STABLE RANK OF RINGS AND DIMENSIONALITY OF TOPOLOGICAL SPACES

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The concept of the stable rank of a ring, introduced by H. Bass, turned out to be very useful in treating the stabilization problem in K-theory. This paper opens with the definition of stable rank and with an investigation of its basic properties (the connection of the stable rank of a ring with the stable ranks of the opposite ring, of the matrix ring, of the factor ring of ideals). We then consider the connection of the stable ranks of certain commutative rings with the dimensions of the spaces of their maximal ideals. For example, the stable rank of the ring of all continuous real-valued functions on n-dimensional topological space and the stable rank of the ring of polynomials in n unknowns with real coefficients both equal n + 1.

FORMULATION OF THE RESULTS

I. Let J be an associative ring. It will be convenient for us to consider J as being a ring embedded as a two-sided ideal in some associative ring L with unity. Such an L could be, for example, the ring J^1, obtained by the formal adjunction of a unity to ring J. The definition of the stable rank of ring J, to be given below, does not depend on the choice of L.

Definition. Column vector $b = (b_i)_{1 \le i \le n}$ is called J-unimodular if $b_1 - 1$, $b_i \in J$ $(i > 1)$ and there exist $a_1 - 1$, $a_i \in J$ $(i > 1)$, such that $\sum_{i=1}^{n} a_i b_i = 1$.

The following lemma shows that our definition coincides with that of [2].

LEMMA 1 (see, Lemma 2.0 of [4]). Let $b_1 - 1$, $b_i \in J$ $(i > 1)$. Then, the following three assertions are equivalent:

a) vector $b = (b_i)_{1 \le i \le n}$ is J-unimodular,

b) $\sum_{i=1}^{n} Lb_i = L$,

c) $\sum_{i=1}^{n} Jb_i = J$.

Definition. By the stable rank of ring J (abbreviated as st. r. (J)) we mean the least natural number m for which the following condition is met:

$(1)_m$ for any J-unimodular vector $(b_i)_{1 \le i \le m+1}$ there exist $v_i \in J$, such that vector $(b_i + v_i b_{m+1})_{1 \le i \le m}$ is J-unimodular. If such a natural m does not exist we then set st. r. (j) = ∞.

Remark. If ring L is generated as the left ideal of its own infinite subset $(b_\lambda)_{\lambda \in \Lambda}$, then, for any $\mu \in \Lambda$, there exist $v_\lambda \in L$, such that L is generated as the left ideal of elements $(b_\lambda + v_\lambda b_\mu)_{\lambda \in \Lambda - \mu}$.

THEOREM 1. If st. r. (J) = m then, when n > m, for any J-unimodular vector $(b_i)_{1 \le i \le n}$ there exist $v_i \in J$ such that vector $(b_i + v_i b_n)_{1 \le i \le n-1}$ is J-unimodular, and $v_i = 0$ when i > m. In particular, $(1)_n$ $(1)_{n+1}$ for any ring J and natural number n.

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According to this theorem, the expression "st. r. (J) $\leq$ m" has here the same meaning as in [3, 4] and, for rings with unity, the same meaning as the expression "m defines the stable rank for GL(J)" in [1].

It is obvious that the following two lemmas flow from the definition of stable rank.

LEMMA 2. If ring J decomposes into the direct product (of any number) of rings J_λ, then st. r. (J) = $\max_\lambda$ (st. r. (J_λ)).

LEMMA 3. St. r. (J) = st. r. (J/rad J), where rad J is the Jacobson radical of ring J.

The following theorem eliminates the lack of equivalence of row vectors and column vectors in the definition of stable rank.

THEOREM 2. St. r. (J) = st. r. (J^0), where J^0 is the inverse ring to ring J.

We denote by $M_n(J)$ the ring of all n × n matrices over ring J. The connection of the stable rank of ring J with the stable rank of ring $M_n(J)$ was quite unexpected by the author.

THEOREM 3. St. r. $(M_n(J))-1=-\left[-\frac{(J)-1}{n}\right]$.

Here, [r] denotes the integral part of the number r, i.e., the greatest integer not exceeding r.

From Theorem 3, by means of Lemmas 2 and 3, follows the evident

COROLLARY. If the ring J/rad J decomposes into the product of (any number of) matrix rings over nonassociative division rings (for example, if J is a finite-dimensional algebra over a field), then st. r. (J) = 1.

To be sure, this assertion can also be proven without the use of Theorem 3 (see [1] or [4]).

It is known (see [3] or [4]) that for any two-sided ideal J_1 in J the inequalities st. r. (J_1) $\leq$ st. r. (J) and st. r.(J/J_1) $\leq$ st. r. (J) hold. In this paper we obtain bounds on the other side for the stable rank of ring J.

THEOREM 4. For any ring J and twosided ideal J_1 in J the following inequalities hold: max (st. r. (J_1), st. r. (J/J_1)) $\leq$ st. r. (J) $\leq$ max (st. r. (J_1), st. r. (J/J_1) + 1).

If, for each (J/J_1)-unimodular vector $b = (b_i)_{1 \leq i \leq m}$, where m = st. r. (J), there exists matrix A $\in$ GL(m, L), such that modulo J_1 vector Ab is congruent with the first column of the unit matrix I_n, then st. r. (J) = max (st. r. (J_1), st. r. (J/J_1)).

Here, as everywhere in the sequel, we denote by GL(n, L) the group of two-sided invertible n × n matrices over ring L, and by I_n the unit element of this group.

COROLLARY. St. r. (J^1) = max (2, st. r. (J)), where J^1 is the ring obtained by the formal adjunction of unity to ring J.

Indeed, J^1/J = Z, the ring of integers.

Remarks on Theorem 4. a) The author knows no counterexamples to the equation st. r. (J) = max (st. r. (J_1), st. r. (J/J_1)).

b). Let us show that for the validity of the inequality st. r. (J_1) $\leq$ st. r. (J) it is essential that J_1 is a two-sided ideal. Let st. r. (J) = m; by our subsequent Theorem 5, m can take on any natural value. Then, by Theorem 3, st. r. ($M_m(J)$) = 2. Consider in ring $M_m(J)$ the left ideal J_1 consisting of matrices differing from the zero matrix only in the first column. Then, J_1/rad J_1 = J/rad J, so that, by Lemma 3, st. r. (J_1) = m.

II. Before formulating the theorems connecting stable ranks of rings of continuous functions on a topological space with the dimensionality of this space, we provide the appropriate definition of dimensionality.

On real n-dimensional space $\mathbf{R}^n$ we consider the ordinary distance $\rho(a, b)=\sqrt{\sum_{i=1}^{n}(a_i-b_i)^2}$ and the corresponding Hausdorff topology.

LEMMA-DEFINITION. Let X be a topological space, and let $a : X \to \mathbf{R}^n$ be a continuous mapping. The point $O = (0, \ldots, 0)$ is called an unstable value of mapping "a" if the following conditions, all equivalent to one another, hold:

a) for any $\varepsilon > 0$ there exists a continuous mapping $b: X \to \mathbf{R}^n$, such that $\rho(a(x), b(x)) \leq \varepsilon$ for all $x \in X$ and $b(x) \not\ni O$;

b) for any $\varepsilon > 0$ there exists a continuous mapping $b: X \to \mathbf{R}^n$, such that $a(x) = b(x)$ when $\rho(a(x), O) \geq \varepsilon$ and $b(x) \not\ni O$;

c) for any $\varepsilon > 0$ there exists a continuous mapping $b: X \to \mathbf{R}^n$, such that $\rho(a(x), b(x)) \leq \varepsilon \leq \rho(b(x), O)$ for all $x \in X$, and $a(x) = b(x)$ when $\rho(a(x), O) \geq \varepsilon$.

Proof of the implication a) ⇒ b). We find b from a) and set

$$b'(x) = \alpha(\rho(a(x), O))\,a(x) + (1 - \alpha(\rho(a(x), O)))\,b(x),$$

where

$$\alpha(r) = \begin{cases} 0 & \text{when } r \leqslant \varepsilon, \\ r/\varepsilon - 1 & \text{when } \varepsilon \leqslant r \leqslant 2\varepsilon, \\ 1 & \text{when } r \geqslant 2\varepsilon. \end{cases}$$

Then $b'(X) \not\ni O$ and $b'(x) = a(x)$ when $a(x) \geq 2\varepsilon$.

Proof of the implication b) ⇒ c). We find b from b) and set

$$b'(x) = \begin{cases} b(x) & \text{when } \rho(b(x), O) \geqslant \varepsilon, \\ \varepsilon b(x)/\rho(b(x), O) & \text{when } \rho(b(x), \leqslant O)\ \varepsilon. \end{cases}$$

The implication c) ⇒ a) is obvious.

Definition. The dimension d(x) of topological space X is the greatest integer d for which there exists a continuous mapping $a: X \to \mathbf{R}^d$ with stable value O. If such a d does not exist we then set $d(X) = \infty$.

We note that if O is a stable value for $a : X \to \mathbf{R}^n$, then this is also true for the composition of "a" with a projection on any linear subspace in $\mathbf{R}^n$. Therefore, for any $n \leq d(X)$, there exists a continuous mapping $X \to \mathbf{R}^d$ with stable value of O. It is easy to verify that d(X) coincides with the dimension defined in [5] by means of the mapping of X, not in $\mathbf{R}^n$, but in the unit cube $I^n \subset \mathbf{R}^n$. It is known that for "good" spaces, e.g., for metrizable separable X (see [5]), d(X) coincides with all the other dimensions (inductive, combinatory, etc.) which for any topological space are different, in general. For example, $d(\mathbf{R}^n) = n$.

We provide one further definition of dimension. We denote by S^n the n-dimensional sphere $\{a \in \mathbf{R}^{n+1} \mid \rho(a, O) = 1\}$ and by $S^{n-1} = \{a \in S^n \mid a_{n+1} = 0\}$ the equator of this sphere.

Definition. Continuous mapping $a : X \to S^n$ is said to be nonessential if there exists a continuous mapping $b : X \to S^{n-1}$, such that $b(x) = a(x)$ when $a(x) \in S^{n-1}$.

It is readily verified that the nonessentiality of "a" is equivalent to the existence of a homotopy $a_t : X \to S^n$ $(0 \leq t \leq 1)$, $a_0 = a$, such that a_t coincides with "a" on $a^{-1}(S^{n-1})$ and $a_1(X) \subset S^{n-1}$.

Definition. The dimension d'(X) of topological space X is the greatest integer d for which there exists an essential mapping $X \to S^d$. If such a d does not exist we then set $d'(X) = \infty$.

THEOREM 5. For any topological space X we denote by $\mathbf{R}^X$ (respectively, by $\mathbf{R}_0^X$) the ring of all (respectively, bounded) continuous realvalued functions on X. Then, st. r. $(\mathbf{R}^X)$ = st. r. $(\mathbf{R}_0^X) = d(X) + 1 = d'(X) + 1$.

THEOREM 6. Let X be a topological space, and let K be the subring in $\mathbf{R}^X$ containing all constants. We assume that for any bounded function $f \in \mathbf{R}_0^X$ and for any $\varepsilon > 0$ there exists function $g \in K$ such that $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$. Then, st. r. (K) $\geq d(X) + 1$. We further assume that if $g \in K$ and $g(x) \geq \varepsilon$ for some $\varepsilon > 0$ and for all $x \in X$, then $g^{-1} \in K$. Then st. r. (K) $= d(X) + 1$.

Example. Let X be the ring of infinitely differentiable functions on $\mathbf{R}^n$. Then, st. r. (K) $= n + 1$.

THEOREM 7. Let X be a topological space, C^X the ring of all continuous complexvalued functions on X, and K a subring in C^X containing all constants. We assume that for any bounded function $f \in \mathbf{R}_0^X$ and

$\varepsilon > 0$ there exists function $g \in K \cap \mathbf{R}^X$, such that $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$. Then st. r. $(K) \geq$ st. r. $(\mathbf{C}^X) = [d(X)/2] + 1$. We further assume that if $g \in K$ and $|g(x)| \geq \varepsilon$ for some $\varepsilon > 0$ and for all $x \in X$, then $g^{-1} \in K$. Moreover, let $g \in K$ for any function $g \in K$. Then st. r. $(K) = [d(X)/2] + 1$.

In this theorem, $[r]$ is the integral part of number r.

III. We recall that topological space X is said to be Noetherian if there does not exist an infinite chain $X_1 \supset X_2 \supset \ldots$ of closed sets strictly embedded within one another. A closed set is called irreducible if it is impossible to represent it as the union of two of its proper closed subsets.

Definition. The dimension $\dim(X)$ of topological space X is the maximal length d of the chain $X_0 \supset X_1 \supset \ldots \supset X_d$ of different embedded closed irreducible sets.

The dimension $\dim(X)$ is usually used only for Noetherian spaces X; for a Hausdorff space X it is always the case that $\dim X = 0$. On the other hand, for a Noetherian space X, it is always true that $d(X) = 0$.

THEOREM OF BASS (see [1] or [4]). Let K be a commutative ring with unity, the space X of maximal ideals of which is a Noetherian space of dimension $\dim(X) = d$. Then, st. r. $(K) \leq d + 1$.

(On the space of maximal ideals, we are considering a topology in which the closed sets are the sets of ideals containing some ring elements.)

It would be desirable, under the conditions of the Bass Theorem, to obtain a bound for st. r. (K) on the other side. In this connection we succeeded in obtaining the following result.

THEOREM 8. Let k be any subfield in the field of real numbers $\mathbf{R}$, and let $K = k[t_1, \ldots, t_n]$ be the polynomial ring in n unknowns with coefficients from k. Then, st. r. $(K) = n + 1$.

The author knows no counterexample to the equation st. r. $(K) = \dim(X) + 1$ under the conditions of the Bass Theorem. If the condition of being Noetherian is dropped, it is then impossible, in general, to characterize the stable rank of the ring only in terms of the topological space of maximal ideals. For example, the spaces of maximal ideals of rings $\mathbf{R}^X$ and $\mathbf{C}^X$ (see Theorems 5 and 6) are isomorphic as topological spaces, while, at the same time, st. r. $(\mathbf{R}^X) = d(X) + 1$ and st. r. $(\mathbf{C}^X) = [d(X)/2] + 1$.

PROOFS OF THE THEOREMS

I. Proof of Theorem 1. Let condition $(1)_m$ hold, $n > m$, and let vector $b = (b_i)_{1 \leq i \leq n}$ be J-unimodular, i.e., $\sum_{i=1}^{n} a_i b_i = 1$ for some $a_i \in L$ (see Lemma 1). We set $b_i' = b_i (1 \leq i \leq m)$ and $b'_{m+1} = \sum_{i=m+1}^{n} a_i b_i \in J$. Then vector $b' = (b_i')_{1 \leq i \leq m+1}$ is J-unimodular and, by condition $(1)_m$, there exist $v_i' \in J$ such that $\sum_{i=1}^{m} L b_i'' = L$, where $b_i'' = b_i' + v_i' b_{m+1} = b_i \ v_i' \sum_{j=m+1}^{n} a_j b_j$. We set $A_{i,j} = v_i' a_j (1 \leq i \leq m < j \leq n-1)$, $v_i = v_i' a_n (1 \leq i \leq m)$ and $v_i = 0$ when $i > m$. Then, $b_i'' = b_i + v_i b_n + \sum_{j=m+1}^{n-1} A_{i,j} b_j (1 \leqslant i \leqslant m)$. We set $b_i'' = b_i$ when $m < i < n$ and $A = 1_{n-1} + \sum_{i=1}^{m} \sum_{j=m+1}^{n-1} A_{i,j} e_{i,j} \in GL(n-1, J)$, where $A_{i,j} e_{i,j}$ is the matrix with $A_{i,j}$ in position i, j and with zeros elsewhere. Since vector $b'' = (b_i'')_{1 \leq i \leq n-1}$ is J-unimodular, vector $A^{-1} b'' = (b_i + v_i b_n)_{1 \leq i \leq n-1}$ is also, q.e.d.

Proof of Theorem 2. Since $(J^0)^0 = J$, it suffices to prove the inequality st. r. $(J^0) \leq$ st. r. (J). Let st. r. $(J) = m$. We need to show that if $\sum_{i=1}^{m+1} a_i b_i = 1$, where $a_1 - 1, b_1 - 1 \in J$ and $a_i, b_i \in J (i > 1)$, there then exist $u_i \in J$, such that $\sum_{i=1}^{m} (a_i + a_{m+1} u_i) J = J$.

Consider the matrix

$$E = \begin{pmatrix} 1 & a \\ O & 1_{m+1} \end{pmatrix} \begin{pmatrix} 1 & O \\ -b & 1_{m+1} \end{pmatrix} = \begin{pmatrix} 0 & a \\ -b & 1_{m+1} \end{pmatrix} \in GL(m+2, L).$$

By condition $(1)_m$ and Lemma 1 there exist $v_i \in J$ and $c_i \in J$ such that $\sum_{i=1}^{m} c_i(b_i + v_i a_{m+1} b_{m+1}) = -b_{m+1}$ Then

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_m & 0 \\ 0 & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -v & 1_m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_m & va_{m+1} \\ 0 & 0 & 1 \end{pmatrix} E$$

has the form

$$A = \begin{pmatrix} 0 & & a \\ * & * & 0 \\ 0 & -u & 1 \end{pmatrix} \in GL(m+2, L),$$

where $v = (v_i)_{1 \le i \le m}$ is a column vector while c and u are row vectors of length n, with $u_i \in J$. Matrix $B = A\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_m & 0 \\ 0 & u & 1 \end{pmatrix}$ has the form $\begin{pmatrix} 0 & a' & a_{m+1} \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $a' = (a_i + a_{m+1}u_i)_{1 \le i \le m}$, so that, consequently, $\begin{pmatrix} 0 & a' \\ * & * \end{pmatrix} \in GL(m+1, L)$, whence $\sum_{i=1}^{m} a_i' L = L$, q.e.d.

Proof of Theorem 3. Definition. Matrix $B = (B_{i,j})_{1 \le i \le n, 1 \le j \le k}$ of dimensions $n \times k$ is said to be J-unimodular if $B_{i,i} - 1, B_{i,j} \in J$ when $i \ne j$ and if there exists a $k \times n$ matrix A such that $A_{i,i} - 1, A_{i,j} \in J$ when $i \ne j$ and $AB = 1_k$.

Example. Each n-dimensional $(M_k(J))$-unimodular vector can be considered as a J-unimodular $(nk \times k)$-matrix.

Theorem 3 is a special case of the following theorem.

THEOREM 3'. Let k be a natural number. Then, condition $(1)_m$ is equivalent to the following condition:

$(1)_m^k$ for any J-unimodular $((m+k) \times k)$-matrix B there exist $v_i \in J$ such that $\begin{pmatrix} 1_{m+k-1} & v \\ 0 & 1 \end{pmatrix} B = \begin{pmatrix} B' \\ u \end{pmatrix}$, where $((m+k-1) \times k)$-matrix B' is J-unimodular and u is the last row of matrix B.

For the proof of Theorem 3' we shall use the following readily verified

LEMMA. Matrix B of the form $\begin{pmatrix} 1 & u \\ 0 & B' \end{pmatrix}$, where u is a row vector with coordinates in J, is j-unimodular if and only if matrix B' is J-unimodular.

Proof of Theorem 3' is by induction on k. When k = 1 condition $(1)_m$ coincides with $(1)_m^1$. We now assume that $k \ge 2$ and that we have already proven the equivalence $(1)_n \Leftrightarrow (1)_n^{k-1}$ for all n. We now show that then $(1)_n \Leftrightarrow (1)_n^k$ for all n.

Initially, assuming (1) and $(1)_{m-1}^{k-1}$ we obtain $(1)_m^k$ (we recall that by Theorem 1 $(1)_m \Rightarrow (1)_n$ for $n > m$). Let B be a J-unimodular matrix of dimensions $(m+k) \times k$. We consider its first column b which, certainly, is J-unimodular. By $(1)_{m+k-1}$ there exist $v_i \in J$ such that vector $b' = (b_i + v_i b_{m+k})_{1 \le i \le m+k-1}$ is J-unimodular. It follows from $(1)_{m+k-2}$ (see (c) on p. 411 of [4]) that $Ab' = e_1$ (the first column of the unit matrix) for some $A \in GL(m+k-1, J)$. The matrix

$$B'' = A\begin{pmatrix} 1_{m+k-1} & v \\ 0 & 1 \end{pmatrix} B \quad \text{has the form} \quad \begin{pmatrix} 1 & u' \\ & 0 \\ b_{m+k} & B' \end{pmatrix}.$$

Assertion $(1)_m^k$ simultaneously holds, or does not hold, for matrices B and $B'\begin{pmatrix} 1 & -u' \\ 0 & 1_{k-1} \end{pmatrix}$. Replacing B by $B'\begin{pmatrix} 1 & -u' \\ 0 & 1_{k-1} \end{pmatrix}$, we shall assume that from the very beginning matrix B had the form $\begin{pmatrix} 1 & 0 \\ 0 & \\ b_{m+k} & B' \end{pmatrix}$.

From the J-unimodularity of matrix B follows the J-unimodularity of matrix B'. By $(1)_{m-1}^{k-1}$ there exist $v_i \in J$, such that $\begin{pmatrix} 1 & O & 0 \\ O & 1_{m+k-2} & v \\ 0 & O & 1 \end{pmatrix} B = \begin{pmatrix} 1 & O \\ * & B'' \\ * & u \end{pmatrix}$; where matrix B" is J-unimodular and, consequently, so also is matrix $\begin{pmatrix} 1 & O \\ * & B'' \end{pmatrix}$.

To obtain the reverse implication $(1)_m^k \Rightarrow (1)_m$, it suffices to apply $(1)_m^k$ to matrix B of the form $\begin{pmatrix} 1_{k-1} & O \\ O & b \end{pmatrix}$, where b is any (m + 1)-dimensional J-unimodular column vector.

Proof of Theorem 4. As has already been mentioned, the inequality st. r. (J) $\geq$ max (st. r. (J_1), st. r. (J/J_1)) was proven in [3, 4]. Since the condition of the second part of the Theorem is automatically met if st. r. $(J/J_1) \leq m-1$ (see [3, 4], it then only remains to show that, assuming this condition to hold, we have the inequality st. r. (J) $\leq$ m when m = max (st. r. (J_1), st. r. (J/J_1)).

Let vector $b = (b_i)_{1 \leq i \leq m+1}$ be J-unimodular. Since st. r. $(J/J_1) \leq m$, we can then find $v_i \in J$, such that vector J_1, (J/J_1)-unimodular. By hypothesis, there exists matrix $A \in GL(m, L)$, such that $Ab' \equiv e_1$ mod J_1. Assertion $(1)_m$, which we shall now prove, is simultaneously valid or invalid for vectors b and $b' = A\begin{pmatrix} 1_m & v \\ O & 1 \end{pmatrix} b$. Replacing b by b" we shall assume that from the beginning $b_1 - 1, b_i \in J_1 (2 \leq i \leq m)$.

Replacing ring L by J^1 we shall assume that J_1 is a twosided ideal in L (this is convenient for the use of Lemma 1). Since $\sum_{i=1}^{m+1} Lb_i = L$ and $b_1 - 1 \in J_1$, then $\sum_{i=1}^{m+1} a_i b_i = 1 - b_1$ for some $a_i \in J_1$, whence $(a_1 + 1)$ $b_1 + \sum_{i=2}^{m+1} a_i b_i = 1$, i.e., vector $\begin{pmatrix} 1_m & O \\ O & a_{m+1} \end{pmatrix} b$ is J_1-unimodular. Since st. r. $(J_1) \leq m$ there exist $v_i \in J_1$ such that vector $(b_i + v_i a_{m+1} b_{m+1})_{1 \leq i \leq m}$ is J_1-unimodular and, in particular, J-unimodular. This completes the proof of condition $(1)_m$.

II. Proof of Theorem 5. We need to show the equivalence of the following four assertions:

(a) $(1)_n$ for $J = \mathbf{R}_0^X$,

(b) $(1)_n$ for $J = \mathbf{R}^X$,

(c) each continuous mapping $b : X \to S^n$ is nonessential,

(d) for each continuous mapping $a : X \to \mathbf{R}^n$ the value O is unstable.

Proof of the implication (a) $\Rightarrow$ (b). Let $b = (b_i)_{1 \leq i \leq n+1}$ be an $\mathbf{R}^X$-unimodular vector and then $f(x) = \rho(b(x), O) > 0$ for all $x \in X$ (this inequality is necessary and sufficient for $\mathbf{R}^X$-unimodularity). Vector $(b_i/f)_{1 \leq i \leq n+1}$ is $\mathbf{R}_0^X$-unimodular so that by condition $(1)_n$, assumed true for $J = \mathbf{R}_0^X$, there exist $v_i \in \mathbf{R}_0^X$, such that vector $(b_i/f + v_i b_{n+1}/f)_{1 \leq i \leq n}$ is $\mathbf{R}_0^X$-unimodular. Then vector $(b_i + v_i b_{n+1})_{1 \leq i \leq n}$ is $\mathbf{R}^X$-unimodular.

Proof of implication (b) $\Rightarrow$ (c). Let $b : X \to S^n$ be a continuous mapping. We denote by $b_i(x)$ the i-th projection of vector $b(x) \in \mathbf{R}^{n+1}$, and we consider the $\mathbf{R}^X$-unimodular vector $b = (b_i)_{1 \leq i \leq n+1}$. By $(1)_n$ for $J = \mathbf{R}^X$ there exist $v_i \in \mathbf{R}^X$, such that vector $a' = (b_i + v_i b_{n+1})_{1 \leq i \leq n}$ is $\mathbf{R}^X$-unimodular. We set $a(x) = a'(x)/\rho(a'(x), O)$. Then, $a : X \to S^{n-1}$ is a continuous mapping coinciding with b on $b^{-1}(S^{n-1})$, i.e., on those x for which $b_{n+1}(x) = 0$.

Proof of implication (c) $\Rightarrow$ (d). Let $b' : X \to \mathbf{R}^n$ be a continuous mapping, and let $\varepsilon > 0$. We set $f(x) = \rho(b'(x), O)$,

$$b_i(x) = \begin{cases} b_i'(x) & \text{when } f(x) \leqslant \varepsilon, \\ \varepsilon b_i'(x)/f(x) & \text{when } f(x) \geqslant \varepsilon, \end{cases} \qquad b_{n+1}(x) = \begin{cases} \sqrt{\varepsilon^2 - f(x)^2} & \text{when } f(x) \leqslant \varepsilon, \\ 0 & \text{when } f(x) \geqslant \varepsilon. \end{cases}$$

Then $\sum_{i=1}^{n+1} b_i^2 = \varepsilon^2$. By (c) there exists a continuous mapping $a' : X \to \varepsilon S^{n-1}$ coinciding with b on $b^{-1}(\varepsilon S^{n-1})$. We set

$$a(x)=\begin{cases} b'(x) & \text{when } f(x)\geqslant \varepsilon, \\ a'(x)/\rho(a'(x),O) & \text{when } f(x)\leqslant \varepsilon. \end{cases}$$

It remains to mention that $a(X) \not\ni O$ and, recalling definition b), is an unstable mapping.

Proof of implication (d) $\Rightarrow$ (a). Let vector $b=(b_i)_{1\le i\le n+1}$ be R_0^X-unimodular which, obviously, is equivalent to the inequality $f(x)=\rho(b(x),O)\le\varepsilon$ for some $\varepsilon>0$ and all $x\in X$. Since for the mapping $b': X\to R^n$, given by the formula $b'(x)=(b_i(x))_{1\le i\le n}$, the value O is unstable [according to our assumed assertion (d)], there then exists a continuous mapping $a: X\to R^n$ for which $\rho(a(x),O)\ge\varepsilon/2$ for all $x\in X$ and $a(x)=b'(x)$ when $\rho(b'(x),O)\ge\varepsilon/2$.

We set

$$v_i(x)=\begin{cases} (a_i(x)-b_i(x))/b_{n+1}(x) & \text{when } \rho(b'(x),O)\leqslant\varepsilon/2, \\ 0 & \text{when } \rho(b'(x),O)\geqslant\varepsilon/2. \end{cases}$$

Then, $a_i=b_i+v_ib_{n+1}$ and vector $a=(b_i+v_ib_{n+1})_{1\le i\le n}$ is R_0^X-unimodular.

Proof of Theorem 6. It is necessary to show that if $(1)_n$ holds when J = K, then the value of O is unstable for each continuous mapping $a: X\to R^n$. Let $1>\varepsilon>0$. We set $f(x)=\rho(a(x),O)$ and

$$a'(x)=\begin{cases} a(x) & \text{when } f(x)\leqslant 2\varepsilon, \\ 2\varepsilon a(x)/f(x) & \text{when } f(x)\geqslant 2\varepsilon. \end{cases}$$

Since ring K is dense in R_0^X there exist $b_i\in K$, such that $\rho(b'(x),a'(x))\le\varepsilon/2$ for all $x\in X$, where $b'=(b_i)_{1\le i\le n}$, and in particular, $\rho(b'(x),a(x))\le\varepsilon/2$ when $\rho(a(x),O)\le 2\varepsilon$. We set $b_{n+1}(x)=(\varepsilon/4)^2-\sum_{i=1}^{n}b_i(x)^2$. Then, vector $b=(b_i)_{1\le i\le n+1}$ is K-unimodular. We can find $v_i\in K$ such that vector $c'=(b_i+v_ib_{n+1})_{1\le i\le n}$ is K-unimodular. We set

$$c''(x)=\begin{cases} \varepsilon c'(x)/4\rho(c'(x),O) & \text{when } b_{n+1}(x)\geqslant 0, \\ b'(x) & \text{when } b_{n+1}(x)\leqslant 0. \end{cases}$$

Then, $c'': X\to R^n$ is a continuous mapping $c''(X)\not\ni O$ and $\rho(c''(x),a(x))\le\varepsilon$ when $\rho(a(x),O)\le 2\varepsilon$. Finally we set

$$c(x)=\alpha(\rho(a(x),O))a(x)+(1-\alpha(\rho(a(x),O)))c''(x),$$

where function α is the same as in the proof of the equivalence of the definitions of unstable mappings. Then, $c(X)\not\ni O$ and $\rho(c(x),a(x))\le\varepsilon$ for all $x\in X$.

Now, let hold the additional condition of the Theorem's second part. We need to show that when $n=\text{st. r.}(R_0^X)$ condition $(1)_n$ holds for J = K. For any K-unimodular vector $b=(b_i)_{1\le i\le n+1}$ the vector b/f, where $f(x)=\rho(b(x),O)$, is R_0^X-unimodular, since $\sum_{i=1}^{n+1}(b_i/f)^2=1$. Consequently, there exist $v_i'\in R_0^X$, such that vector $b'=(b_i/f+v_i'b_{n+1}/f)_{1\le i\le n}$ is R_0^X-unimodular, i.e., $\sum_{i=1}^{n}b_i'(x)^2\geqslant\varepsilon$ for some $\varepsilon>0$ and for all $x\in X$. We can find $v_i\in K$ such that the inequality $\sum_{i=1}^{n}(v_i(x)-v_i'(x))^2\leqslant(\varepsilon/2)^2$ (for all $x\in X$) holds. We set $c=(b_i+v_ib_{n+1})_{1\le i\le n}$. Then, $g(x)=\rho(c(x)/f(x),O)\ge\rho(b'(x),O)-\rho(b'(x),c(x)/f(x))\ge\varepsilon/2$ for all $x\in X$. Consequently, $\sum_{i=1}^{n}(c_i/f^2)\cdot c_i=g^2\geqslant\varepsilon^2/4$, whence $g^{-2}\in K$, if $f^{-2}\in K$. For the proof of the K-unimodularity of vector c it remains to show that $f^{-2}\in K$. We recall that $f^2=\sum_{i=1}^{n+1}b_i^2$. Since vector b is K-unimodular then $\sum_{i=1}^{n+1}a_ib_i=1$ for some $a_i\in K$. By the Cauchy-Bunyakovskii Inequality, $f^2s^2\ge 1$, where $s^2=\sum_{i=1}^{n+1}a_i^2$, whence $(fs)^{-2}\in K$ and $f^{-2}\in K$.

Proof of Theorem 7. We shall show that if condition $(1)_n$ holds for J = K then $d(X)\le 2n-1$, i.e., for each continuous mapping $a: X\to R^{2n}$ the value of O is unstable. Let $\varepsilon>0$. We set $f(x)=\rho(a,(x),O)$ and

$$a'(x)=\begin{cases} a(x) & \text{when } f(x)\leqslant 2\varepsilon, \\ 2\varepsilon a(x)/f(x) & \text{when } f(x)\geqslant 2\varepsilon. \end{cases}$$

By hypothesis there exist $b_i' \in K \cap \mathbf{R}^X$ such that $\rho(b'(x), a'(x)) \leq \varepsilon/2$ for all $x \in X$, where $b' = (b_i')_{1\leq i\leq 2n}$, in particular, $\rho(b'(x), a(x)) \leq \varepsilon/2$ when $\rho(a(x), O) \leq 2\varepsilon$. We set $b_i = b_i + \sqrt{-1}\, b_{n+i}' \in K$, $b_{n+1} = (\varepsilon/4)^2 - \sum_{i=1}^{2n} (b_i')^2 \in K$ Then, vector $b = (b_i)_{1\leq i\leq n+1}$ is K-unimodular. We can find $v_i \in K$ such that vector $c' = (b_i + v_i b_{n+1})_{1\leq i\leq n}$ is K-unimodular.

We set

$$c''(x) = \begin{cases} \varepsilon c'(x)/4\rho(c'(x), O) & \text{when } b_{n+1}(x) \geqslant 0, \\ b'(x) & \text{when } b_{n+1}(x) \leqslant 0. \end{cases}$$

Then, $c'' : X \to \mathbf{R}^{2n}$ is a continuous mapping, $c''(X) \not\ni O$ and $\rho(a(x), c''(x)) \leq \varepsilon$ when $\rho(a(x), O) \leq 2\varepsilon$. Finally, we define the continuous mapping $c : X \to \mathbf{R}^{2n}$ the same as at the end of the proof of the first half of Theorem 6.

Now, let the additional condition of the second half of the theorem hold. We need to show that when $2n \geq$ st. r. $(\mathbf{R}_0^X)$ for $J = K$ then condition $(1)_n$ is met. For any K-unimodular vector $b = (b_i)_{1\leq i\leq n+1}$ we set $f(x) = \rho(b(x), O)$, where b is considered as a continuous mapping $X \to \mathbf{R}^{2n}$. We set $b_i/f = f_i + \sqrt{-1}\, f_{n+i}$, where $f_i \in \mathbf{R}_0^X (1 \leq i \leq 2n+2)$. Then $\sum_{i=1}^{2n+2} f_i^2 = 1$. Since st. r. $(\mathbf{R}_0^X) \leq 2n$ there then exist $v_i' \in \mathbf{R}_0^X$ such that vector $b' = (f_i + v_i'(f_{n+1}^2 + f_{2n+2}^2))_{1\leq i\leq 2n}$ is $\mathbf{R}_0^X$-unimodular, i.e., $\sum_{i=1}^{2n} b_i'(x)^2 \geqslant \varepsilon$ for some $\varepsilon > 0$ and all $x \in X$. We can find $v_i \in K$ such that, for all $x \in X$, the following inequality holds:

$$\sum_{i=1}^{n} |(v_i' + \sqrt{-1}\, v_{n+i}')(f_{n+1} - \sqrt{-1}\, f_{2n+2}) - v_i|^2 \leqslant \varepsilon^2/4.$$

We set $c = (b_i + v_i b_{n+1})_{1\leq i\leq n}$. Then, $g(x) = \rho(c(x)/f(x), O) \geq \varepsilon/2$ for all x, whence $g^2 = \sum_{i=1}^{n} (c_i/f^2) \cdot c_i \geqslant \varepsilon^2/4$. Therefore, $g^{-2} \in K$, if $f^{-2} \in K$. To prove the K-unimodularity of vector c it remains to show that $f^{-2} \in K$. We recall that $f^2 = \sum_{i=1}^{n+1} b_i \bar{b}_i$.

Since vector b is K-unimodular then $\sum_{i=1}^{n+1} a_i b_i = 1$ for some $a_i \in K$. By the Cauchy–Bunyakovskii Inequality, $f^2 s^2 \geq 1$, where $s^2 = \sum_{i=1}^{n+1} a_i \bar{a}_i$, whence $(fs)^{-2} \in K$ and $f^{-2} \in K$.

<u>Remark.</u> It can be shown analogously that the stable rank of ring K of continuous quaternion-valued function on topological space X equals $[d(X)/4] + 1$.

<u>III. Proof of Theorem 8.</u> The inequality st. r. $(K) \leq n + 1$ is contained in the Bass Theorem. It remains for us to show that when $J = K$ condition $(1)_n$ is not met. We set $b_i = t_i$ $(1 \leq i \leq n)$ and $b_{n+1} = 1 - \sum_{i=1}^{n} t_i^2$. Vector $b = (b_i)_{1\leq i\leq n+1}$, obviously, is K-unimodular. Were there to exist $v_i \in K$ such that vector $b' = (b_i + v_i b_{n+1})_{1\leq i\leq n}$ were K-unimodular, then we would be able to define a continuous mapping $a : \mathbf{R}^n \to \mathbf{R}^n$ by the formula

$$a(x) = \begin{cases} b'(x)/\rho(b'(x), O) & \text{when } b_{n+1}(x) \geqslant 0, \\ x & \text{when } b_{n+1}(x) \leqslant 0. \end{cases}$$

Then, $a(\mathbf{R}^n) \not\ni O$ and $\rho(a(x), x) \leq 2$ for all $x \in \mathbf{R}^n$. Whence, obviously, would follow the instability of the value of O for the identical mapping of space $\mathbf{R}^n$ which, as is well known, is false.

<u>Remarks.</u> a) When $k = \mathbf{R}$ the inequality st. r. $(K) \geq n + 1$ in Theorem 8 follows from Theorem 6 as applied to the unit ball X in $\mathbf{R}^n$.

b) Applying Theorem 8 when $k = \mathbf{R}$, we can obtain the inequality st. r. $(\mathbf{C}[t_1, \ldots, t_n]) \geq [n/2] + 1$, where $\mathbf{C}$ is the field of complex numbers.

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