MULTIDIMENSIONAL INVERSE SPECTRAL PROBLEM FOR THE EQUATION $-\Delta\psi + (v(x) - Eu(x))\psi = 0$

R. G. Novikov

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1. Introduction

The equation of the form

$$-\Delta\psi + (v(x) - Eu(x))\psi = 0 \quad (1.1)$$

arises in many branches of physics: in quantum mechanics, theory of elasticity, electrodynamics. The reconstruction of the potentials v(x) and u(x) in Eq. (1.1) from spectral data is a classical problem. For the one-dimensional case this problem has been investigated well (see [1-3]). For the multidimensional case without the assumption of spherical symmetry the first results were obtained in [4-7]. A fundamental study of the multidimensional inverse problem was made by Faddeev [8, 9]. Some of the results of [9] were independently obtained by Newton [10]. In [11], Dubrovin, Krichever, and Novikov obtained the first substantial results in a multidimensional inverse problem with a fixed value of a spectral parameter. With all these and subsequent results (see [12-25]), the study of multidimensional inverse problems is far from being complete. Specifically, the following inverse problem, posed by Gel'fand [26] has remained unsolved.

Let in a bounded region $D \in \mathbf{R}^n$, $n \geqslant 2$, the equation

$$-\Delta\psi + v(x)\psi = E\psi \quad (1.2)$$

holds and the operator $\hat{\Phi}(E)$, acting on ∂D in a space of functions, is defined thus:

$$\left(\frac{\partial}{\partial\nu}\psi\Big|_{\partial D}\right) = \hat{\Phi}(E)(\psi|_{\partial D}), \quad (1.3)$$

where ψ is a solution of (1.2) and ν is the outward normal to ∂D. It is required to determine the potential v(x) from the operator $\hat{\Phi}(E)$. For Eq. (1.1), this problem is of even greater interest.

It seems that there have been no works devoted to solving the multidimensional inverse problem in exactly this formulation. However, it follows from [4, 2, 27] that under some restrictions on u(x) and v(x) in Eq. (1.1), the operator $\hat{\Phi}(E)$ defined for all E uniquely determines v(x) and u(x).

In the present paper, it is shown that for a fixed E the operator $\hat{\Phi}(E)$ uniquely determines the potential v(x) − Eu(x). This potential is found from the operator $\hat{\Phi}(E)$, E = const, on the basis of the solution of Fredholm linear integral equations. In essence, a characterization of the kernel of the operator $\hat{\Phi}(E)$ – a function $\Phi(x, y, E)$, $x, y \in \partial D$ – is obtained. The interrelation between the operator $\hat{\Phi}(E)$ and other scattering data is established. Specifically, the solution of the following problem is given. Let for Eq. (1.1) in D with u(x) > 0 the spectrum $E_1, E_2, \ldots$ of a Dirichlet boundary-value problem and the normal derivatives $\frac{\partial}{\partial\nu}\psi_j(x)\Big|_{\partial D}$ of its orthonormal eigenfunctions on ∂D are given. It is required to determine from these data the potentials u(x) and v(x) in Eq. (1.1). For the cases v(x) ≡ 0 and u(x) ≡ 1 the problem is solved in explicit formulas. Also, in the paper a procedure is offered for reconstructing the finite potential in a Schrödinger equation from the scattering amplitude for a fixed energy and the uniqueness of this reconstruction is proved (for $n \geqslant 3$) without the assumption of the smallness of the norm of the potential. Earlier, in [21] (n = 2) and [23] ($n \geqslant 3$), it was proved (under the condition of the smallness of the norm of the potential) that Eq. (1.2) cannot have two different potentials exponentially decreasing and having the same scattering amplitudes for a fixed energy. Formulations of the results of the present paper and sketches of the proofs were presented in the survey given in [23].

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It has become known recently that independently from [21, 23] Sylvester and Uhlmann obtained the following uniqueness theorem, in [28] for n = 2 and in [29] for $n \geqslant 3$. Let in a bounded region $D \in \mathbf{R}^n$ $(n \geqslant 2)$ a function ψ satisfy the equation

$$\nabla \cdot (\gamma(x) \nabla \psi) = 0, \tag{1.4}$$

where $\gamma(x)$ is a smooth real-valued function on $\bar{D}$, $\gamma(x) > \varepsilon > 0$. Then there do not exist two different functions $\gamma_1(x)$ and $\gamma_2(x)$ with one and the same operator $\hat{\Phi}$ such that $\left(\frac{\partial}{\partial \nu} \psi \Big|_{\partial D}\right) = \hat{\Phi}$ $(\psi|_{\partial D})$, where ψ is a solution of Eq. (1.4). For the case n = 2 a proof of this assertion is obtained under the condition of smallness of the norm of the function $1 - \gamma(x)$.

The result of [28, 29] follows from the results for Eq. (1.1) mentioned above, for by the substitution $\psi = \psi_0/\sqrt{\gamma(x)}$ Eq. (1.4) reduces to the equation $-\Delta\psi_0 + v(x)\psi_0 = 0$, where $v(x) = -\Delta\sqrt{\gamma(x)}/\sqrt{\gamma(x)}$.

It is of interest that some properties of exponentially growing solutions of Eq. (1.4) in $\mathbf{R}^n$ $n \geqslant 3$, were used in proving the uniqueness theorem in [29]. In an inverse problem, the properties of the solutions of this kind for a Schrödinger equation were first utilized by Faddeev [8, 9]. Later the properties of the solutions of that type were used in detail in [14-25].

As is indicated in [29], the problem of determining the function $\gamma(x)$ from the operator $\hat{\Phi}$ was posed by Calderon in [30]. The results of the present paper solve this problem. Finally, let us observe that for the case of a real analytic function $\gamma(x)$ the uniqueness theorem in Calderon's problem was earlier proved in [31].

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2. Some Initial Results for the Multidimensional Inverse Scattering Problem in the Entire Space

Let Eq. (1.2) or (1.1) hold in the entire space $\mathbf{R}^n$, $n \geqslant 2$. The functions $v(x)$ and $w(x) = u(x) - 1$ are assumed to be real-valued, bounded, and rapidly decreasing at infinity. The scattering amplitude $f(k, l)$, $k, l \in \mathbf{R}^n$, $k^2 = l^2 = E$, can be determined by the equality

$$f(k, l) = \left(\frac{1}{2\pi}\right)^n \int\limits_{x \in \mathbf{R}^n} e^{-ilx}\psi^+(x, k)(v(x) - Ew(x))\,dx, \tag{2.1}$$

where $\psi^+(x, k)$ is a solution of the integral equation

$$\psi^+(x, k) = e^{ikx} + \int\limits_{y \in \mathbf{R}^n} G^+(x - y, k)(v(y) - Ew(y))\psi^+(y, k)\,dy, \tag{2.2}$$

$$G^+(x, k) = -\left(\frac{1}{2\pi}\right)^n \int\limits_{\xi \in \mathbf{R}^n} \frac{e^{i\xi x}\,d\xi}{\xi^2 - k^2 - i0}. \tag{2.3}$$

The central role in solving the inverse scattering problem is played by the family of solutions $\psi(x, k)$ of Eq. (1.1) introduced into the study by Faddeev [8, 9] for the case of Schrödinger equation. Faddeev defined these functions as the solutions of the integral equation

$$\psi(x, k) = e^{ikx} + \int\limits_{y \in \mathbf{R}^n} G(x - y, k)v(y)\psi(y, k)\,dy, \tag{2.4}$$

where

$$G(x, k) = -\left(\frac{1}{2\pi}\right)^n \int \frac{e^{i\xi x}\,d\xi}{\xi^2 + 2k\xi} e^{ikx}. \tag{2.5}$$

For a suitable class of functions, Eq. (2.4) is a Fredholm one and for $k \in \mathbf{C}^n \setminus (\mathbf{R}^n \cup \mathscr{E})$ uniquely solvable. The basic properties of the set of exceptional points $\mathscr{E}$, where Eq. (2.4) loses its unique solvability are obtained in [23].

It has been found as a result of a discussion with P. G. Grinevich that for $k \in \mathbf{C}^n \setminus (\mathbf{R}^n \cup \mathscr{E})$ the solutions $\psi(x, k)$ can also be defined as the solutions of Eq. (1.1) with the property

$$\psi(x, k) = e^{ikx}\left(1 + O\left(\left(\frac{1}{|x|}\right)^{\frac{n-1}{2}}\right)\right).$$

Faddeev introduced into consideration the generalized scattering data of the form

$$h(k, l) = \left(\frac{1}{2\pi}\right)^n \int_{x \in \mathbf{R}^n} e^{-ilx} \psi(x, k) v(x)\, dx, \tag{2.6}$$

where $k, l \in \mathbf{C}^n$, $k^2 = l^2$, $\operatorname{Im} k = \operatorname{Im} l$.

For the subsequent formulations, it is convenient to make the change of variables k = k, p = Re k − Re ℓ and to consider the function $H(k, p) = h(k, k - p)$, $k \in \mathbf{C}^n$, $p \in \mathbf{R}^n$, $p^2 = 2kp$. We have

$$H(k, p) = \left(\frac{1}{2\pi}\right)^n \int_{x \in \mathbf{R}^n} e^{ipx} \mu(x, k) v(x)\, dx, \tag{2.7}$$

where $\mu(x, k) = \psi(x, k) e^{-ikx}$.

For real γ, k, l, $k^2 = l^2 = E$, $|\gamma| = 1$, there exist the following limits ([9], see also [23]):

$$G_\gamma(x, k) = G(x, k + i0\gamma), \quad \psi_\gamma(x, k) = \psi(x, k + i0\gamma), \quad h_\gamma(k, l) = h(k + i0\gamma, l + i0\gamma),$$

where $\gamma, k, l \in \mathbf{R}^n$, $|\gamma| = 1$, $k^2 = l^2 = E$ and the equalities

$$G^+(x, k) = G\left(x, k + i0\frac{k}{|k|}\right), \quad \psi^+(x, k) = \psi\left(x, k + i0\frac{k}{|k|}\right),$$
$$f(k, l) = h\left(k + i0\frac{k}{|k|}, l + i0\frac{l}{|l|}\right)$$

hold. The functions μ(x, k) and H(k, p) satisfy the following equations [16, 17, 23] ($k \in \mathbf{C}^n$, $p \in \mathbf{R}^n$):

$$\frac{\partial}{\partial \bar{k}_j} \mu(x, k) = -2\pi \int_{\xi \in \mathbf{R}^n} \xi_j H(k, -\xi) e^{i\xi x} \delta(\xi^2 + 2k\xi) \mu(x, k + \xi)\, d\xi, \tag{2.8}$$

$$\frac{\partial}{\partial \bar{k}_j} H(k, p) = -2\pi \int_{\xi \in \mathbf{R}^n} \xi_j H(k, -\xi) H(k + \xi, p + \xi) \delta(\xi^2 + 2k\xi)\, d\xi. \tag{2.9}$$

If the function $H(k, p)$ ($n \geqslant 3$) is known for an energy level $k^2 = E$, $p^2 = 2kp$, then the inverse problem can be solved, for example, with the use of the formula [23]

$$\hat{v}(p) - \hat{w}(p) E = \lim_{\substack{k \to \infty,\, k^2 = E, \\ p^2 = 2kp}} H(k, p), \tag{2.10}$$

where

$$\hat{v}(p) - E\hat{w}(p) = \left(\frac{1}{2\pi}\right)^n \int_{x \in \mathbf{R}^n} e^{ipx} (v(x) - Ew(x))\, dx. \tag{2.11}$$

The inverse problem can also be solved with the use of the formula [17, 23]

$$v(x) - Eu(x) = \frac{\triangle \psi(x, k)}{\psi(x, k)}, \tag{2.12}$$

where $\psi = e^{ikx} \mu(x, k)$ is a solution of $\bar{\partial}$-equation (2.8) restricted to the energy level $k^2 = E$, with the property $\mu(x, k) \to 1$, $|k| \to \infty$.

For the two-dimensional case with a fixed energy level formula (2.10) is not valid, but formula (2.12) can be used (see [18-20; 25]). Note that in the two-dimensional case with $k^2 = E \in \mathbf{R}_+$, to find the function μ(x, k) for $k^2 = E$, besides $\bar{\partial}$-equation (2.8) for the energy level $k^2 = E$, the relation $\psi_1 = \hat{R}(E)\psi_2$ should also be taken into account, where ψ_1 and ψ_2 are the limits of the function $\psi(x, k)$, $k \in \mathbf{C}^2$, $k^2 = E$ on a real circumference $k \in \mathbf{R}^2$, $k^2 = E$, and $\hat{R}(E)$ is an operator expressible in terms of a scattering operator $\hat{S}(E)$ (see [19]). For the two-dimensional case a characterization of scattering data at an energy level was obtianed in [18-20; 25], for the three-dimensional, in [17, 23].

3. Main Results

The kernel of the integral operator $\hat{\Phi}(E)$ corresponding to Eq. (1.2) or (1.1) will be denoted by $\Phi(x, y, E)$, where $x, y \in \partial D$, $E \in \mathbf{C}$. Throughout this section we shall assume the functions u(x) and v(x) to be bounded in the region D. Let $\Phi_0(x, y, E)$ be the kernel of an integral operator $\hat{\Phi}_0(E)$ of the form (1.3) for the equation

$$-\triangle \psi = E\psi \tag{3.1}$$

in the region D. Let the equalities $v(x) \equiv 0$, $u(x) \equiv 1$ hold outside the bounded region D with a smooth boundary (throughout this section by a smooth boundary we shall mean a twice continuously differentiable boundary). Then the following results are valid:

THEOREM 1. The function $h(k, \ell)$ defined by equality (2.6) can be obtained from the operator $\hat{\Phi}(E)$ by the formula

$$h(k, l) = \left(\frac{1}{2\pi}\right)^n \iint\limits_{x\in\partial D,\ y\in\partial D} e^{-ilx} (\Phi - \Phi_0)(x, y, k^2)\, \psi(y, k)\, \sigma(dx)\, \sigma(dy), \tag{3.2}$$

$k, l \in \mathbf{C}^n$, $k^2 = l^2$, $\operatorname{Im} k = \operatorname{Im} l$. At that, the function $\psi(y, k)$ defined by Eq. (2.4) satisfies the equation

$$(\psi(x, k)|_{\partial D}) = e^{ikx} + \int\limits_{y\in\partial D} A(x, y, k)\, (\psi(y, k)|_{\partial D})\, \sigma(dy), \tag{3.3}$$

where

$$A(x, y, k) = \int\limits_{x,\, y\in\partial D\ z\in\partial D} G(x - z, k)\, (\Phi - \Phi_0)(z, y, k^2)\, \sigma(dz).$$

Equation (3.3) can be restricted to the real space. Therewith integral equations for the functions $\psi_\gamma(x, k)$ and $\psi^+(x, k)$ are produced. The following formulas for the functions $h_\gamma(k, \ell)$ and $f(k, \ell)$ are valid:

$$h_\gamma(k, l) = \left(\frac{1}{2\pi}\right)^n \iint\limits_{x,\, y\in\partial D} e^{-ilx} (\Phi - \Phi_0)(x, y, k^2)\, \psi_\gamma(y, k)\, \sigma(dx)\, \sigma(dy), \tag{3.4}$$

$$f(k, l) = \left(\frac{1}{2\pi}\right)^n \iint\limits_{x,\, y\in\partial D} e^{-ilx} (\Phi - \Phi_0)(x, y, k^2)\, \psi^+(y, k)\, \sigma(dx)\, \sigma(dy), \qquad k, l, \gamma \in \mathbf{R}^n,\quad k^2 = l^2,\quad \gamma^2 = 1. \tag{3.5}$$

Remark 1. Formula (3.5) and Eq. (3.3) for $k = (1 + i0)\operatorname{Re} k$ remain valid in the one-dimensional case as well, where they reduce the inverse spectral problem on a closed interval to the inverse problem with a scattering matrix on the entire axis.

We denote by s the spectrum of the Dirichlet problem for Eq. (1.1) and by s_0 the spectrum of the Dirichlet problem for Eq. (3.1).

Proposition 1. Let $E \notin s \cup s_0$, Then the operator $\hat{\Phi}(E) - \hat{\Phi}_0(E)$ is completely continuous in the space of bounded functions on ∂D. Furthermore, equalities (4.8) and (4.9) hold.

It follows from Proposition 1 that Eq. (3.3) is a Fredholm equation of the second kind in the space of bounded functions on ∂D.

Proposition 2. For $k \in (\mathbf{C}^n \setminus \mathbf{R}^n)$ and $k^2 \notin s \cup s_0$ Eqs. (3.3) and (2.4) are simultaneously uniquely solvable in the space of bounded functions on D.

Theorem 1 implies the following corollaries:

COROLLARY 1. To reconstruct from the operator $\hat{\Phi}(E)$ the potential $v(x) - Eu(x)$ in the case $n \geqslant 2$), it suffices to find the function h at the energy level $k^2 = E$ on the basis of formulas (3.2)-(3.5) and then to make use of the methods presented in [23] for solving the problem. (Some of these methods have been given in Sec. 2.)

COROLLARY 2. In the case $n \geqslant 2$ the operator $\hat{\Phi}(E)$ uniquely determines the potential $v(x) - Eu(x)$.

In the case $n \geqslant 3$ the assertion of this corollary is proved at least for any bounded potential.

As for the case $n = 2$, an accurate proof of Corollary 2 is so far obtained under some additional restrictions on the potential, for example, with the proviso that the norm of the function $v(x) - Ew(x)$ is small in comparison with $|E|$.

Let further it be required to determine a potential $q(x)$ from an operator $\hat{\Phi}$ such that $\frac{\partial}{\partial \nu}\psi|_{\partial D} = \hat{\Phi}\psi|_{\partial D}$, where ψ is a solution in D of the equation $-\Delta\psi + q\psi = 0$. Then, rewriting this equation in the form $-\Delta\psi + (q + E)\psi = E\psi$, where E is an arbitrary complex number, we can regard the operator $\hat{\Phi}$ as the data $\hat{\Phi}(E)$ of the inverse problem for the potential $q + E$ at the chosen "energy" E.

It is of interest to note the way of solving the direct and inverse problems in a bounded region in the Born approximation. Suppose, for example, that in Eq. (1.2) the potential v(x) has a very small norm. In this case the two following formulas hold:

$$\Phi(x, y, E) - \Phi_0(x, y, E) = \int\limits_{z\in D} \frac{\partial}{\partial \nu_x} G_0(x, z, E)\, v(z) \frac{\partial}{\partial \nu_y} G_0(z, y, E)\, dz + \bar{o}(\|v\|),$$

$$\hat{v}(p) = \iint\limits_{\substack{x,\, y\in\partial D,\\ p^2=2kp,\, k^2=E}} e^{ipx} e^{-ikx} (\Phi - \Phi_0)(x, y, E)\, e^{iky}\, dx\, dy + \bar{o}(\|v\|), \tag{3.6}$$

where $G_0(x, y, E)$ is Green's function of the Dirichlet problem for Eq. (3.1).

We now turn to characterizing the function $\Phi(x, y, E)$. In this connection the following proposition is of interest.

Proposition 3. The function $H(k, p)$, $k \in \mathbf{C}^n$, $p \in \mathbf{R}^n$, constructed according to formulas (3.2) and (3.3) from an arbitrary function $\tilde{\Phi}(x, y, E)$, $x, y \in \partial D$, analytic in the variable E, satisfies $\bar{\partial}$-equation (2.9).

Theorem 1, Proposition 3, and the results of [23] imply the following corollary.

COROLLARY 3. Let $n \geqslant 3$, and let D be a convex region. For a function $\Phi(x, y, E)$, $x, y \in \partial D$, E = const, to correspond to some potential $v(x)$ $x \in D$, such that $|\hat{v}(p)| \leqslant C\cdot(1+|p|)^{-\left(\frac{n+1}{2}\right)}$, it is necessary and sufficient that the function $\mu(x, k) = e^{-ikx}\psi(x, k)$ constructed according to formula (3.2) tend to 1 as $|k| \to \infty$, $k^2 = E$, and the function $H(k, p)$, $k^2 = E$, $p^2 = 2kp$, $p \in \mathbf{R}^n$, constructed according to formula (3.2) satisfy the inequality $|H(k, p)| < C(1+|p|)^{-\left(\frac{n+1}{2}\right)}$ as $|k| \to \infty$.

Note that a real-valued potential with $E \in \mathbf{R}$ possesses two symmetries: $\Phi(x, y, E) = \bar{\Phi}(x, y, E)$ and $\Phi(x, y, E) = \Phi(y, x, E)$. Note also that in the two-dimensional case the function $\Phi(x, y, E)$, $E =$ const, $x, y \in \partial D$, as well as the potential, depends on two variables, i.e., in the two-dimensional case the inverse problem is not overdetermined when E = const.

If it is admissible for the potential $v(x)$, $x \in D \in \mathbf{R}^2$, to be a generalized function, then the two indicated symmetries are necessary and sufficient properties of the function $\Phi(x, y, E)$, $x, y \in \partial D$, $E \in \mathbf{R}$.

Many other versions of an inverse problem for Eqs. (1.1) and (1.2) given in a region D reduce to the inverse problem with the operator $\hat{\Phi}$.

Consider, e.g., for Eq. (1.1) with $u(x) \geqslant u_0 > 0$ in a region D the Dirichlet boundary-value problem $\psi|_{\partial D} = 0$. By the data of the inverse problem we shall mean the eigenvalues E_j, j = 1, 2, 3,... of the Dirichlet problem and the sequence of functions $\frac{\partial}{\partial \nu}\psi_{j,r}(x, E_j)|_{\partial D}$, where $\psi_{j,r}(x, E_j)$ are normalized eigenfunctions:

$$\int\limits_D \psi_{j,r}(x, E_j)\, \psi_{j',r'}(x, E_{j'})\, u(x)\, dx = \begin{cases} 1 & \text{for} \quad j = j',\ r = r', \\ 0 & \text{for} \quad j \neq j' \ \text{or} \ r \neq r'. \end{cases}$$

The following formula reduces this inverse problem to the already solved inverse problem with the operator $\hat{\Phi}(E)$:

$$\Phi(x, y, E) = \sum_{j,r} \frac{\frac{\partial}{\partial \nu}\psi_{j,r}(x) \frac{\partial}{\partial \nu}\psi_{j,r}(y)}{E - E_j}\, x,\ y \in \partial D. \tag{3.7}$$

It is not hard to consider the boundary condition $\alpha(x)\psi|_{\partial D} + \beta(x)\frac{\partial}{\partial \nu}\psi|_{\partial D} = 0$ instead of a Dirichlet boundary condition. For example, in the case of the Neumann boundary condition $\frac{\partial}{\partial \nu}\psi|_{\partial D} = 0$ the following formula holds for the kernel Q(x, y, E) of the operator $\hat{Q}(E) = (\hat{\Phi}(E))^{-1}$:

$$Q(x, y, E) = \sum \frac{\psi_{j,r}(x)\, \psi_{j,r}(y)}{E - E_j}, \quad x, y \in \partial D.$$

Let us now show a method of determining the operator $\hat{\Phi}(E)$ from the scattering amplitude $f(k, \ell)$ at a fixed energy $k^2 = \ell^2 = E$.

For potentials u(x) and v(x) such that $u(x) \equiv 1$ and $v(x) \equiv 0$ for $x \in \mathbf{R}^n \setminus D$, where D is a region with a connected boundary, the formulas

$$f(k, l) = \int_{\partial D} \psi^+(x, k) K(x, l) \sigma(dx) \quad \psi^+(x, k) = \int_{l^2=E} f(k, l) \tilde{K}(l, x) dl \tag{3.8}$$

hold [4], where the function $K(x, \ell)$ and $\tilde{K}(\ell, x)$ are completely determined by the region D. Substituting (3.8) into (3.5), we obtain an equation for determining the function $\Phi(x, y, E)$, $x, y \in \partial D$, from the scattering amplitude $f(k, \ell)$, $k^2 = \ell^2 = E$.

Let D be a bounded simply connected region with a smooth boundary, and let u and v be bounded in D. The following propositions hold.

Proposition 4. The operator $\hat{\Phi}(E)$ and the scattering amplitude $f(k, \ell)$, $k^2 = \ell^2 = E$ (E fixed) are uniquely determined each by the other.

Proposition 5. In the case $n \geqslant 3$ a potential v(x) in Eq. (1.2) vanishing outside D is uniquely determined by its scattering amplitude $f(k, \ell)$, $k^2 = \ell^2 = E$, defined for any fixed $E > 0$ [the smallness of the norm of the function v(x) is not assumed]. In fact, a little adaptation of the arguments in [21, 23] leads to the proof of the uniqueness of the reconstruction of an exponentially decreasing potential from a scattering amplitude at a fixed energy for $n \geqslant 3$, without the assumption of the smallness of its norm. The only thing which is required is to carry out for the functions $\Delta(k)$ and $H \cdot \Delta$ the same arguments that were used for H in [21, 23].

In [4], there was established the interrelation between the inverse problem with the scattering amplitude for a finite potential and many other possible versions of an inverse problem.

It is important to note that the inverse problems for the equations $-\Delta\psi + v\psi = E\psi$ and $-\Delta\psi = Eu\psi$ admit simplifications in comparison with the general Eq. (1.1). In these particular cases the inverse spectral problem can be solved in explicit formulas on the basis of formula (3.7) and the following proposition.

Proposition 6. Let the following equations hold in a bounded region D: (a) $-\Delta\psi = \lambda u(x)\psi$ (respectively), (b) $-\Delta\psi + v(x)\psi = \lambda\psi$. Then the following formulas hold:

(a) $(n \geqslant 2)$

$$\hat{v}(p) = \lim_{\lambda \to 0} \frac{1}{\lambda} \iint_{\partial D \times \partial D} e^{ipx + ik(y-x)} (\Phi(x, y, \lambda) - \Phi_0(x, y, 0)) \sigma(dx) \sigma(dy),$$

$k^2 = 0$, $p^2 = 2kp$ (respectively);

(b) $(n \geqslant 3)$

$$\hat{v}(p) = \lim_{\tau \to +\infty} \iint_{\partial D \times \partial D} e^{ipx + ik(\tau)(y-x)} (\Phi(x, y, \lambda(\tau)) - \Phi_0(x, y, \lambda(\tau))) \sigma(dx) \sigma(dy),$$

$$p^2 = 2kp,\ k^2(\tau) = \lambda(\tau),\ \operatorname{Re} \lambda(\tau) \to +\infty,\ \operatorname{Im} \lambda(\tau) = c_1 \neq 0,\ |\operatorname{Im} k(\tau)| \leqslant c_2.$$

A similar formula can be written for the case n = 2 also.

Proposition 6(b) implies that the potential $v(x)$, $x \in \mathbf{R}^n$, $n \geqslant 2$, in the Schrödinger equation is uniquely determined by the data E_j; $\frac{\partial}{\partial \nu} \psi_{j,r}(x)|_{\partial D}$, given beginning with an arbitrarily large number j.

Note also that in the case of Schrödinger equation there can be given a characterization of the function $\Phi(x, y, E)$, $x, y \in \partial D$, $E \in \mathbf{R}_+$, analogous to the characterization of a scattering amplitude $f(k, l)$, $k^2 = l^2 = E \in \mathbf{R}_+$, obtained in [9, 23].

With some applications in geophysical inverse problems in mind, let us formulate some rseults for a region D which is a half-space.

Let in the half-space $D = \{x \in \mathbf{R}^n, x_n < 0\}$ there be a given Eq. (1.1) with real-valued bounded coefficients u and v such that $u(x) \to 1$, $v(x) \to 0$ sufficiently fast as $x \to \infty$. The spectrum of a problem (1.1) with a Dirichlet boundary condition in $L_2(D, u(x), dx)$ consists of the entire positive semiaxis R_+ and and isolated points on the negative semiaxis. If E is not a point of the spectrum, then the operator $\hat{\Phi}(E)$ can be determined from formula (1.3) with the use of solutions of Eq. (1.1) in $L_2(D, u(x)dx)$. For $E \in \mathbf{R}_+$ there exist the limits

$$\hat{\Phi}^{\pm}(E) = \hat{\Phi}(E \pm i0).$$

In order to make it possible that a bounded solution ψ of Eq. (1.1) in D for $E \in \mathbf{R}_+$ with a given condition $\psi|_{\partial D} = f$ be uniquely determined, a radiation condition at infinity should be additionally prescribed. The solution ψ with a radiation condition is to be found from the integral equation

$$\psi(x) = \int_{\partial D} \frac{\partial}{\partial \nu_y} G_0(x, y, E) f(y) \sigma(dy) + \int_D G_0(x, y, E)(v(y) - Ew(y)) \psi(y) dy,$$

where $G_0(x, y, E) = G^+(x - y, E) - G^+(x - y^*, E), y_1 = y_1^*, \ldots, y_n = -y_n^*$.

[The operator $\hat{\Phi}^+(E)$, $E \in \mathbf{R}_+$, can be determined from the formula (1.3) with the use of the solutions satisfying the radiation condition.]

Extend the functions u and v to $\mathbf{R}^n \setminus D$ as $u(x) \equiv 1$, $v(x) \equiv 0$ and consider the solutions of Eq. (1.1) in the entire $\mathbf{R}^n$. The following proposition holds.

Proposition 7. Let $k \in D$, Then the function $\psi^+(x, k)|_{\partial D}$ satisfies the equation

$$\psi^+(x, k)|_{\partial D} + \hat{G}^+(k)(\hat{\Phi}^+(k^2) - \hat{\Phi}_0^+(k^2))(\psi^+|_{\partial D}) = e^{ikx},$$

where $G^+(k)$ is an integral operator with a kernel $G^+(x - y, k)$, $x, y \in \partial D$. At that, the scattering amplitude $f(k, \ell)$ for $k \in D$, $l \in \mathbf{R}^n \setminus D$ is determined by the formula

$$f(k, l) = \left(\frac{1}{2\pi}\right)^n \iint e^{-ilx} (\Phi^+ - \Phi_0^+)(x, y, k^2) \psi^+(y, k) \sigma(dx) \sigma(dy).$$

If the potentials v and u are such that $v(x) \equiv 0$, $u(x) \equiv 1$ for sufficiently large $|x|$, then the function $f(k, \ell)$, $k^2 = \ell^2 = E$, admits an analytic continuation with respect to the variables $\theta = k/|k|$ and $\theta' = \ell/|\ell|$. On the basis of this continuation, the scattering amplitude can be found for any directions of k and ℓ.

Proposition 7 reduces the inverse scattering problem in a half-space to the inverse problem with scattering amplitude in the entire space.

4. Proof of the Main Theorem and Propositions 1-7

Proof of the Theorem. Equation (3.3) is obtained from Eq. (2.4) with the use of Green's formula

$$\int_D (g \triangle f - f \triangle g) dx = \int_{\partial D} \left(g \frac{\partial}{\partial \nu} f - f \frac{\partial}{\partial \nu} g\right) \sigma(dx). \tag{4.1}$$

Using formula (4.1), we find the validity of the following chain of equalities:

$$\int_D G(x - y, k)(v(y) - k^2 w(y)) \psi(y) dy = \int_D G(x - y, k)(\triangle + k^2) \psi(y, k) dy =$$
$$= \int_D \psi(y, k)(\triangle + k^2) G(x - y, k) dy + \int_{\partial D} \left(G(x - y, k) \frac{\partial}{\partial \nu} \psi -\right.$$
$$\left.- \psi \frac{\partial}{\partial \nu} G(x - y, k)\right) \sigma(dy) = \int_D \psi(y, k) \delta(x - y) dy + \int_{\partial D} \left(G(x - y, k) \frac{\partial}{\partial \nu} \psi - \psi \frac{\partial}{\partial \nu} G(x - y, k)\right) \sigma(dy), \tag{4.2}$$

where ψ is a solution of the equation $\Delta \psi + (v - k^2 w) \psi = k^2 \psi$.

Applying (4.2) to Eq. (2.4), for $x \in \mathbf{R}^n \setminus D$ we have

$$\psi(x, k) = e^{ikx} + \int_{\partial D} \left(G(x - y, k) \frac{\partial}{\partial \nu} \psi - \psi \frac{\partial}{\partial \nu} G(x - y, k)\right) \sigma(dy) =$$
$$= e^{ikx} + \int_{\partial D} \left(G(x - y, k)(\hat{\Phi}(k^2) \psi|_{\partial D}) - \psi \frac{\partial}{\partial \nu} G(x - y, k)\right) \sigma(dy) =$$
$$= e^{ikx} + \int_{\partial D} G(x - y, k)(\hat{\Phi}(k^2) - \hat{\Phi}_0(k^2))(\psi|_{\partial D}) \sigma(dy) +$$
$$+ \int_{\partial D} \left(G(x - y, k)(\hat{\Phi}_0(k^2) \psi|_{\partial D}) - \psi \frac{\partial}{\partial \nu} G(x - y, k)\right) \sigma(dy), \tag{4.3}$$

where

$$\int_{\partial D}\left(G(x-y,k)(\hat{\Phi}_0\psi|_{\partial D})-\psi\frac{\partial}{\partial\nu}G(x-y,k)\right)\sigma(dy)=0.\quad \hat{\Phi}(k^2)\psi=\int_{\partial D}\Phi(y,z,k^2)\psi(z)\sigma(dz), \tag{4.4}$$

(4.4) follows from formula (4.1) and the existence of a solution φ of the equation $\Delta\varphi = k^2\varphi$, such that $\psi(x,k)|_{\partial D} = \varphi(x)|_{\partial D}$. Now Eq. (3.3) follows from (4.3) and (4.4). Derive formula (3.2) from formula (2.6). Indeed,

$$\int_D e^{-ilx}\psi(x,k)v(x)\,dx=\int_D e^{-ilx}(\triangle+k^2)\psi\,dx=(k^2-l^2)\int_D e^{-ilx}\psi(x,k)\,dx+\int_{\partial D}\left(e^{-ilx}\frac{\partial}{\partial\nu}\psi-\psi\frac{\partial}{\partial\nu}e^{-ilx}\right)\sigma(dx). \tag{4.5}$$

It suffices to know the function h(k, ℓ) under the conditions $k^2 = l^2 = 0$, $\operatorname{Im} k = \operatorname{Im} l$, $k, l \in \mathbf{C}^n$. Further,

$$\int_{\partial D}\left(e^{-ilx}\frac{\partial}{\partial\nu}\psi-\psi\frac{\partial}{\partial\nu}e^{-ilx}\right)\sigma(dx)=\int_{\partial D}e^{-ilx}(\hat{\Phi}-\hat{\Phi}_0)\psi\sigma(dx)+\int_{\partial D}\left(e^{-ilx}(\hat{\Phi}_0\psi)-\psi\frac{\partial}{\partial\nu}e^{-ilx}\right)\sigma(dx), \tag{4.6}$$

where

$$\int_{\partial D}\left(e^{-ilx}(\hat{\Phi}_0\psi)-\psi(x,k)\frac{\partial}{\partial\nu}e^{-ilx}\right)\sigma(dx)=0 \tag{4.7}$$

for $\ell^2 = k^2$.

Equality (4.7) is true for the same reasons as equality (4.4) is. Now equality (3.2) follows from formulas (4.5)-(4.7). Theorem 1 is proved.

To prove Proposition 7, it is necessary to resort to the following additional consideration. When $k \in D$, the function $\psi^+(x, k)$ describes the propagation of oscillations from the boundary into the interior of the region D. This property is possessed by all the solutions of Eq. (1) in the region D with a radiation condition at infinity, i.e., the solutions with the property

$$\varphi(x)=e^{i|k||x|}|x|^{\frac{1-n}{2}}g\left(\frac{x}{|x|}\right)+o(|x|^{\frac{1-n}{2}}).$$

It is by just these solutions that the operator $\hat{\Phi}^+(k^2)$ is determined. The function $\psi^+(x, k)$ for $k \in D$ can (and for $k \notin D$ cannot) be represented as the limit of a sequence of functions each of which satisfies the radiation condition. Therefore for $k \in D$ the formula $\frac{\partial}{\partial\nu}\psi^+|_{\partial D}=\hat{\Phi}^+(k^2)(\psi^+|_{\partial D})$ holds, which must necessarily be used in writing an equality of the type (4.3) for a half-space. Equalities of the type (4.5)-(4.7) admit to be written only for $l \in \mathbf{R}^n \setminus D$, $k \in D$.

Proof of Proposition 1. Without restriction of generality, we can take E equal to zero. Then for the function $c(x,y) = \Phi(x,y) - \Phi_0(x,y)$, $x, y \in \partial D$ in the three-dimensional case the equality

$$c(x,y)|x-y|\Big|_{y=x;\,x,\,y\in\partial D}=\frac{1}{4\pi}v(x)\Big|_{x\in\partial D}, \tag{4.8}$$

and in the two-dimensional, the equality

$$c(x,y)(\ln|x-y|)^{-1}|_{y=x;\,x,\,y\in\partial D}=\mathrm{const}\cdot v(x)|_{x\in\partial D} \tag{4.9}$$

hold. At that, the function c(x, y) is bounded, except on the diagonal x = y. Hence Proposition 1 follows. The idea of the proof of equalities (4.8) and (4.9) consists in the following. Consider in $\mathbf{R}^3$ the half-space $D = \{x : x_3 < 0\}$. Let the operator $\Phi(x, y)$ correspond in D to the equation $-\Delta\psi + v\psi = 0$, where v = const. The operator $\Phi_0(x, y)$ corresponds to the equation $-\Delta\psi = 0$. We have

$$\Phi_0(x,y)=\frac{\partial^2}{\partial\nu_x\partial\nu_y}\left(-\frac{1}{4\pi}\right)\left(\frac{1}{|x-y|}-\frac{1}{|x-y^*|}\right)\Big|_{x,\,y\in D}=-\frac{1}{2\pi}\left(\frac{1}{|x-y|^3}-\frac{3x_3^2}{|x-y|^5}\right)\Big|_{x,\,y\in\partial D},$$

$$c(x,y)=(\Phi-\Phi_0)(x,y)=$$

$$=\left(-\frac{1}{4\pi}\right)\frac{\partial^2}{\partial\nu_x\partial\nu_y}\left(\frac{e^{i\sqrt{-v}|x-y|}-1}{|x-y|}-\frac{e^{i\sqrt{-v}|x-y^*|}-1}{|x-y^*|}\right)\Big|_{x,\,y\in\partial D}=\frac{1}{4\pi}\cdot\frac{v}{|x-y|}\Big|_{x,\,y\in\partial D}+O(1).$$

The proof of Proposition 3 follows the same scheme as the derivation of Eq. (2.9) in [23]. That is, differentiating equality (3.3) with respect to k_j, we arrive at Eq. (2.8), where $x \in \partial D$. Then, differentiating equality (3.2) with respect to $\bar{k}_j$ and taking into account Eq. (2.8), we arrive at Eq. (2.9).

The scheme of the proof of Proposition 2 is as follows. Let Eq. (2.4) have several solutions. Then, repeating the proof of Theorem 1 separately for each solution, we find that the restriction to ∂D of each of these solutions satisfies Eq. (3.3). Thus, Eq. (3.3) has at least as many solutions as Eq. (2.4). To prove the converse and thereby to prove Proposition 2, it suffices to show that each solution of Eq. (1.1) turning on the boundary into a solution of Eq. (3.3) is a solution of Eq. (2.4). This can be done on the basis of the equalities (4.3), (4.2) and the following fact. If formula (4.3) holds for $x \in \mathbf{R}^n \setminus D$, then the formula

$$e^{ikx} + \int\limits_{\partial D} \left(G(x-y, k)\frac{\partial}{\partial \nu}\psi - \psi\frac{\partial}{\partial \nu}G(x-y, k)\right)\sigma(dy) = 0$$

holds for $x \in D$.

Proposition 5 follows from Proposition 4 and Corollary 2.

The scheme of the proof of Proposition 4 is as follows. From the scattering amplitude find the operator $\hat{\Phi}(E)$. For a fixed vector k and $x \in \mathbf{R}^n \setminus D$ the function $\psi^+(x, k)$ is uniquely found from the scattering amplitude $f(k, \ell)$ (see [4] and (3.8) in the present paper). Thus $\forall k \in \mathbf{R}^n, k^2 = E$, the two functions $\psi^+(x, k)|_{\partial D}$ and $\frac{\partial}{\partial \nu}\psi^+(x, k)|_{\partial D}$ are known. It suffices to show that the operator $\hat{\Phi}(E)$ is uniquely determined by its action on the functions $\psi^+(x, k)|_{\partial D}$ $\left(\frac{\partial}{\partial \nu}\psi^+|_{\partial D} = \hat{\Phi}(E)(\psi^+|_{\partial D})\right)$. For this, it suffices to prove that any solution ψ of Eq. (1.2) in D can be approximated by linear combinations of the functions $\psi^+(x, k)$, $k^2 = E$. The last assertion follows from Eq. (2.2), the representability of any solution ψ of Eq. (1.2) in D in the form $\psi = \psi_0 + G^+ * (v - Ew)\psi$, where ψ_0 is a solution of Eq. (3.1), and the assertion that ψ_0 admits approximation by the functions e^{ikx}, $k^2 = E$, $k \in \mathbf{R}^n$.

From the operator $\hat{\Phi}(E)$ the scattering amplitude is uniquely found by Theorem 1.

The proof of Proposition 6 follows from formula (3.6) of the solution of the inverse problem in the Born approximation.

Note at the Proofreading. Recently A. G. Ramm [*; **], proceeding from [23, 29], made several attempts to prove Proposition 5. His article [*] contains a substantial error, consisting in the identification of the solution ψ from formula (3) in [*] with the solution u from Proposition 1 in [*]. However, at least one of his subsquent attempts proved to be successful (see [**, Sec. II, Subsec. 3]).

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