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CURVATURE OF GROUPS OF DIFFEOMORPHISMS PRESERVING THE MEASURE OF THE 2-SPHERE

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In this paper, the curvatures of the groups S Diff(S^2) (diffeomorphisms of the 2-sphere S^2 preserving the standard density) equipped with the natural right-invariant Riemannian metric (weak metric) are calculated. It was shown by Arnol'd [1, 2] that the geodesics on groups of this type express flows of an ideal incompressible fluid, and negativity of the sectional curvature along two-dimensional directions is a criterion for exponential instability of flows. Steady flow on a two-dimensional torus having the velocity field sin y ∂x (i.e., a Passat flow) was, in particular, studied in [2] in detail. In this paper, the following analog of the Passat flow is studied for S^2: viz., the vector field $g = z(-y\partial x + x\partial y)$; in many two-dimensional planes cutting the field g, the curvatures turn out to be negative. The curvature values obtained are used to estimate the interval of time during which long-term dynamic weather forecasting is not possible, and results close to those of [2] are obtained. The vector field $h = -y\partial x + x\partial y$ (the curl on S^2) is also studied, for which the sectional curvatures are nonnegative. The author sincerely thanks V. I. Arnol'd for valuable advice, and also A. L. Onishchik for helpful discussions.

1. Statement of the Results

Let S^2 be defined in $\mathbf{R}^3$ by the equation $x^2 + y^2 + z^2 = 1$. We denote by SV(S^2) the Lie algebra of the group S Diff(S^2) consisting of vector fields with zero divergence. The right-invariant metric on S Diff(S^2) is defined at the identity by

$$\langle u, v\rangle = \int_{S^2} (u(x), \bar{v}(x))\, d\mu(x) \qquad (u, v \in SV(S^2)^{\mathbf{C}}).$$

It is convenient to represent vector fields on SV(S^2) by their flow functions: $v = T(f_v) = I(\operatorname{grad} f_v)$ (where I is the operator given by clockwise rotation by 90°).

We choose in the space of flow functions a basis consisting of the spherical functions (φ, θ being the standard spherical coordinates on S^2)

$$Y_m^l = \left[\frac{(l-m)!}{(l+m)!}\frac{2l+1}{4\pi}\right]^{1/2}\frac{1}{2^l l!}(e^{i\varphi}\sin\varphi)^m\frac{d^{l+m}(\sin^l\theta)}{d(\cos\theta)^{l+m}} \quad (l \in \mathrm{N},\ m = -l, \ldots, l).$$

We remark that $\| T(Y_m^l)\|^2 = -\lambda_\Delta \| Y_m^l\|^2 = l(l+1)$, from which an orthonormal basis in SV(S^2) is formed by the vector fields $e_m^l = T(1/\sqrt{l(l+1)}Y_m^l)$.

We agree to denote by K(u, v) the curvature taken at the identity element of S Diff(S^2) along the two-dimensional plane L{u, v}.

THEOREM 1. The sectional curvatures along two-dimensional planes containing the vector field $h = -y\partial x + x\partial y$ are given by the formulas

1) $K(h, e_m^l) = \frac{3}{8\pi}\frac{m^2}{l^2(l+1)^2}$;

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TABLE 1

(l, m)	(1,1)	(2,1)	(2,2)	(3,1)	(3,2)	(3,3)
$K(g, e^l_m)$	$1/32\pi$	$15/224\pi$	$-15/28\pi$	$1/128\pi$	$-1/8\pi$	$-111/128\pi$

2) if $v = \sum_{l,m} v^l_m e^l_m$, then $K(h, v) = \frac{3}{8\pi} \sum_{l,m} |v^l_m|^2 \frac{m^2}{l^2(l+1)^2}$.

COROLLARY 1. The curvatures $K(h, v) \geqslant 0$, and K(h, v) = 0 only for vector fields v commuting with h, $\operatorname{Max}_v K(h, v) = 3/32\,\pi = K$ (the curvature of SO(3), the orthogonal subgroup) and is attained when $v \in SO(3)$.

THEOREM 2. For curvatures in two-dimensional planes containing the Passat flow, we have

1) $K(g, e^l_m) = (15\,m^2/32\,\pi)\,[(1 - c_l)^2\,(a^l_m b_l + a^{l+1}_m/b_{l+1}) + 2(1 + c_l) \times (a^l_m + a^{l+1}_m) - 3\,(a^l_m/b_l + a^{l+1}_m b_{l+1})]$. Here $a^l_m = (l^2 - m^2)/(4l^2 - 1)$, $b_l = (l+1)/(l-1)$, $c_l = 6/l\,(l+1)$.

2) For $|m| > 1$, the curvature $K(g, e^l_m) < 0$.

3) As $l \to \infty$ we have $K(g, e^l_{\pm l}) \to -15/8\,\pi$.

Table 1 gives values of the curvatures $K(g, e^l_m)$ for small (l, m); we remark that $K(g, e^l_0) = 0$ and $K(g, e^l_{-m}) = K(g, e^l_m)$.

THEOREM 3. 1) In the case of the Passat flow, the curvature functional has the form

$$K(g, v) = \sum_{l \geqslant m \geqslant 0} (2K^l_m |v^l_m|^2 + L^l_m \operatorname{Re}(v^{l-1}_m \bar{v}^{l+1}_m));$$

here $K^l_m = K(g, e^l_m)$, $L^l_m = (15m^2/16\pi)\sqrt{a^l_m a^{l+1}_m / b_l b_{l+1}}\,((1 - c_l)^2 b_{l+1} + (1 + c_l)(1 + b_l b_{l+1}) - 3b_l)$.

2) For vector fields v having $v^l_m = 0$ for $l < L$, the curvature is given asymptotically (as $L \to \infty$) by the following expression:

$$K(g, v) = -\frac{15}{8\pi} \sum_{l,m} \frac{(1 - m^2)\,m^2}{(l^2 - 1)((l+1)^2 - 1)} |v^l_m|^2 + O\left(\frac{1}{L^2}\right).$$

3) For vector fields v having $v^l_m = 0$ for $|m| \leqslant 2$ we have K(g, v) < 0.

Remark. If we assume that the state of atmospheric flows is like that of a Passat flow (i.e., under assumptions analogous to the ones made in [2]) and choose as the average value of the curvature $K_{av} = \alpha \inf_v K(g, v)$ for some $\alpha \in (0, 1)$, we obtain the following:

if ε is the error in the initial conditions for dynamic weather forecasting, then after n months the size of the error will be $10^{kn}\varepsilon$ (here $k \approx (30 \cdot 24/400)\,4\pi\sqrt{\alpha}\,\lg e \approx 10\sqrt{\alpha}$).

In particular, if we take $K_{av} = (1/4) \inf K$, we obtain that the increase in the error in determining the state of the weather amounts to 10^5 times over one month; if we take $K_{av} = (1/16) \inf K$, a 10^5-fold increase occurs over a 2-month period.

We remark that in Arnol'd's model in [2], a similar increase of the error (for $K_{av} \approx (1/4) \inf K$) takes place during two months.

It is of interest to note that in models with a torus or sphere for the time interval during which long-term weather forecasting is in practice impossible, estimates of the same order of magnitude are obtained.

2. Proofs

In order to calculate the curvatures of the group S Diff(S^2), we make use of a method introduced by Arnol'd [1]. We denote by B(u, v) the operator adjoint to ad u(v) = [u, v] (for fixed u). The curvature in the two-dimensional plane defined by orthonormal vector fields ξ, η is given by (see [1])

$$K(\xi, \eta) = \langle\delta, \delta\rangle + 2\langle\alpha, \beta\rangle - 3\langle\alpha, \alpha\rangle - 4\langle B_\xi, B_\eta\rangle,$$

where $2\delta = B(\xi, \eta) + B(\eta, \xi)$, $2\beta = B(\xi, \eta) - B(\eta, \xi)$, $2\alpha = [\xi, \eta]$, $2B_\mu = B(\mu, \mu)$.

In order to determine the form of B(u, v), it is necessary to find the structure constants of the Lie algebra $SV(S^2)$. We introduce the functions

$$f_{m,k} = (x + iy)^m z^k \quad (m, k \in \mathbf{Z}).$$

We note that the spherical functions can be expanded in the $f_{m,k}$.

LEMMA 1. The Poisson bracket for the functions $f_{m,k}$ has the form

$$\{f_{m,k}, f_{n,l}\} = i(kn - lm) f_{m+n,k+l-1},$$
$$\{f(z)(x + iy)^m, \varphi(z)(x + iy)^n\} = (inf'\varphi - imf\varphi')(x + iy)^{m+n}.$$

For the proof, we remark that $T(f_{m,k}) = imf_{m-1,k}e + kf_{m,k-1}h$, where $e = T(-i(x + iy))$. We have

$$[h, e] = ie, \quad h(f_{m,k}) = imf_{m,k}, \quad e(f_{m,k}) = -kf_{m+1,k-1},$$

from which the assertion of the lemma is easily derived.

The spherical functions Y^l_m can be expressed by homogeneous polynomials of degree l which are harmonic in R^3 (see [3]). We put $P_l = L\{Y^l_m \mid m = -l, \ldots, l\}$ and $V_l = T(P_l)$. The subspaces V_l are simple submodules of the SO(3)-module of spherical vector fields on S^2 (see [4]). Using a formula for B(u, v) in [1], it is easily shown that for $u \in V_k$, $v \in V_l$ we have $B(v, u) = (-l(l+1) / k(k+1)) B(u, v)$. In particular, for vector fields with homogeneous flow functions (i.e., $v \in V_l$) we have B(v, v) = 0, i.e., all such flows are steady (see [1]). For the case of a steady field ξ, we have $B_\xi = 0$ and the curvature formula simplifies to

$$K(\xi, \eta) = \langle\delta, \delta\rangle + 2\langle\alpha, \beta\rangle - 3\langle\alpha, \alpha\rangle. \qquad (*)$$

In the case of a vector field h, we remark that $[h, e^l_m] = im\, e^l_m$. Putting $h' = \sqrt{3/8\pi}\, h$ ($\| h' \| = 1$), we have

$$B(e^l_m, h') = -im\sqrt{3/8\pi}\, e^l_m \text{ and } B(h', e^l_m) = (-2/l(l+1))\, B(e^l_m, h').$$

Using (*), we obtain Theorem 1 from this.

In the case of a Passat flow g, we obtain $g' = \sqrt{15/8\pi}\, g$ ($\| g' \| = 1$). The flow function for the vector field g' has the form $f = (1/2)\sqrt{15/8\pi}\, z^2$. From Lemma 1 we get $\{f, \varphi(z)(x + iy)^m\} = im\sqrt{15/8\pi}\, z\varphi(z)(x + iy)^m$. Passing to spherical coordinates, we have

$$\{f, Y^l_m\} = im\sqrt{15/8\pi} \cos\theta Y^l_m.$$

Using the representation of functions in P_l by homogeneous harmonic polynomials (see [3]), it can be shown that $zP_l \subset P_{l-1} + P_{l+1}$. It is verified directly that

$$\cos\theta p^l_m = -\frac{l - m + 1}{(2l+1)(2l+2)} p^l_m - \frac{2l(l+m)}{2l+1} p^{l-1}_m \quad \left(p^l_m = \frac{d^{l+m}(\sin^l\theta)}{d(\cos\theta)^{l+m}}\right).$$

In the basis e^l_m we have (using the notation of Theorem 2)

$$\operatorname{ad} g'(e^l_m) = -im\sqrt{15/8\pi}\left(\sqrt{a^l_m/b_l}\, e^{l-1}_m + \sqrt{a^{l+1}_m b_{l+1}}\, e^{l+1}_m\right).$$

The operator B(v, g') is adjoint to ad g', from which we get

$$B(e^l_m, g') = im\sqrt{15/8\pi}\left(\sqrt{a^l_m b_l}\, e^{l-1}_m + \sqrt{a^{l+1}_m / b_{l+1}}\, e^{l+1}_m\right).$$

Moreover, $B(g', e^l_m) = (-6/l(l+1)) B(e^l_m, g')$. Further, using (*) formulas can be obtained for $K(g, e^l_m)$ (Theorem 2.1)) and K(g, v) (Theorem 3.1)).

In order to estimate the sign of K(g, v), it is useful to transform the expressions for K^l_m, L^l_m.

LEMMA 2. In the case of a Passat flow we have (in the notation of Theorem 3)

$$K^l_m = \frac{15m^2}{8\pi}\left[(1 - m^2)\left(1 - \frac{3\rho_l}{l(l+1)}\right) \Big/ (l^2 - 1)((l+1)^2 - 1) + \frac{27\varkappa_l}{4l^5(l+1)}\right],$$
$$L^l_m = \frac{15m^2}{8\pi}\left(\frac{9}{l-1} + \frac{9}{l^2}\right)\nu_l\sqrt{(l^2 - m^2)((l+1)^2 - m^2)/l^3(l+1)^2}.$$

Here

$$l \geqslant 2, \quad \rho_l = (8l^3 + 12l^2 - 32l - 18)/(2l-1)(2l+1)(2l+3),$$
$$\varkappa_l = 4l^3(2l^2 + 5l + 2)/(l+1)(l+2)(2l-1)(2l+1)(2l+3),$$
$$\nu_l = 4l^2\sqrt{(l-1)\,l/(l+1)(l+2)(4l^2-1)(4(l+1)^2-1)},$$

and $0 < \rho_l, \varkappa_l, \ \nu_l < 1$.

This is proved by tedious but straightforward calculations.

Using Lemma 2, it is easy to obtain the assertions concerning the sign of $K(g, e_l^m)$ (Theorem 2.2)) and the asymptotic formula for K(g, v) (Theorem 3.2)). In order to prove Theorem 3.3), it is useful to transform the curvature functional K(g, v) for v with $v_m^l = 0$ for $|m| < 3$ as follows:

$$K(g,v) = \sum_{m\geqslant 3}\left[(2K_m^m|v_m^m|^2 + L_m^{m+1}\operatorname{Re}(v_m^m\bar{v}_m^{m+2}) + K_m^{m+2}|v_m^{m+2}|^2) + \right.$$
$$+ (2K_m^{m+1}|v_m^{m+1}|^2 + L_m^{m+2}\operatorname{Re}(v_m^{m+1}\bar{v}_m^{m+3}) + K_m^{m+3}|v_m^{m+3}|^2) +$$
$$\left. + \sum_{l\geqslant m+2}(K_m^l|v_m^l|^2 + L_m^{l+1}\operatorname{Re}(v_m^l\bar{v}_m^{l+2}) + K_m^{l+2}|v_m^{l+2}|^2)\right],$$

after which it can be shown using the estimates of Lemma 2 that the expressions in parentheses are negative definite forms.

We remark that values close to the minimum $K(g, v) \approx -15/8\pi$, are taken on vector fields $e_{\pm l}^l$ for large l, or, if real fields are considered, on $e_+^l = (1/2)(e_l^l + e_{-l}^l)$, $e_-^l = (1/2i)(e_l^l - e_{-l}^l)$ (these fields have as flow functions $f_+^l = \lambda_l \operatorname{Re}(x+iy)^l$, $f_-^l = (\lambda_l/i)\operatorname{Im}(x+iy)^l$). The functions $\{f_+^l, f_-^l \mid l \in \mathrm{N}\}$ span on S^2 a subspace of functions depending only on x, y, and it is easy to see that the curvature K(g, v) for vector fields with such flow functions has the form

$$K(g,v) = \sum_{l\geqslant 1} K_l^l(|v_l^l|^2 + |v_{-l}^l|^2).$$

The character of the increase of the error in weather forecasting is determined analogously to [2]. One need only take into account that the length of a circle along the latitudes at which maximum velocity of a Passat flow is obtained (i.e., at $\theta = \pi/4, 3\pi/4$) is smaller than the length of the equator by a factor of $\sqrt{2}$ and also that $\inf_v K(g, v) \approx -15/8\pi$.

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