Now take a = v in (2) and any b, c in L. It follows from Lemma 5 that f = h = 0. So 0 = g = $\langle u, v\rangle H_b c = \mu H_b c$. The theorem is proved.

The author is deeply grateful to G. I. Ol'shanskii: the first variant of the proof was erroneous and the second one too "coordinate."

LITERATURE CITED

1. K. H. Hofmann and J. D. Lawson, Semigroup Forum, 23, 343-357 (1981).
2. A. V. Levichev, in: Group-Theoretic Methods in Physics [in Russian], Vol. 2, Nauka, Moscow (1983), pp. 140-143.
3. G. I. Ol'shanskii, Funktsion. Anal. Ego Prilozhen., 16, No. 4, 53-66 (1982).
4. M. B. Menskii, Method of Induced Representations: Space—Time and the Conception of Particles [in Russian], Nauka, Moscow (1976).
5. J. Hilgert and K. H. Hofmann, Preprint TH No. 779, Darmstadt (1983).

HIGHER BRUHAT ORDERS, RELATED TO THE SYMMETRIC GROUP

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1. As it is known, the triangle or Yang—Baxter equations express a "factorizability" condition for a certain matrix-valued function S on the symmetric group Σ_n, $n \geqslant 3$ (see [1]). This interpretation is a consequence of the fact that as a complete system of relations in Σ_n one can take the Coxeter relations. In the present note we derive objects which play the role of Σ_n relative to the equations of Zamolodchikov's d-simplexes [2, 5] (the latter form a multidimensional generalization of the Yang—Baxter equations). More exactly, for all $n \geqslant k \geqslant 1$ we define partially ordered sets B(n, k) such that B(n, 1) is isomorphic to Σ_n with the weak Bruhat order [3] and B(n, k) is the quotient set of the set of maximal chains in B(n, k − 1). The fundamental result of the paper is the purely combinatorial theorem of Sec. 2, in which for B(n, k) one establishes the analogues of the classical properties of Σ_n. The "Coxeter relations" in B(n, k) have length k + 2 (see Example 3). This stipulates the fact that the equation of the (k + 2)-simplexes has in the left- and right-hand sides a product of k + 2 operators. The solutions of these equations, constructed in [5], give examples of linear representations of B(n, k). The sets B(n, k) are closely related with the combinatorial structure of the convex hull of the general orbit Σ_n in $\mathbf{R}^n$. It seems that the investigation of the sets B(n, k) and their representations may have an independent interest.

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2. Let n, k be integers, $n \geqslant k \geqslant 1$. Let C(n, k) denote the set of k-element subsets of the set {1, 2,...,n}. The elements of C(n, k) will be denoted by $(i_1 i_2 \ldots i_k)$, $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. For $c = (i_1 \ldots i_k) \in C(n, k)$ we denote by $c_{\hat{j}} \in C(n, k-1)$ $(1 \leqslant j \leqslant k)$ the subset of c obtained by removing the element i_j.

Definition 1. A(n, k) is the set of those total orders on C(n, k) such that for each $d \in C(n, k+1)$ either $d_{\hat{1}} < d_{\hat{2}} < \ldots < d_{\widehat{k+1}}$ or $d_{\hat{1}} > d_{\hat{2}} > \ldots > d_{\widehat{k+1}}$.

The elements $a \in A(n, k)$ will be written in the form of chains $a = c_1 \ldots c_N$, $c_i \in C(n, k)$, $c_1 < c_2 < \ldots < c_N$, $N = \binom{n}{k}$. By the inversion of an element $a \in A(n, k)$ we mean a subset $d \in C(n, k+1)$ such that $d_{\hat{1}} < d_{\hat{2}} < \ldots < d_{\widehat{k+1}}$. The set of the inversions of a is denoted by Inv (a) and the number of inversions by inv (a).

Example 1. If $a_{\min}$ is the lexicographic order in C(n, k), then Inv ($a_{\min}$) = $\emptyset$; if $a_{\max}$ is the order opposite to the lexicographic one, then Inv ($a_{\max}$) = C(n, k + 1).

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We say that for $a, a' \in A(n, k)$ we have $a \sim a'$ if a' is obtained from $a = c_1 \dots c_N$ by the permutation of two adjacent subsets c_j and c_{j+1}, containing in the aggregate at least k + 2 elements.

Definition 2. B(n, k) is the quotient of A(n, k) with respect to the equivalence generated by $\sim$.

Let $\pi: A(n, k) \to B(n, k)$ be the projection. Clearly, the definition of an inversion can be carried over in a natural manner to the set B(n, k).

Let $a \in A(n, k)$. We assume that $d \in C(n, k+1)$ is not an inversion of a and that the subsets $d_{\hat{1}}, d_{\hat{2}}, \dots, d_{\widehat{k+1}}$ are in a row relative to the order a. By a rearrangement of a with the aid of d, denoted $p_d(a)$, we mean an element $a' \in A(n, k)$ obtained from a by permuting the subsets $d_{\hat{i}}$ into the inverse order. For $b, b' \in B(n, k)$ we shall write $b' = p_d(b)$ if there exist $a, a' \in A(n, k)$ such that $b = \pi(a)$, $b' = \pi(a')$ and $a' = p_d(a)$. Clearly, $\mathrm{Inv}(b') = \mathrm{Inv}(b) \cup \{d\}$.

Now we introduce a partial order on B(n, k).

Definition 3. We say that $b \leqslant b'$ if b' is obtained from b by some sequence of rearrangements.

Example 2. B(n, 1) = A(n, 1) is isomorphic to the symmetric group with the weak Bruhat order.

We recall that a partially ordered set X is said to be ranked with rank function l: X → (integers) if for all $x, y \in X, x \leqslant y$, and all incompressible chains $x = x_0 < x_1 < \dots < x_n = y$ we have $n = l(y) - l(x)$.

THEOREM. a) Relative to the introduced order, B(n, k) is a ranked order set, where the rank is the inv function. In B(n, k) one has unique minimal $b_{\min} = \pi(a_{\min})$ and maximal $b_{\max} = \pi(a_{\max})$ elements (see Example 1).

b) Let $a = d_1 d_2 \dots d_M \in A(n, k+1)$. Then the elements $b_{\min}, p_{d_1}(b_{\min}), p_{d_2}p_{d_1}(b_{\min}), \dots, p_{d_M}p_{d_{M-1}} \dots p_{d_1}(b_{\min}) = b_{\max}$ form a maximal chain in B(n, k). The constructed correspondence defines a bijection between A(n, k + 1) and the set of maximal chains in B(n, k).

c) Each element $b \in B(n, k)$ is uniquely determined by the set of its inversions $\mathrm{Inv}(b) \in C(n, k+1)$.

The *proof* is carried out by induction on n + k. For this we make use of the following

Proposition. For each $b \in B(n, k)$ there exists a representative $a = c_1 \dots c_N \in A(n, k)$ in which all c_i, containing 1, are in a row.

Example 3. B(n, n) = A(n, n) consists of one element $a = (12 \dots n)$; B(n, n − 1) = A(n, n − 1) consists of two elements: $a_{\hat{1}} \dots a_{\hat{n}}$ and $a_{\hat{n}} \dots a_{\hat{1}}$. B(n, n − 2) consists of 2n elements and represents 2 disjoint chains of length n, joining $b_{\min}$ and $b_{\max}$.

3. Let $x = (x_1, \dots, x_n)$ be a point in $\mathbf{R}^n$ with distinct coordinates. We denote by S_n the convex hull of the points $\sigma x = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, $\sigma \in \Sigma_n$ (see [4]). (We are interested only in the combinatorial structure of S_n, which does not depend on the selection of x.) It is plausible that for all k, $1 \leqslant k \leqslant n - 1$, one has a unique bijection between the set of rearrangements, i.e., the incompressible chains of length 1 in B(n, k), and the set of k-dimensional faces of S_n, combinatorially isomorphic to S_{k+1}. For k = 1 this is the known interpretation of the weak Bruhat order.

4. Relation with Hyperplane Configurations. Let $\mathscr{H} = (H_1, \dots, H_n)$ be a set of affine hyperplanes in general position in $\mathbf{R}^k$, $n \geqslant k$. We set $H(i_1, \dots, i_p) = H_{i_1} \cap \dots \cap H_{i_p}$ We shall call $\mathscr{H}$ an initial arrangement if for every $(i_1 \dots i_{k+2}) \in C(n, k+2)$: a) all points $a_j = H(i_1, \dots, \hat{i_j}, \dots, i_{k+1})$ for $j = 1, \dots, k+1$ lie on the same side of H_{k+2}; b) moving in the direction of these points parallel to itself, H_{k+2} meets them in the order $a_1, a_2, \dots, a_{k+1}$. We fix the projection $T: \mathbf{R}^k \to \mathbf{R}^1$ ("time"). Let $H^t = T^{-1}(t)$, $t \in \mathbf{R}^1$. Let $\mathscr{B}(n, k)$ be the space whose points are general arrangements $\mathscr{H}$ such that a) none of the lines $H(i_1, \dots, i_{k-1})$ for $(i_1 \dots i_{k-1}) \in C(n, k-1)$ is parallel to H^0; b) for sufficiently small t the arrangement $(H_1 \cap H^t, \dots, H_n \cap H^t)$ is initial. Let $\mathscr{A}(n, k) \subset \mathscr{B}(n, k)$ be the subspace of such arrangements for which all $T(H \times (i_1, \dots, i_k))$ for $(i_1 \dots i_k) \in C(n, k)$ are distinct. Let $\mathscr{H} \in \mathscr{A}(n, k)$. We order the elements $(i_1, \dots, i_k) \in C(n, k)$ according to increasing $T(H(i_1, \dots, i_k))$. We obtain the element A(n, k). This defines the mapping $\varphi(n, k): \pi_0 \mathscr{A}(n, k) \to A(n, k)$. In a similar manner one defines $\psi(n, k): \pi_0 \mathscr{B}(n, k) \to B(n, k)$.

Conjecture. For all (n, k), $\varphi(n,k)$ and $\psi(n,k)$ are bijections. From the previous results one can derive that this conjecture is true if $n-k \leqslant 2$ or $k = 1$ or $(n, k) = (5, 2)$.

LITERATURE CITED

1. I. V. Cherednik, Teor. Mat. Fiz., 61, No. 1, 35-44 (1984).
2. A. B. Zamolodchikov, Zh. Eksp. Teor. Fiz., 79, No. 2(8), 641-664 (1980).
3. R. P. Stanley, "On the number of reduced decompositions of elements of Coxeter groups," Preprint, Princeton Univ. (1985).
4. F. A. Berezin and I. M. Gel'fand, Tr. Mosk. Mat. Obshch., 5, 312-351 (1956).
5. Yu. I. Manin and V. V. Shekhtman, Proc. Int. Sem. "Group-theoretic methods in physics," Nauka, Moscow (1986).

REPRESENTATIONS OF THE SYMMETRIC GROUP IN THE FREE LIE (SUPER-) ALGEBRA AND IN THE SPACE OF HARMONIC POLYNOMIALS

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This note contains the following results:

1) The character of the natural representation of the symmetric group S_n in the free Lie algebra $\mathfrak{G}_n$ with n generators is computed (since this representation is infinite-dimensional, the notion of a character needs to be clarified which is done in part 1). This result is a generalization of two known results: the formula for dimensions of homogeneous components of $\mathfrak{G}_n$ [1] and the theorem on the structure of the subspace of polylinear elements in $\mathfrak{G}_n$ as an S_n-module [2].

2) All these results are extended to the action of S_n in the free Lie superalgebra $\mathfrak{G}_n^{\prime}$ with n odd generators.

3) The representation of S_n on the space of harmonic polynomials $\mathcal{H}_n = \bigoplus_{k=0}^{n(n-1)/2} \mathcal{H}_n^k$, where $\mathcal{H}_n^k$ are homogeneous harmonic polynomials of degree k is studied. It is known that the representation of S_n in $\mathcal{H}_n$ is isomorphic to the regular representation (cf. [3, 4]). Here, a generalization of this fact is obtained describing the structure of representations of S_n in subspaces of $\mathcal{H}_n$ of the form $\bigoplus_{k\equiv m (\mathrm{mod}\ r)} \mathcal{H}_n^k$, where r divides n, m is a residue modulo r.

These results imply a peculiar corollary saying that the spaces of polylinear elements in $\mathfrak{G}_n$ and $\mathfrak{G}_n^{\prime}$ are isomorphic as modules over S_n to natural subspaces in $\mathcal{H}_n$. It would be interesting to construct these isomorphisms explicitly.

1. Let $V = \bigoplus_l V^l$ be a $\mathbf{Z}_+^n$-graded space, where l denotes an n-tuple of numbers $(l_1, \ldots, l_n) \in \mathbf{Z}_+^n$. The symmetric group S_n acts in $\mathbf{Z}_+^n$ by permutations of coordinates: if $\sigma \in S_n$, then $\sigma(l_1, \ldots, l_n) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(n)})$. Suppose that V has an S_n-module structure with $\sigma(V^l) \subset V^{\sigma(l)}$. The character of the the module V is the power series over the n-tuple of auxiliary variables $t = (t_1, \ldots, t_n)$:

$$\mathrm{ch}_V(\sigma, t) = \sum \mathrm{tr}\, \sigma|_{V^l} \cdot t^l, \tag{1}$$

where the summation is performed for $l \in \mathbf{Z}_+^n$ such that $\sigma(l) = l$. One example of an S_n-module of the described type is the free Lie algebra $\mathfrak{G}_n$ with generators $x_1, \ldots, x_n$. Suppose that S_n is given as the group of permutations of $x_1, \ldots, x_n$. Then S_n naturally acts in $\mathfrak{G}_n$ by automorphisms. For each n-tuple $l \in \mathbf{Z}_+^n$ we denote by $\mathfrak{G}_n^l \subset \mathfrak{G}_n$ the subspace of elements of degree l_i in each generator x_i. Then, clearly $\sigma(\mathfrak{G}_n^l) \subset \mathfrak{G}_n^{\sigma(l)}$.

THEOREM 1.

$$\mathrm{ch}_{\mathfrak{G}_n}(\sigma, t) = \sum_{d \geqslant 1} \sum_{l \in \mathbf{Z}_+^n} \frac{\mu(d \cdot |\sigma|)}{d \cdot |\sigma|} t^{d(l + \sigma(l) + \cdots + \sigma^{|\sigma|-1}(l))} \frac{(|l|-1)!}{l!}, \tag{2}$$

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