# CONTINUATION OF DIFFEOMORPHISMS RETAINING VOLUME

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**1.** In this paper we shall denote by $M^m$ a connected, smooth, orientable, closed manifold of class $C^\infty$, by $W^m$ its submanifold with smooth boundary $\partial W^m$, and by $\omega_0$ the volume on $M^m$. On $M^m$ some metric is fixed, $W_i$ are the connected components of $W^m$, $N_j$ are the connected components of $M \setminus W$, and id is the identity mapping of M (with this, if A is a subset then id$|A$ is the identity embedding $A \to M$). For any region $A \subset M$ we write $\langle \omega, A\rangle = \int_A \omega$ ; here, $\omega$ is a differential m-form. The different families of mappings ($f_t(x)$, $F_t(x)$, etc.) are smooth functions of t and x of class $C^\infty$; the exception is the family of diffeomorphisms entering into the lemma.

Let there be a continuous family of embeddings $f_t : \partial W \to M$, with $f_0 = \mathrm{id}|\partial W$; it is known [4] that then $f_t$ can be continued to a family of diffeomorphisms $\bar{f}_t : M \to M$, coinciding with $f_t$ on $\partial W$ and with id when $t = 0$; we set $W_t = \bar{f}_t W$, $W_{it} = \bar{f}_t W_i$, and $N_{jt} + \bar{f}_t N_j$.

<u>THEOREM 1.</u> Let the family of embeddings $f_t : \partial W \to M$ possess the properties: $f_0 = \mathrm{id}|\partial W$, $\langle\omega_0, W_i\rangle = \langle\omega_0, W_{i1}\rangle$ and $\langle\omega_0, N_j\rangle = \langle\omega_0, N_{j1}\rangle$ for all i and j. Then, there exists a family of diffeomorphisms $F_t$: $M \to M$, such that $F_t^*\omega_0 = \omega_0$, with $F_0 = \mathrm{id}$ and $F_1|\partial W = f_1$.

<u>LEMMA.</u> There exists a continuous mapping $\nu$ of the set of positive definite m-forms $\omega$ with identical $\langle\omega, M\rangle = \langle\omega_0, M\rangle$ into Diff(M) such that $\nu(\omega)^*\omega = \omega$.

<u>Proof of the Lemma.</u> (Actually, we shall repeat the arguments of J. Moser, [1], Theorem 2, although this assertion does not formally appear there.) $\omega_t = (1-t)\omega_0 + t\omega$ is the family of forms. We specify a vector field $V_t$ such that the family of diffeomorphisms $\varphi_t$: $M \to M$ defined by it has the property $\varphi_t^*\omega_t = \omega_0$.

We perform the following computations:

$$0 = \frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*(\dot{\omega}_t + L_{V_t}\omega_t) = \varphi_t^*(\dot{\omega}_t + di_{V_t}\omega_t)$$

(i denotes inner multiplication and L the Lie derivative). Hence, $di_{V_t}\omega_t = -\dot{\omega}_t$. Moreover, $\langle\dot{\omega}_t, M\rangle = 0$, so that $\dot{\omega}_t$ is an exact form. By using the expansion given in [2] (§31), we obtain $di_{V_t}\omega_t = -d\delta G\dot{\omega}_t$. From the equation $i_{V_t}\omega_t = \delta G\dot{\omega}_t$ we uniquely find the field $V_t$ plus the corresponding family of diffeomorphisms $\varphi_t$. We have the equation $\varphi_1^*\omega = \omega_0$, i.e., mapping $\nu(\omega) = \varphi_1$ is the one we seek. This mapping is continuous as a mapping of the space of forms with the topology of $C^n$ into the space of diffeomorphisms with the topology $C^n$, $n = 1, 2, \ldots, \infty$. Indeed, G is continuous as a mapping of the space of differential forms of class $C^n$ into the space of forms of class $C^{n+1}$, so that the mapping $\omega_t \to V_t$ is continuous as a mapping of a form into a vector field of class $C^n$; finally, the continuity of mapping $V_t \to \varphi_1$ is, in essence, the theorem on the continuous dependence of the solutions of differential equations on the right-hand sides of those equations.

We note in passing that when $n = \infty$ the mapping we obtain is a global section of the fibration constituting the diffeomorphism $\varphi$ of form $\varphi^*\omega_0$. When $n < \infty$ $\nu$ is not a section of this fibration since if $\nu$ is of class $C^n$ we then have the formula $\varphi^*\omega_0$, and this means that $\nu(\varphi^*\omega_0)$ will, in general, be forms of class $C^{n-1}$. It is possible to consider $\nu$ as a section of the fibration the space of which consists of the diffeomorphisms of class $C^n$ with Jacobians of class $C^n$. It is not hard to deduce from this that this space is the direct product of the space of forms by Diff $(M, \omega_0)$ (the space of diffeomorphisms retain volume $\omega_0$).

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Proof of Theorem 1. We set $\omega_t = (1-t)f_t^{-1*}\omega_0 + t(\bar{f}_1 \circ \bar{f}_t^{-1})^*\omega_0$, where $\bar{f}_t$ is as above; then, $\langle\omega_t, W_{it}\rangle = \langle\omega_0, W_i\rangle, \langle\omega_t, N_{jt}\rangle$ and $\omega_1 = \omega_0$. We set $\varphi_t = \nu(\omega_t)^{-1} \circ \bar{f}_t$, $\bar{N}_{jt} = \varphi_t N_j$, $\bar{W}_{it} = \varphi_t W_i$. The family of embeddings $\varphi_t|\partial W$ has the properties $\varphi_0 = \bar{f}_0 = \mathrm{id}$, $\varphi_1 = \bar{f}_1$, so that $\varphi_1|\partial W = f_1$, $\langle\omega_0, \bar{W}_{it}\rangle = \langle\omega_0, \bar{N}_{jt}\rangle = \langle\omega_0, N_j\rangle$. Indeed,

$$\langle\omega_0, \bar{N}_{jt}\rangle = \langle\omega_0, \nu(\omega_t)^{-1}N_{jt}\rangle = \langle\nu(\omega_t)^{-1*}\omega_0, N_{jt}\rangle = \langle\omega_t, N_{jt}\rangle = \langle\omega_0, N_j\rangle.$$

We now proceed to the construction of $F_t$. On $\varphi_t\,\partial W$ we define the vector field $V_t(\varphi_t x) = \frac{d}{ds}\Big|_{s=0} \varphi_{t+s}x$. In a $\varepsilon$-neighborhood of $\partial W$ we introduce coordinate $y$ transversal to $\partial W$. We continue field $V_0$ in this neighborhood in the following way: $V_{0x}(x, y) = V_{0x}(x)$, and we find $V_{0y}(x, y)$ from the equation $\mathrm{div}_{V_0}\omega_0 = 0$ with the initial condition $V_{0y}(x)$. By an analogous construction we proceed in $\varphi_t U_\varepsilon\, \partial W$. We obtain the closed $(n-1)$-form $i_{V_t}\omega_0$ in the corresponding neighborhoods of manifolds $\varphi_t\partial W$. It remains to construct the family of closed forms $w_t$ on M coinciding with $i_{V_t}\omega_0$ in some neighborhood of manifold $\varphi_t\,\partial W$.

Let $\{z_h\}$ be the nonzero generators of $H_{n-1}(M, \mathbb{R})$ lying in $\varphi_t U_\varepsilon \partial W$ (i.e., of $H_{n-1}(\partial W, \mathbb{R})$). There exists on M a unique harmonic form $\omega_k$, such that $\langle\omega_h, z_h\rangle = 1$ and $\langle\omega_h, z\rangle = 0$ for all other cycles not depending on $z_h$. The cycles of $H_{n-1}(\varphi_t U_\varepsilon\, \partial W, \mathbb{R})$, not depending on $\{z_h\}$, are formed by the boundaries $\bar{N}_{jt}$ and $\bar{W}_{it}$. The values of $i_{V_t}\omega_0$ on them equal zero since

$$\langle i_{V_t}\omega_0, \partial\bar{N}_{jt}\rangle = \langle di_{V_t}\omega_0, \bar{N}_{jt}\rangle\ \langle L_{V_t}\omega_0, \bar{N}_{jt}\rangle = \frac{d}{ds}\Big|_{s=t} \langle(\varphi_s \circ \varphi_t^{-1})^*\omega_0, \bar{N}_{jt}\rangle = 0$$

The form $\alpha_t = i_{V_t}\omega_0 - \sum_h \langle i_{V_t}\omega_0, z_h\rangle\,\omega_h$ has zero period and, consequently, is exact on $\varphi_t U_\varepsilon\, \partial W$. Let $g_s : \varphi_t U_\varepsilon \partial W \to \varphi_t U_\varepsilon\, \partial W$ be a mapping which, in terms of the coordinates introduced above, is such that $g_s(x, y) = (x, sy)$. We construct an operator on the forms $\psi$

$$k\psi = \int_0^1 \frac{y}{s}\, i_{d/dy} g_s^*\psi\, ds.$$

Standard computations in the local coordinates show that $kd\alpha_t + dk\alpha_t = \alpha_t - g_0^*\bar{\alpha}_t$, where $\bar{\alpha}_t = (\mathrm{id}|\varphi_t\partial W)^*\alpha_t$ is an $(m-1)$-form of zero periods on the closed manifold $\varphi_t\,\partial W$. Using the expansion of [2] (§ 31), we obtain $\alpha_t = dk\alpha_t + dg_0^*\delta G\bar{\alpha}_t$. Let $p(y)$ be a smooth function equal to 1 when $|y| \le (\varepsilon/2)$ and equal to 0 when $|y| \ge \varepsilon$, so that then the $(m-1)$-form

$$w_t = \sum_h \langle i_{V_t}\omega_0, z_h\rangle\,\omega_h + d\,[p(y)\,(k\alpha_t + g_0^*\delta G\bar{\alpha}_t)]$$

is closed, is defined on M, and coincides with $i_{V_t}\omega_0$ in $\varphi_t U_{\varepsilon/2}\partial W$. In terms of it we uniquely reconstitute vector field $V_t$ and "motions" $F_t$, coinciding with $\varphi_t$ on $\partial W$ and retaining volume $\omega_0$ by virtue of the closure of $w_t$.

By analogous and, in part, simpler arguments we prove

THEOREM 2. Let the family of embeddings $f_t : W \to M$ have the properties: $f_0 = \mathrm{id}|W$, $f_t^*\omega_0 = \omega_0$ and $\langle\omega_0, N_{jt}\rangle = \langle\omega_0, N_j\rangle$. Then, there exists a family of diffeomorphisms $F_t : M \to M$ such that $F_t^*\omega_0 = \omega_0$ and $F_t|W = f_t$.

Finally, for some connected components of manifold W one can specify "motions" $f_t : W_i \to M$, of their boundaries $f_t : \partial W_i \to M$.

Remark. An important special case is when W is the set of "balls" $D_i^m$ on closed manifold $M^m$, with $m > 1$. Theorem 1 asserts that for any two sets, $D_i$ and $\bar{D}_i$, with identical volumes of the corresponding "balls" there exists a motion M retaining volume and translating $D_i$ into $\bar{D}_i$.

This assertion is also true in the case of nonclosed M, which strengthens Lemma 1.1 of [5]. Indeed, let $\varphi_t : \partial W \to M$ be an arbitrary family of embeddings taking $\partial D_i$ into $\partial\bar{D}_i$. In M we choose a compact manifold with boundary N such that $N \supset D_{it}$ for all i and t. Splicing boundary N and that of the manifold N' we can obtain a smooth closed manifold (double). On the union of N' and $\cup\, \partial D_i$ we specify a family of embeddings $f_t$ as follows: $f_t|N' = \mathrm{id}$ and $f_t|\partial D_i = \varphi_t$. In view of what was said above there exist a motion $\Phi_t$ of the closed manifold such that $\Phi_t^*\omega_0 = \omega_0$, $\Phi_1 D_i = \bar{D}_i$ and $\Phi_t|N' = \mathrm{id}|N'$. Then, the $F_t$ we have been seeking is defined by: $F_t$ on N coincides with $\Phi_t$ and $F_t = \mathrm{id}$ outside N.

2. We set $\langle\omega_0, W_i\rangle = a_i$, $\langle\omega_0, N_j\rangle = b_j$, $a = \{a_i\}$, $b = \{b_j\}$. If the embeddings $f : W \to M$ or $f : \partial W \to M$ are diffeotopically identical we have then uniquely defined $W_i(f) = \bar{f} W_i$, $N_j(f) = \bar{f} N_j$, where $\bar{f} : M \to M$ is related to id by means of the diffeotopy continuing the diffeotopy connecting $f$ with id|W or with id|∂W. (It is clear then $N_j(f)$ does not depend on the choice of the continuation.)

We introduce the following topological spaces with $C^\infty$-topology:

$E_{ab}(W, M)$ is the set of embeddings $f : W \to M$ diffeotopic to id|W and with properties 1) $\langle\omega_0, W_i(f)\rangle = a_i$; 2) $\langle\omega_0, N_j(f)\rangle = b_j$;

$E_{ab}(W, M, \omega_0)$ is the set of embeddings $f : W \to M$, diffeotopic to id|W and with properties 1, 2, and 3) $f^*\omega_0 = \omega_0$;

$E_{ab}(\partial W, M)$ is the set of embeddings $f : \partial W \to M$, diffeotopic to id|∂W and satisfying conditions 1) and 2);

$E_a(W, M)$ and $E_a(\partial W, M)$ are the corresponding sets of $E_{ab}(W, M)$ and $E_{ab}(\partial W, M)$ extended by dropping condition 2).

The following mappings arise:

$$\begin{array}{ccccc} \mathrm{Diff}_0(M,\omega_0) & \mathrm{Diff}(M,\omega_0) & E_{ab}(W,M) & E_{ab}(W,M,\omega_0) & E_a(W,M) \\ \downarrow\pi\ , & \downarrow\pi\ , & \downarrow\pi\ , & \downarrow\pi\ , & \downarrow\pi \\ E_{ab}(\partial W,M) & E_{ab}(W,M,\omega_0) & E_{ab}(\partial W,M) & E_{ab}(\partial W,M) & E_a(\partial W,M) \end{array}$$

(here, $\mathrm{Diff}_0$ is the connected component of id in Diff, and $\pi$ restriction mapping on, respectively, W or ∂W).

<u>THEOREM 3.</u> The triple $(\mathrm{Diff}_0(M, \omega_0), E_{ab}(\partial W, M), \pi)$ is a locally trivial fibration.

<u>Proof.</u> Our goal is the construction of a local section, i.e., a continuous mapping $\eta$ of some neighborhood $f_0 \in E_{ab}(\partial W, M)$ in $\mathrm{Diff}_0(M, \omega_0)$ such that $\eta(f) \circ f = f_0$ for all $f$ of $Uf_0$. With no loss of generality we can assume that $f_0 = \mathrm{id}|\partial W$. Denoting the exponential geodesic mapping by exp, we construct the homotopy $f_t(x) = \exp_x(t \exp_x^{-1}(f(x)))$, joining $f_0$ and $f_1 = f$. It is known from Theorem 1 that there exists a family of diffeomorphisms $F_t$ retaining $\omega_0$ and such that $F_0 = \mathrm{id}$ and $F_1|\partial W = f$. In the construction of $F_t$ there is an indeterminacy in the choice of $\bar{f}_t : M \to M$, the continuation of $f_t$. It can be eliminated by using the local section $\chi : Uf_0 \to \mathrm{Diff}_0(M)$, constructed by R. Palais in [3] (section 4). We set $\bar{f}_t = \chi(f_t)$, by which there will be defined on $Uf_0$ the continuous mapping $\eta(f) = F_1$, q.e.d.

It immediately follows from Theorem 3 that the triples $(E_{ab}(W, M), E_{ab}(\partial W, M), \pi)$ and $(E_{ab}(W, M, \omega_0), E_{ab}(\partial W, M), \pi)$ are locally trivial fibrations. By similar methods we prove the local triviality of the fibration $(\mathrm{Diff}_0(M, \omega_0), E_{ab}(W, M, \omega_0), \pi)$.

<u>THEOREM 4.</u> The triple $(E_a(W, M), E_a(\partial W, M), \pi)$ is a locally trivial fibration.

<u>Proof.</u> For a sufficiently small neighborhood $Uf_0$ there exist "balls" $D_{\rho j}$ of radius $\rho$ such that $D_{\rho j} \subset N_j(f)$ for all $f$ of $Uf_0$. We put into correspondence with each $f \in Uf_0$

$$\omega(f) = \begin{cases} \omega_0 & \text{outside } \bigcup_j D_{\rho j}, \\ \omega_0 + c_j(f)\, p_j(x)\, \omega_0 & \text{in } D_{\rho j}, \end{cases}$$

where $p_j(x)$ is a smooth function, equal to 1 in $D_{\frac{\rho}{2} j}$ and equal to 0 outside $D_{\rho j}$, while the $c_j(f)$ are found from the equation $\langle\omega(f), N_j(f)\rangle = \langle\omega_0, N_j(f_0)\rangle$. Using the lemma, we obtain the mapping $\zeta : Uf_0 \to \mathrm{Diff}_0(M)$, where $\zeta(f) = \nu(\omega(f))$ and $\zeta(f)^*\omega(f) = \omega_0$. Embedding $\zeta(f)^{-1} \circ f_0$ belongs to $E_{ab}(\partial W, M)$. Using the local section $\eta$, constructed for the proof of Theorem 3, we get $\eta(\zeta(f)^{-1} \circ f) \circ \zeta(f_0)^{-1} \circ f_0 = \zeta(f)^{-1} \circ f$ or $\xi(f) \circ f_0 = f$, where $\xi(f) = \zeta(f) \circ \eta(\zeta(f)^{-1} \circ f) \circ \zeta(f_0)^{-1}$ is a mapping of $Uf_0$ in the set of diffeomorphisms M leaving volume W invariant. Thus, the local section is constructed.

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## LITERATURE CITED

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