to B$ is called a lagrangian fibration if its fiber is lagrangian.

Definition 10.4. A lagrangian map $\pi: L \to B$ is the map induced by the projection $\pi = pi$ of a lagrangian manifold onto the base of a lagrangian fibration.

Examples of lagrangian fibrations are the cotangent fibrations T^*B of smooth manifolds B. More particular examples are coordinate 2n-dimensional space R^{2n} with coordinates x_i, y_i ($i = 1, \ldots, n$), with form $\omega^2 = \sum dx_i \wedge dy_i$ and projections $\pi(x, y) = y$.

PROPOSITION 10.5. Any lagrangian fibration is locally isomorphic to the standard one just described, and the affine structure on the fiber in a neighborhood of each point is defined invariantly.

The proof is based on an application of Darboux's theorem.

An example of a lagrangian submanifold is a submanifold $L = \{x, y : y = \partial S/\partial x\}$, where S is a function on $R^n = \{x\}$, of the standard 2n-dimensional space R^{2n}. The corresponding lagrangian map $\pi: L \to B$ is called the gradient. If x_i are taken as coordinates on L, and y_i on B, then π is given by the formula

$$\pi(x_1, \ldots, x_n) = \left(\frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n}\right). \tag{10.1}$$

Every lagrangian submanifold L^n in R^{2n} is locally given by its "derivative function" S(x) in a neighborhood of each of its points, in which the tangent space to L^n is transversal to the y-space.

PROPOSITION 10.6. In a neighborhood of each point without exception a lagrangian manifold is given by at least one of the 2^n formulas of the following form:

$$y_i = \frac{\partial F}{\partial x_i}, \quad x_j = -\frac{\partial F}{\partial y_j} \qquad (i \in I,\ j \in J), \tag{10.2}$$

where $F = F(x_I, y_J)$ is the "derivative function" and $I = (i_1, \ldots, i_k)$ is one of the 2^n subsets of the set (1, ..., n), and J is its complement. Here, if the kernel of the projection of the tangent plane onto the y-space is k-dimensional, then as I one can take a set of k elements.

The proof is found, for example, in [19].

Parallel to the general theory of singularities of smooth maps there is a theory of singularities of lagrangian maps, which are about as frequently met in applications (caustics, envelopes, Huygens principle, Hamilton-Jacobi equations, etc.).

Definition 10.7. A lagrangian equivalence of lagrangian maps $\pi_{1,2} : L_{1,2} \to B_{1,2}$ is a symplectic diffeomorphism $s: M_1 \to M_2$, $d: B_1 \to B_2$ of the lagrangian fibrations $p_{1,2}: M_{1,2} \to B_{1,2}$, such that L_1 is carried into L_2. If such an equivalence exists, then the maps π_1, π_2 are said to be lagrangian equivalent.

Proposition 10.8. From lagrangian equivalence follows equivalence in the sense of the ordinary theory of singularities of smooth maps ($f_{1,2}: U \to V$ are equivalent if there exist diffeomorphisms $h: U \to U$ and $k: V \to V$ such that $kf_1 = f_2h$).

The proof is obvious. The converse proposition is not true, as is shown by the example of smooth equivalent but lagrangian inequivalent germs at zero $f_1 = x^3$ and $f_2 = x^3 + x^4$.

Definition 10.9. A lagrangian map is said to be lagrange stable if every close lagrangian map is lagrangian equivalent to it.

Remark. In the definition of lagrange stability one must choose among several possible concepts. Here is one of them.

Definition 10.10. A lagrangian map is said to be weakly lagrange stable if every close lagrangian map is equivalent to it in the sense of the ordinary theory of singularities.

Lagrange stable maps are weakly lagrange stable according to proposition 10.8. I do not know examples of weakly lagrange stable germs of maps which are not lagrange stable.*

*There exist germs of smooth maps $R^2 \to R^2$ which are not equivalent to gradients in the sense of the ordinary theory of singularities (and also local rings which are not realized by gradients). The author thanks V. P. Palamodova, for showing the example: $y_1 = x_1^3$, $y_2 = x_1^2 x_2 + x_2^3$.

Remark. The concepts of lagrangian equivalence and stability in the large are not very reasonable. For example, the lagrangian curves $(x^2 + y^2 - 4)((x - \lambda)^2 + y^2 - 1) = 0$ in the plane (x, y) for different λ_1, $\lambda_2 (0 \leq \lambda_{1,2} < 1)$ are lagrange inequivalent because of the invariance of the affine structure on the fiber of a lagrangian fibration.

Local variants of definitions of lagrangian stability and weak lagrangian stability of germs are constructed on the model of the ordinary theory of singularities (see [1]). Below "stability" always means "lagrangian stability," if the contrary is not asserted.

PROPOSITION 10.11. Every germ of a lagrangian map is lagrange equivalent to the germ of a gradient map.

For the proof, a slight bending of the coordinate system suffices. In fact, every germ of a lagrangian map can be written in the form (10.2). The symplectic diffeomorphism given by $x'_i = x_i$, $y'_i = y_i$ $(i \in I)$, $x'_j = x_j + \lambda y_j$, $y'_j = y_j$ $(j \notin I)$, preserves fibers and defines a lagrangian equivalence of the initial map with some new lagrangian map, depending on the number λ. It is easy to verify that this new germ is a gradient [i.e., can be written in the form (10.1)] for almost all λ.

In fact, a derivative function of the form (10.2) for the new germ will be $F - \frac{1}{2}\lambda \sum y_j^2$ $(j \in J)$. The condition of local solvability of the new equation (10.2) with respect to y_J will be $|\partial^2 F/\partial y_J^2 - \lambda E| \neq 0$. If λ is not an eigenvalue of the hessian of F with respect to y_J, then the new germ is a gradient; thus, the exceptional values of λ are finite in number (not more than n).

Proposition 10.11 formally reduces the study of all germs of lagrangian maps to the study of gradient germs. However actually in working with lagrangian singularities it is more convenient to use the charts (10.2).

We proceed now to the infinitesimal analogs of the concepts introduced.

Definition 10.12. An infinitesimal lagrangian equivalence of a lagrangian fibration is a vector field on the total space which preserves both the symplectic structure and the fiber structure.

Just like any vector field which preserves the symplectic structure on M, the field of an infinitesimal lagrangian equivalence X is locally given by a real function (the Hamiltonian function) H by the formula $\omega^2(X(x), \xi) = dH(\xi)$ $(\forall \xi \in T_x M)$.

LEMMA 10.13. The Hamiltonian function of an infinitesimal lagrangian equivalence is linear (inhomogeneous) along each fiber. Conversely, any function which is linear (inhomogeneous) along each fiber gives an infinitesimal lagrangian equivalence.

Proof. By virtue of proposition 10.5, it suffices to consider the coordinate fibration $(x, y) \mapsto (y)$. Then the field X has components $\partial H/\partial y, -\partial H/\partial x$. For the field to be a lagrangian equivalence, the second component should not depend on x. Consequently, $H = a(y)x + b(y)$, which is what was needed.

Remark. Lagrangian equivalences are classically called "extended point transformations."

Definition 10.14. An infinitesimal lagrangian deformation of a lagrangian manifold is an element of the tangent space to the manifold of a lagrangian manifold.

Definition 10.15. A lagrangian map is called infinitesimally stable if every infinitesimal lagrangian deformation of its lagrangian manifold is induced by some infinitesimal lagrangian equivalence.

The definition of the corresponding local concepts is analogous: it is only necessary to replace manifolds and maps by their germs everywhere.

Remark. In order to make definition 10.14 for germs completely precise, we note that if the germ of a lagrangian manifold is given by formula (10.1) or (10.2), then the tangent space to the manifold germ is identified in the germs considered with the space of germs of smooth functions of x (or of x_I, y^J). From Lemma 10.13 it is easy to deduce

PROPOSITION 10.16. For infinitesimal stability of a gradient germ (10.1) at zero it is necessary and sufficient that the local ring of the gradient map be generated by linear functions.

Here, the local ring of a gradient map is the quotient ring of the germs of smooth functions of x at 0 by the ideal spanned by the n germs of the partial derivatives $\partial S/\partial x_i$.

Thus, the condition of infinitesimal stability consists in the fact that for every germ α of a smooth function at 0 there exists a decomposition $\alpha(x) = \sum \frac{\partial S}{\partial x_i} h_i(x) + c_0 + c_1x_1 + \ldots + c_nx_n$, where h_i are germs of smooth functions at zero, and c_i are numbers.

Now we shall formulate the condition of infinitesimal stability for germs of the form (10.2) at zero. We assume that at zero $F = 0$ and $\partial F/\partial x_I = 0$ (the latter can be achieved by translation of the origin of coordinates y_I). We introduce the functions $f(x) = F(x, 0)$ and $\varphi_j(x) = (\partial F/\partial y_j)(x, 0)$ $(j \in J, x = x_I)$.

From Lemma 10.13 follows

THEOREM 10.17. Infinitesimal stability of the germ (10.2) at zero is equivalent to versality of the n + 1-parameter deformation G of the germ of the function f at 0 which is obtained from F if the y_j are considered as parameters and a general linear inhomogeneous function of x is added:

$$G(x, \lambda) = f(x) + \sum \lambda_j \varphi_j(x) + \lambda_0 + \sum \lambda_i x_i \quad (i \in I,\ j \in J).$$

The proof is based on the fact that the condition of infinitesimal stability (existence of a decomposition of any germ $\alpha(x, y)$ into a sum $\alpha(x, y) = A_0 + \sum A_i x_i + \sum A_j(\partial F/\partial y_j)$, $i \in I$, $j \in J$, where A_m are germs of functions of $\partial F/\partial x_i$ and of y_J at zero) coincides with the condition of infinitesimal versality from Par. 8.2 (existence of a decomposition of any germ $\alpha(x)$ into a sum $(x) = \sum h_i \partial f/\partial x_i + \sum c_j \varphi_j(x_I) + c_0 + \sum c_i x_i$ $(i \in I,\ j \in J)$ (where h are germs of functions of x_I at zero, and c_m are numbers) by the preparation theorem of Weierstrass-Malgrange.

The classification of lagrangian germs is obtained by combining the theorems on functions from §§ 2-8, Theorem 10.17 and the following proposition.

THEOREM 10.18. Every infinitesimally stable germ of a lagrangian map is stable.

This follows from general theorems on actions of infinite-dimensional groups, which are not formulated in [1], and can be proved on the model of the proofs of the stability theorems in [1] or on the model of the proofs of J. Mather in [7].

§11. Classification of Simple Lagrangian Germs

Now we shall define and classify simple stable germs of lagrangian maps. Germs of lagrangian maps of an n-dimensional lagrangian manifold in general position are stable and simple for $n < 6$. Hence our classification reduces to normal form the germs of lagrangian maps in general position for $n < 6$.

Definition 11.1. A germ of a lagrangian map at the point x_0 is called simple, if there exists a finite set of germs of lagrangian maps such that every finite-parameter family of lagrangian maps containing a map with the given germ for the value zero of the parameters has for all close values of the parameters at all points sufficiently close to x_0, only germs which are lagrangian equivalent to germs of the given set.

Remark 11.12. Stable germs are not necessarily simple. In fact, the manifold of singularities of a stable germ can contain curves along which the lagrangian type of the germ varies continuously. Simple germs can be unstable (an example is the gradient map of the line with $S(x) = x^4$). The basic result of this paragraph is the classification of simple stable germs.

THEOREM 11.3. Every stable simple germ of a lagrangian map of an n-dimensional lagrangian manifold is lagrangian equivalent to one of the germs of the following list:

$$A_k:\ F = \pm x_1^{k+1} + y_{k-1}x_1^{k-1} + \ldots + y_2x_1^2 \qquad (k \leqslant n+1, I = \{1\}),$$
$$D_k:\ F = \pm x_1^2x_2 \pm x_2^{k-1} + y_{k-1}x_2^{k-2} + \ldots + y_3x_2^2 \qquad (k \leqslant n+1, I = \{1, 2\}),$$
$$E_6:\ F = \pm x_1^3 \pm x_2^4 + y_5x_1x_2^2 + y_4x_1x_2 + y_3x_2^2 \qquad (5 \leqslant n, I = \{1, 2\}),$$
$$E_7:\ F = \pm x_1^3 \pm x_1x_2^3 + y_6x_1x_2 + y_5x_2^4 + y_4x_2^3 + y_3x_2^2 \quad (6 \leqslant n, I = \{1, 2\}),$$
$$E_8:\ F = \pm x_1^3 \pm x_2^5 + y_7x_1x_2^3 + y_6x_1x_2^2 + y_5x_1x_2 + y_4x_2^3 + y_3x_2^2 \qquad (7 \leqslant n,\ I = \{1, 2\}).$$

Here F is a function which gives the equations of a lagrangian submanifold of the space $R^{2n} = \{(x, y)\}$ by the formulas $y_i = \partial F/\partial x_i$, $x_j = -\partial F/\partial y_j$ $(i \in I, j \notin I)$. The lagrangian map is given by the projection $(x, y) \mapsto y$.

Remark 11.4. The normal forms 11.3 are written by throwing out summands of degree 0 and 1 in x from the formulas for versal deformations in Par. 8.4. The stability of the germs follows from Theorems 10.17 and 10.18; the completeness of the list is proved (with the help of these theorems) by the same arguments which in §§3–6 proved the completeness of the list of simple germs of functions.

COROLLARY 11.5. Every stable simple germ of a lagrangian map of an n-dimensional lagrangian manifold is lagrangian equivalent to one of the germs of gradient maps given by the following functions:

$$A_k:\ S = \pm x_1^{k+1} + (x_{k-1} + x_1^{k-1})^2 + \ldots + (x_2 + x_1^2)^2 + Q,$$
$$D_k:\ S = \pm x_1^2 x_2 \pm x_2^{k-1} + (x_{k-1} + x_2^{k-2})^2 + \ldots + (x_3 + x_2^2)^2 + Q,$$
$$E_6:\ S = \pm x_1^3 \pm x_2^4 + (x_5 + x_1 x_2^2)^2 + (x_4 + x_1 x_2)^2 + (x_3 + x_2^2)^2 + Q,$$
$$E_7:\ S = \pm x_1^3 \pm x_1 x_2^3 + (x_6 + x_1 x_2)^2 + (x_5 + x_2^4)^2 + (x_4 + x_2^3)^2 + (x_3 + x_2^2)^2 + Q,$$
$$E_8:\ S = \pm x_1^3 \pm x_2^5 + (x_7 + x_1 x_2^3)^2 + (x_6 + x_1 x_2^2)^2 + (x_5 + x_1 x_2)^2 + (x_4 + x_2^3)^2 + (x_3 + x_2^2)^2 + Q,$$

where $Q = x_k^2 + \ldots + x_n^2$.

The corollary is proved with the help of proposition 10.11 and the normal forms given here are obtained from the normal forms of Theorem 11.3 by the transformations indicated in the proof of proposition 10.11.

THEOREM 11.6. For $n < 6$ a lagrangian map of an n-dimensional lagrangian manifold in general position has at each point a simple stable germ.

COROLLARY 11.7. A lagrangian map of a lagrangian manifold of dimension $n < 6$ can, by a small perturbation (in the class of lagrangian maps), be transformed into one such that in a neighborhood of each of its points it will reduce by a lagrangian equivalence to one of the following normal forms:

for n = 1

$$A_1:\ F = x_1^2, \quad A_2:\ F = \pm x_1^3;$$

for n = 2, in addition,

$$A_3:\ F = \pm x_1^4 + y_2 x_1^2;$$

for n = 3, in addition,

$$A_4:\ F = \pm x_1^5 + y_3 x_1^3 + y_2 x_1^2,$$
$$D_4:\ F = \pm x_1^2 x_2 \pm x_2^3 + y_3 x_2^2;$$

for n = 4, in addition,

$$A_5:\ F = \pm x_1^6 + y_4 x_1^4 + y_3 x_1^3 + y_2 x_1^2,$$
$$D_5:\ F = \pm x_1^2 x_2 \pm x_2^4 + y_4 x_2^3 + y_3 x_2^2;$$

for n = 5, in addition,

$$A_6:\ F = \pm x_1^7 + y_5 x_1^5 + \ldots + y_2 x_1^2,$$
$$D_6:\ F = \pm x_1^2 x_2 \pm x_2^5 + y_5 x_2^4 + y_4 x_2^3 + y_3 x_2^2,$$
$$E_6:\ F = \pm x_1^3 \pm x_2^4 + y_5 x_1 x_2^2 + y_4 x_1 x_2 + y_3 x_2^2.$$

Here the lagrangian manifold is given in the space $R^{2n} = \{(x, y)\}$ by the equations $y_i = \partial F/\partial x_i$, $x_j = -\partial F/\partial y_i$, where $i = 1$, $j \neq 1$ for the cases A_k and $i = 1$ or 2, $j \neq 1$ and 2 for D_k and E_k. The lagrangian map is the projection onto the y-space.

For example, a typical lagrangian map of a three-dimensional manifold has at isolated points singularities of type A_4 and D_4 (three tangents come together, two of them can be minimal), on curves joining these points there can be singularities of type A_3 (cusps) and on surfaces passing through these curves there can be singularities of type A_2 (folds).

We remark here further that the codimension $c(I) = c(i_1, \ldots, i_s)$ of the Boardman class Σ^I (see [1] or [20], [21]) in the space of jets of lagrangian maps of an n-dimensional manifold can be computed by the following formula: $c(I) = \nu_{n,1}(n, I) - n$, where ν is the codimension of the Boardman class $\Sigma^{n, i_1, \ldots, i_s}$ in the space of jets of functions $R^n \to R^1$. To compute ν, in [1] or [20] there are formulas. For example, $A_k = \Sigma^{1_{k-1}, 0}$ and has codimension $k - 1$, Σ^2 has codimension 3 and consists (with a residual part of codimension 4) of jets of type D_4, $\Sigma^{2,1}$ has codimension 5 and consists (with a residual part of codimension 6) of jets of type E_6, and $\Sigma^{2,1,1}$ has codimension 7 and contains E_8.

COROLLARY 11.8. A lagrangian map of a typical lagrangian manifold of dimension $n < 6$ can, by a small perturbation (in the class of lagrangian maps), be transformed into one such that in a neighborhood of any point it can be reduced by a lagrangian equivalence to one of the following normal forms $x \mapsto \partial S/\partial x$:

for n = 1

$$A_1\colon\ S = x_1^2, \quad A_2\colon\ S = \pm x_1^3,$$

for n = 2

$$A_1\colon\ S = x_1^2 + x_2^2, \quad A_2\colon\ S = \pm x_1^3 + x_2^2, \quad A_3\colon\ S = \pm x_1^4 + (x_2 + x_1^2)^2.$$

for n = 3

$$\begin{aligned}
A_1&\colon\ S = x_1^2 + x_2^2 + x_3^2,\\
A_2&\colon\ S = \pm x_1^3 + x_2^2 + x_3^2,\\
A_3&\colon\ S = \pm x_1^4 + (x_2 + x_1^2)^2 + x_3^2,\\
A_4&\colon\ S = \pm x_1^5 + (x_3 + x_1^3)^2 + (x_2 + x_1^2)^2,\\
D_4&\colon\ S = \pm x_1^2 x_2 \pm x_2^3 + (x_3 + x_2^2)^2,
\end{aligned}$$

for n = 4

$$\begin{aligned}
A_1&\colon\ S = x_1^2 + x_2^2 + x_3^2 + x_4^2,\\
A_2&\colon\ S = \pm x_1^3 + x_2^2 + x_3^2 + x_4^2,\\
A_3&\colon\ S = \pm x_1^4 + (x_2 + x_1^2)^2 + x_3^2 + x_4^2,\\
A_4&\colon\ S = \pm x_1^5 + (x_3 + x_1^3)^2 + (x_2 + x_1^2)^2 + x_4^2,\\
A_5&\colon\ S = \pm x_1^6 + (x_4 + x_1^4)^2 + (x_3 + x_1^3)^2 + (x_2 + x_1^2)^2,\\
D_4&\colon\ S = \pm x_1^2 x_2 \pm x_2^3 + (x_3 + x_2^2)^2 + x_4^2,\\
D_5&\colon\ S = \pm x_1^2 x_2 \pm x_2^4 + (x_4 + x_2^3)^2 + (x_3 + x_2^2)^2,
\end{aligned}$$

for n = 5

$$\begin{aligned}
A_1&\colon\ S = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2,\\
A_2&\colon\ S = \pm x_1^3 + x_2^2 + x_3^2 + x_4^2 + x_5^2,\\
A_3&\colon\ S = \pm x_1^4 + (x_2 + x_1^2)^2 + x_3^2 + x_4^2 + x_5^2,\\
A_4&\colon\ S = \pm x_1^5 + (x_3 + x_1^3)^2 + (x_2 + x_1^2)^2 + x_4^2 + x_5^2,\\
A_5&\colon\ S = \pm x_1^6 + (x_4 + x_1^4)^2 + (x_3 + x_1^3)^2 + (x_2 + x_1^2)^2 + x_5^2,\\
A_6&\colon\ S = \pm x_1^7 + (x_5 + x_1^5)^2 + (x_4 + x_1^4)^2 + (x_3 + x_1^3)^2 + (x_2 + x_1^2)^2,\\
D_4&\colon\ S = \pm x_1^2 x_2 \pm x_2^3 + (x_3 + x_2^2)^2 + x_4^2 + x_5^2,\\
D_5&\colon\ S = \pm x_1^2 x_2 \pm x_2^4 + (x_4 + x_2^3)^2 + (x_3 + x_2^2)^2 + x_5^2,\\
D_6&\colon\ S = \pm x_1^2 x_2 \pm x_2^5 + (x_5 + x_2^4)^2 + (x_4 + x_2^3)^2 + (x_3 + x_2^2)^2,\\
E_6&\colon\ S = \pm x_1^3 \pm x_2^4 + (x_5 + x_1 x_2^2)^2 + (x_4 + x_1 x_2)^2 + (x_3 + x_2^2)^2.
\end{aligned}$$

Remark 11.9. For $n \geq 6$ there exist lagrangian maps which cannot be approximated by lagrangian maps whose germs at each point are lagrange stable (or simple, or even weakly lagrange stable). All this follows from the existence of projectively invariant cubical curves on the projective plane and from the fact that the codimension of Σ^3 in the lagrangian case is equal to 6.

Note Added in Proof. At the time of publication of this paper the author received the preprint of J. Guckenheim on "Catastrophes and partial differential equations," pp. 1-29 (Princeton, 1972); among the results announced by J. Guckenheim are some of the propositions of § 10 of the present paper; he also indicated that some of them were found by L. Hörmander and A. Weinstein.

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