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INTEGRAL GEOMETRY FOR $\bar{\partial}$-COHOMOLOGY IN q-LINEAR CONCAVE DOMAINS IN CP^n

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Introduction

A domain D in CP^n is called $(n-q-1)$-linearly concave if for any point $z \in D$ there exists a q-dimensional analytic plane ζ, such that $z \in \zeta \subset D$. For $q = n-1$, this is equivalent to the fact that the compactum $CP^n \setminus D$ is linearly convex in the sense of Martineau [11]. From $(n-q-1)$-linear concavity follows $(n-q-1)$-pseudoconcavity in the sense of Andreotti—Grauert [1]. Characteristic examples of $(n-q-1)$-concave domains in CP^n are $CP^n \setminus CP^q$ and the domain, bounded by the quadratic (in homogeneous coordinates):

$$\{z: |z^0|^2 + |z^1|^2 + \ldots + |z^q|^2 - |z^{q+1}|^2 - \ldots - |z^n|^2 > 0\}.$$

A basic analytic fact relating to $(n-q-1)$-pseudoconcave domains is that the space $H^s(D, \Omega^p)$ (the s-dimensional cohomology with coefficients in the sheaf of holomorphic p-forms) is finite-dimensional for $0 \leqslant s < q$ (Andreotti—Grauert [1]), and, in general, infinite-dimensional for $s = q$ (Andreotti—Norguet [2]). From the point of view of complex analysis, the space $H^q(D, \Omega^p)$ plays the same role in a precise sense for $(n-q-1)$-pseudoconcave domains that the space of holomorphic functions does for pseudoconvex ones. By Dolbeault's theorem [7], $H^q(D, \Omega^p)$ can be realized as the quotient-space of the $\bar{\partial}$-closed forms of type (p, q) by the $\bar{\partial}$-exact ones. However, for many problems of complex analysis, it is important to canonically connect $H^q(D, \Omega^p)$ with some spaces of holomorphic forms (not only smooth ones as in Dolbeault's theorem).

This paper is devoted to the proof of the fact that for an $(n-q-1)$-linearly concave domain D, there exists a holomorphic bundle E over D, such that all elements of $H^q(D, \Omega^n)$ can be obtained by restricting some holomorphic closed forms in E to any fixed section of E.

The inspirations for our construction were the results of Leray [8] and Martineau [11] on the reconstruction of analytic functionals by means of their Fantapié indicatrices (see Sec. 1 here).

The results of the present paper also develop the papers of Andreotti and Norguet [2, 3], where the map ρ_0 of the space $H^q(D, \Omega^q)$ into the space of holomorphic functions on the set of compact holomorphic cycles of D (obtained by integration over these cycles of differential forms from $H^q(D, \Omega^q)$ is studied. Andreotti and Norguet proved, in particular, that for $(n-q-1)$-concave domains, the image and kernel of the map ρ_0, in general, are infinite-dimensional. To get more precise results, it is important to describe the kernel and image of the map ρ_0.

In the present paper these problems are solved for the spaces $H^q(D, \Omega^p)$ in $(n-q-1)$-linearly concave domains D under the condition that $p = n$. The case of any $p > q$ might be considered analogously. It turns out that under certain natural assumptions the map ρ_0 does not have a kernel, and the image is explicitly described by the system of differential equations indicated.

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The advantage of $(n-q-1)$-linearly convex domains is that in this case one can take in D not all compact q-dimensional analytic submanifolds, but only q-dimensional analytic planes. Then the proof of the absence of a kernel can be carried out with the aid of the technique of integral representations [7], and to describe the image, an analogy with the problems of integral geometry, considered by Gel'fand, Graev, and Shapiro in [5] turns out to be decisive. The system of differential equations mentioned above, which describes the image, coincides with the system of ultrahyperbolic differential equations appearing in their papers.
hyperbolic differential equations appearing in their papers.

It is interesting to compare the results of the present paper for the space $H^1(D, \Omega^3)$ with the domain $D = \{z \in CP^3\colon\ |z^0|^2 + |z^1|^2 - |z^2|^2 - |z^3|^2 > 0\}$ with the recent results of Penrose [13] and Lerner [9] (see also [15]) on realizing solutions of Maxwell's equations in the form of elements of the space $H^1(D, \Omega^3)$.

We conclude with a more detailed survey of the contents of the paper. Let D be an $(n-q-1)$-linearly concave domain in CP^n, D^* be the set of q-dimensional analytic planes $\zeta \subset D$, $F(D^*)$ be the set of flags (ζ, ζ'), where $\zeta \in D^*$ and $\zeta' \subset \zeta$ is an analytic plane of dimension $q-1$. If φ is an (n, q)-form on D, then its Radon transform $\mathcal{R}$ is its integral over $\zeta \in D^*$. The result of integrating can be understood in two ways: either as an $(n-q)$-form on D^* or as a section of a one-dimensional bundle over $F(D^*)$. From the $\bar\partial$-closedness of φ it follows that $\mathcal{R}\varphi$ is holomorphic. Here we use the second interpretation of $\mathcal{R}\varphi$. As bundle E over D, one takes the bundle $\{(z, \zeta), z \in D, \zeta \in D^*, z \in \zeta\}$ or the larger bundle of pairs (z, v), where v is a collection of points $v_1, \ldots, v_q$ such that the plane $\{z, v_1, \ldots, v_q\}$ lies in D. By analogy with [5] one constructs a differential operator $\mathcal{R}\varphi \mapsto \varkappa\mathcal{R}\varphi$ in the space of holomorphic closed $(n+q)$-forms on E. One proves that if the fiber of the bundle is contractible, then the restriction of $\varkappa\mathcal{R}\varphi$ to any section of E over D is cohomologous with φ. It is characteristic that in contrast to [5], in the construction of φ from $\varkappa R\varphi$, integration of the form is not involved but rather its restriction.

Sections 1 and 2 of the paper are devoted to the case $q = n-1$. Here Sec. 1 is essentially a recounting of the results of Martineau [11, 12] with some additions. In Sec. 3 the Radon transformation is constructed and an inversion formula for (n, q)-forms is proved. In Secs. 4 and 5 it is proved that under some additional restrictions the kernel of the Radon transformation consists of $\bar\partial$-exact forms (i.e., on cohomology classes there is no kernel).

1. Radon Transformation of $(n, n-1)$-Forms in Linearly Concave Domains in CP^n and the Fantapié Indicatrix of Analytic Functionals

Let D be a linearly concave domain in CP^n, i.e., for each point $z \in D$ there exists an $(n-1)$-dimensional analytic plane ζ_z with the properties $z \in \zeta_z \subset D$. Then the compactum $\Omega = CP^n \setminus D$ is linearly convex in the sense of Martineau [11]. The set of $(n-1)$-dimensional planes ζ, contained in D corresponds to a domain D^* in the Grassman manifold $G_{n+1,n} \simeq (CP^n)^*$.

We denote by $H(CP^n \setminus D)$ the space of holomorphic functions of a variable $z \in CP^n \setminus D$ and by $H^*(CP^n \setminus D)$ the space of linear functionals on $H(CP^n \setminus D)$.

Let η be some hyperplane in D. We consider the space $H^{n-1}(D \setminus \eta, \Omega^n)$ (the $(n-1)$-dimensional cohomology of the domain $D \setminus \eta$ with coefficients in the sheaf Ω^n of holomorphic n-forms).

With each closed $(n, n-1)$-form φ which by virtue of Dolbeault's theorem represents some cohomology class in $H^{n-1}(D \setminus \eta, \Omega^n)$ one can associate a functional $\varphi^* \in H^*(CP^n \setminus D)$ by means of the formula

$$(\varphi^*, h) = \int_{\partial\Omega_\eta} \varphi \cdot h, \tag{1.1}$$

where $h \in H(CP^n \setminus D)$, Ω_η is a neighborhood of the hyperplane η, which is relatively compact in D, such that the compactum $\partial\Omega_\eta$ lies in the domain of holomorphy of the function h.

This definition is proper since, firstly, by virtue of Stokes' formula the integral of (1.1) is independent of the choice of neighborhood Ω_η, and secondly, for a $\bar\partial$-exact form φ, the integral of (1.1) is equal to zero (by virtue of the formula for integration by parts). One of the results of Martineau [11] can be formulated in the following way.

Proposition 1.1 (Martineau). For any linearly concave domain D and any hyperplane $\eta \subset D$, (1.1) establishes an isomorphism between the spaces $H^{n-1}(D \setminus \eta, \Omega^n)$ and $H^*(\mathbf{C}P^n \setminus D)$.

Along with Proposition 1.1, one also has

Proposition 1.2. For any linearly concave domain D, the map $\rho: H^{n-1}(D, \Omega^n) \to H^*(\mathbf{C}P^n \setminus D)$, defined by (1.1), has zero kernel, and the image consists of those functionals $\varphi^* \in H^*(\mathbf{C}P^n \setminus D)$ for which $(\varphi^*, 1) = 0$.

In the domains D and D*, we introduce homogeneous coordinates $Z = (z^0, \ldots, z^n)$ and $\xi = (\xi_0, \ldots, \xi_n)$, respectively. Then

$$z = z(Z) \in \mathbf{C}P^n \text{ и } \zeta = \zeta(\xi) = \{z\colon \langle \xi, Z\rangle = 0\}, \text{where} \langle \xi, Z\rangle = \sum_{k=0}^{n} \xi_k z^k.$$

Definition. Let the functional $\varphi^* \in H^*(\mathbf{C}P^n \setminus D)$. By the Fantapié indicatrix of the functional φ^* we mean the functional of the form

$$\mathcal{F}\varphi^*(\xi, \eta) = \left(\varphi^*, \frac{\langle \eta, Z\rangle}{\langle \xi, Z\rangle}\right),$$

where $\zeta(\xi) \subset D$.

Directly from the definition follow the relations

$$\mathcal{F}\varphi^*(\lambda\xi, \mu\eta) = \frac{\mu}{\lambda}\mathcal{F}\varphi^*(\xi, \eta), \quad \mathcal{F}\varphi^*(\xi, \xi) = (\varphi^*, 1), \tag{1.2}$$

where $\lambda, \mu \in \mathbf{C}$.

If φ is a closed (n, n − 1)-form in D, then for fixed ξ and η the form $\frac{\varphi\langle \eta, Z\rangle}{\langle \xi, Z\rangle}$ is closed in the domain $D \setminus \zeta(\xi)$ and has a first-order pole on the hyperplane $\zeta(\xi)$. We denote Leray's residue-form [8] for this form on $\zeta(\xi)$ by $\langle \xi, dZ\rangle \lrcorner \varphi \cdot \langle \eta, Z\rangle$ (see also (1.4)). In this definition, we use the sign $\lrcorner$ since the ideal of the residue-form corresponds well with the ideal of the interior product of a form and a vector field.

Definition. Let φ be a smooth (n, n − 1)-form in D. By the Radon transformation of the form φ we mean the function of the form

$$\mathcal{R}\varphi(\xi, \eta) = \int_{z \in \zeta(\xi)} \langle \xi, dZ\rangle \lrcorner \varphi \cdot \langle \eta, Z\rangle,$$

where $\zeta(\xi) \subset D$, $dZ = (dz^0, dz^1, \ldots, dz^n)$.

Directly from the definition follow the relations

$$\mathcal{R}\varphi(\lambda\xi, \mu\eta) = \frac{\mu}{\lambda}\mathcal{R}\varphi(\xi, \eta), \quad \mathcal{R}\varphi(\xi, \xi) = 0. \tag{1.3}$$

It follows from (1.3) that $\mathcal{R}\varphi$ is a section of some one-dimensional bundle over the set of flags (ζ, ζ'), where $\zeta \subset D$ and ζ' is an (n − 2)-dimensional analytic plane in ζ.

Proposition 1.3. Let φ be a smooth, $\bar\partial$-closed (n, n − 1)-form on D, and φ^* be the functional in $H^*(\mathbf{C}P^n \setminus D)$ which corresponds to it by virtue of (1.1). Then for any ξ, η such that $\zeta(\xi) \subset D$ the Radon transformation $\mathcal{R}\varphi$ and the Fantapié indicatrix $\mathcal{F}\varphi^*$ are connected by the relation

$$\mathcal{R}\varphi(\xi, \eta) = \frac{1}{2\pi i}\mathcal{F}\varphi^*(\xi, \eta).$$

COROLLARY. For a $\bar\partial$-closed (n, n − 1)-form φ, the Radon transformation $\mathcal{R}\varphi$ depends holomorphically on ξ, η. For a $\bar\partial$-exact form φ the Radon transformation is equal to zero.

Proof. If $\xi \equiv \eta$, then by virtue of Proposition 1.2 and (1.2), (1.3) we have

$$\mathcal{R}\varphi(\xi, \xi) = \frac{1}{2\pi i}\mathcal{F}\varphi^*(\xi, \xi) = \frac{1}{2\pi i}(\varphi^*, 1) = 0.$$

Now let $\xi \neq \eta$ and let Ω_η be some neighborhood of the hyperplane η, relatively compact in D. Without loss of generality, one can assume that the hyperplane $\{z\colon \langle \eta, Z\rangle = 0\}$ has the form $z^0 = 0$, and the point $\{z^1 = \ldots = z^n = 0\} \in \mathbf{C}P^n \setminus \bar{D}$. Then we have the equation

$$\mathcal{R}\varphi(\xi,\eta) = \int\limits_{z\in\zeta(\xi)} \langle \xi, dZ\rangle \,\lrcorner\, \varphi(z)\cdot z^0 = \int\limits_{z\in\zeta(\xi)} \left\langle \xi, d\frac{Z}{z^0}\right\rangle \lrcorner\, \varphi(z).$$

Now using Cauchy's formula, we get (by virtue of the definition of the interior and exterior products)

$$\mathcal{R}\varphi(\xi,\eta) = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\{z:\ |\langle\xi, Z/z^0\rangle| = \varepsilon|\xi_0|\}} \left\langle \xi, d\frac{Z}{z^0}\right\rangle \lrcorner\, \varphi(z)\wedge\left\langle \xi, d\frac{Z}{z^0}\right\rangle \Big/ \left\langle \xi, \frac{Z}{z^0}\right\rangle =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\{z:\ \langle\xi, Z/z^0\rangle = \varepsilon|\xi_0|\}} \varphi(z)\Big/\left\langle \xi, \frac{Z}{z^0}\right\rangle. \tag{1.4}$$

We note now that for any sufficiently small $\varepsilon > 0$, the form $\varphi\Big/\left\langle\frac{\xi}{\xi_0}, \frac{Z}{z^0}\right\rangle$ is closed and has no singularities in the domain $\Omega_\eta\setminus\left\{z: \left|\left\langle\frac{\xi}{\xi_0}, \frac{Z}{z^0}\right\rangle\right| \leqslant \varepsilon\right\}$. Hence, Stokes' formula applies to the form $\varphi\Big/\left\langle\frac{\xi}{\xi_0}, \frac{Z}{z^0}\right\rangle$ in this domain, by virtue of which we have

$$\frac{1}{2\pi i}\int\limits_{\left\{z:\left\langle\frac{\xi}{\xi_0}, \frac{Z}{z^0}\right\rangle = \varepsilon\right\}} \varphi(z)\Big/\left\langle \xi, \frac{Z}{z^0}\right\rangle = \frac{1}{2\pi i}\int\limits_{z\in\partial\Omega_\eta} \varphi(z)\Big/\left\langle \xi, \frac{Z}{z^0}\right\rangle.$$

Whence and from (1.4), (1.1) follows the equation

$$\mathcal{R}\varphi(\xi,\eta) = \frac{1}{2\pi i}\int\limits_{z\in\partial\Omega_\eta} \varphi(z)\cdot z^0/\langle\xi, Z\rangle = \frac{1}{2\pi i}\mathcal{F}\varphi^*(\xi,\eta).$$

Proposition 1.3 is proved.

We denote by $H(D^*)$ the space of holomorphic sections ψ of the one-dimensional bundle over D^* of the form $\psi(\lambda\xi) = \frac{\psi(\xi)}{\lambda}$, where $\zeta(\xi)\in D^*$, $\lambda\in\mathbf{C}$. Further, we fix $\eta\in D^*$ and we denote by $H_\eta(D^*)$ the subspace of those sections ψ in $H(D^*)$ for which $\psi(\eta) = 0$. Let the homogeneous coordinates in the domain D be such that $\{z^1 = z^2 = \ldots = z^n = 0\}\in\mathbf{C}P^n\setminus\bar{D}$, and the hyperplane η has the form $\eta = \{z: z^0 = 0\}$.

Using these coordinates, for each function $\psi\in H_\eta(D^*)$ on the square $\{(z,\zeta)\in[D\setminus\eta]\times D^*:\ \langle\xi, Z\rangle = 0\}$ we define, following Leray [8] and Martineau [12], a holomorphic and closed $(2n-1)$-form of the form

$$L\psi(\zeta, z) = \frac{(-1)^n}{(2\pi i)^{n-1}}\frac{\partial^{n-1}\psi(\xi)}{\partial\xi_0^{n-1}}\,\omega'(\xi)\wedge\omega\left(\frac{Z}{z^0}\right),$$

where $\omega'(\xi) = \sum\limits_{k=1}^{n}(-1)^k\xi_k\bigwedge\limits_{j\neq k} d\xi_j$, $\omega(Z) = \bigwedge\limits_{j=1}^{n} dz^j$.

<u>THEOREM I.</u> Let the linearly concave domain $D\subset\mathbf{C}P^n$ be such that any hyperplane section of the domain $D^*\subset(\mathbf{C}P^n)^*$ is contractible. Then:

A) for each function $\psi\in H(D^*)$ and any smooth map $z\mapsto\zeta_z$, $z\in D\setminus\eta$, $\zeta_z\in D^*$, $z\in\zeta_z$, the restriction $L\psi|_\gamma$ of the form $L\psi$ to the graph γ of the map $z\mapsto\zeta_z$ is an $(n, n-1)$-form in $D\setminus\eta$, such that for the functional $L\psi|_\gamma^*$ in $H^*(\mathbf{C}P^n\setminus D)$ corresponding to it one has

$$\frac{1}{2\pi i}\mathcal{F}(L\psi|_\gamma^*)(\xi,\eta) = \psi(\xi),\ \text{where}\ \zeta(\xi)\in D^*.$$

B) If here $\psi(\eta) = 0$, then the form $L\psi|_\gamma$ lies in the image of the map

$$\rho:\ H^{n-1}(D,\Omega^n)\to H^*(\mathbf{C}P^n\setminus D)\simeq H^{n-1}(D\setminus\eta,\ \Omega^n)$$

and for any $\zeta(\xi)\in D^*$ one has

$$\mathcal{R}(\rho^{-1}L\psi|_\gamma)(\xi,\eta) = \psi(\xi).$$

<u>Remark.</u> Assertion A) in this theorem for the case when $\psi(\xi) = 1/\xi_0$, is equivalent with the integral formula of Cauchy–Fantapié–Leray [8].

Proof. Let Ω_η be a neighborhood of the hyperplane $\eta \subset D$, such that the compactum $CP^n \setminus \Omega_\eta$ is convex. With each point $z \in \partial\Omega_\eta$ we associate the complex hyperplane $\tilde{\zeta}_z$, tangent to $\partial\Omega_\eta$ passing through the point z. By virtue of the convexity of the compactum $CP^n \setminus \Omega_\eta$ we have: $\tilde{\zeta}_z \subset D$ for all $z \in \partial\Omega_\eta$. Since by hypothesis the set of hyperplanes $\{\zeta_z\}$, passing through the point $z \in D$ and lying in D, is contractible, the graphs γ and $\tilde{\gamma}$ of the maps $z \mapsto \zeta_z$ and $z \mapsto \tilde{\zeta}_z$, where $z \in \partial\Omega_\eta$, are homologous. Whence, and from the closedness on the square $\{\langle \xi, Z\rangle = 0\}$ of the form $L\psi$, it follows that $L\psi|_\gamma$ and $L\psi|_{\tilde{\gamma}}$ are cohomologous (n, n − 1)-forms on $\partial\Omega_\eta$. Hence for the Fantapié indicatrix of the functional $L\psi|_\gamma^*$, corresponding by (1.1) to the form $L\psi|_\gamma$, we have

$$\mathcal{F} L\psi|_\gamma^* = \int_{\partial\Omega_\eta} L\psi|_\gamma \cdot \frac{z^0}{\langle \xi, Z\rangle} = \int_{\partial\Omega_\eta} L\psi|_{\tilde{\gamma}} \cdot \frac{z^0}{\langle \xi, Z\rangle}. \tag{1.5}$$

By Martineau's theorem [12] describing the analytic functionals over convex compacta in terms of Fantapié indicatrices, we have

$$\int_{z \in \partial\Omega_\eta} L\psi|_{\tilde{\gamma}}(z) \frac{z^0}{\langle \xi, Z\rangle} = \psi(\xi) \cdot 2\pi i \tag{1.6}$$

for all those ξ, such that $\zeta(\xi) \subset \Omega_\eta$. This same equation is valid for all $\zeta(\xi) \subset D$, since the function $\psi(\xi)$ is holomorphic and the domain D^*, by virtue of the hypotheses of the proposition is connected. Theorem I A) is proved.

Now if $\psi(\eta) = 0$, then from (1.6) we have

$$\int_{\partial\Omega_\eta} L\psi|_{\tilde{\gamma}} \cdot 1 = \int_{\partial\Omega_\eta} L\psi|_{\tilde{\gamma}} \frac{z^0}{z^0} = \psi(\eta) \cdot 2\pi i = 0.$$

By virtue of Proposition 1.2, this equation means that the form defined in the domain $D \setminus \eta$, lies in the image of the map

$$\rho\colon H^{n-1}(D, \Omega^n) \to H^*(CP^n \setminus D) \simeq H^{n-1}(D \setminus \eta, \Omega^n).$$

Now by virtue of Proposition 1.3 and (1.5) and (1.6), for all $\zeta(\xi) \subset \Omega_\eta$ we get the equation

$$\mathcal{R}(\rho^{-1} L\psi|_\gamma)(\xi, \eta) = \psi(\xi).$$

The validity of this equation for all $\zeta(\xi) \subset D$ follows from the holomorphy of the function $\psi(\xi)$ and the connectedness of the domain D^*. Theorem I is completely proved.

As the first application of Theorem I we have the following refinement of a theorem of Martineau [11].

COROLLARY. Let D be a linearly concave domain in CP^n, such that any hyperplane section of the domain $D^* \subset (CP^n)^*$ is contractible. Then for any fixed $\eta \in D^*$ the Radon transformation $\varphi \mapsto \mathcal{R}\varphi(\cdot, \eta)$, where $\varphi \in H^{n-1}(D, \Omega^n)$, is an isomorphism of the spaces $H^{n-1}(D, \Omega^n)$ and $H_\eta(D^*)$, and the Fantapié transformation $\varphi^* \mapsto \mathcal{F}\varphi^*(\cdot, \eta)$, where $\varphi^* \in H^*(CP^n \setminus D)$, is an isomorphism of the spaces $H^*(CP^n \setminus D)$ and $H(D^*)$.

Remark. Independently, S. V. Znamenskii obtained a result from which it follows that the assumption of the corollary is not only sufficient but also necessary for the validity of the assertion of this corollary.

We note also that the second assertion of the corollary in the case when the compactum $(CP^n \setminus D)$ is convex, is precisely a theorem of Martineau [11, 12] (see also [4]).

2. Radon Transform of (n, n − 1)-Forms (Change of Parametrization and a Generalized Inversion Formula)

As in Sec. 1, it will be assumed that inhomogeneous coordinates are introduced in CP^n: $C_Z^{n+1} \setminus \{0\} \to CP_z^n$. It is convenient for us to rewrite the formula defining the plane ζ not in terms of elements of the conjugate space C_ξ^{n+1}, but in terms of frames in ζ. To each n-frame $(w, v_1, \ldots, v_{n-1})$ in C^{n+1} we associate the flag $(\zeta(w, v), \zeta'(v))$, where

$$\zeta'(v) = \left\{Z = \sum_{k=1}^{n-1} t^k v_k\right\}, \quad \zeta(w, v) = \left\{Z = t^0 w + \sum_{k=1}^{n-1} t^k v_k\right\}.$$

Definition. Let $\varphi \in H^{n-1}(D, \Omega^n)$ and $\zeta(w, v) \in D^*$. We get

$$\mathcal{R}\varphi(w, v) = \frac{1}{[w, v, u]} \int\limits_{z \in \zeta(w, v)} \frac{[w, v, dZ]}{[Z, v, u]} \lrcorner\, \varphi, \tag{2.1}$$

where u is any vector such that {w, v, u} is a basis in $\mathbf{C}^{n+1}$, $[Z, v, u]$ is the determinant of order n + 1 made up of the columns Z, v_1, . . ., v_{n-1}, u.

LEMMA 2.1 (i). The right-hand side of (2.1) is independent of the choice of u;

(ii) we have

$$\mathcal{R}\varphi(\lambda w, \mu v) = \lambda^{-2} (\det \mu)^{-1} \mathcal{R}^1\varphi(w, v), \quad \lambda \in \mathbf{C},\ \mu \in \mathrm{GL}(n, \mathbf{C}), \tag{2.2}$$

which means that $\mathcal{R}\varphi(w, v)$ drops to a section of a one-dimensional bundle over F(D*);

(iii) the one-dimensional bundle sections $\mathcal{R}\varphi(\xi, \eta)$ and $\mathcal{R}\varphi(w, v)$ are compatible.

Proof. (2.2) is verified directly. The compatibility of the sections in (iii) means that if $\zeta(w, v) = \zeta(\xi)$, $\zeta'(v) = \zeta'(\xi, \eta)$, then $\mathcal{R}\varphi(\xi, \eta)$ and $\mathcal{R}\varphi(w, v)$ differ by a factor independent of φ. We find $\xi(w, v)$ and $\eta_u(v)$ from the conditions $\langle \xi, Z\rangle = [w, v, Z]$, $\langle \eta_u, Z\rangle = [Z, v, u]$. Then $\zeta(w, v) = \zeta(\xi)$, $\zeta'(v) = \zeta'(\xi, \eta_u)$ and $\mathcal{R}\varphi(w, v) = [w, v, u]\mathcal{R}\varphi(\xi, \eta_u)$. Moreover, (2.1) is not changed upon adding to u combinations of w and v (we use the fact that $\mathcal{R}\varphi(\eta, \eta) = 0$) which means (i) is proved. We note that thanks to the factor 1/[w, v, u] in the parametrization (w, v) there is no analog of the condition $\mathcal{R}\varphi(\eta, \eta) = 0$.

Remark. For given w and v one can choose a basis in $\mathbf{C}^{n+1}$ in which (2.1) appears particularly simple. Thus, one can assume that $w^0 = 1$, and $v_i = \delta_{i+1}$, $1 \leqslant i \leqslant n-1$, where δ_{i+1} is the vector whose (i + 1)-st coordinate is equal to 1, and the others to zero. Then by virtue of (2.2) one can restrict oneself to the section $z^0 = 1$ (passing to inhomogeneous coordinates), and the plane $\zeta(w, \delta)$, $\delta = (\delta_2, \ldots, \delta_n)$, defined by the equation $z^1 = w^1$. Setting $u = \delta_1$, we get

$$\mathcal{R}\varphi(w, \delta) = \int\limits_{\substack{z^0=1 \\ z^1=w^1}} dz^1 \lrcorner\, \varphi. \tag{2.3}$$

Proposition 2.2. The function $\mathcal{R}\varphi(w, v)$ satisfies the system of differential equations

$$\frac{\partial^2 \mathcal{R}\varphi}{\partial w^j \partial v_i^k} - \frac{\partial^2 \mathcal{R}\varphi}{\partial w^k \partial v_i^j} = 0, \qquad 1 \leqslant i \leqslant n-1,\ 0 \leqslant j,\ k \leqslant n. \tag{2.4}$$

Proof. One can assume that $i = 1$, $j = 1$, $k = 2$, $v_r = \delta_{r+1}$ for r > 1, $w^0 = 1$, $w^r = 0$ for $r > 2$, $v_1^0 = 0$, $v_1^1 = v^1$, $v_1^2 = v^2$, $v_1^r = 0$ for r > 2. Then on the section $z^0 = 1$, the hyperplane $\zeta(w, v)$ has the form $z^1 = w^1 + tv^1$, $z^2 = w^2 + tv^2$, $t \in \mathbf{C}_v$, and one can assume that

$$\varphi = (\Phi_1(z) d\bar z^1 \wedge d\bar z^3 \wedge \ldots \wedge d\bar z^n + \Phi_2(z) d\bar z^2 \wedge d\bar z^3 \wedge \ldots \wedge d\bar z^n) \wedge dz^1 \wedge \ldots \wedge dz^n.$$

Then, passing in (2.1) from $(z^1, z^2, z^3, \ldots, z^n)$ to $(w^1, w^2, t, z^3, \ldots, z^n)$, we get that $\mathcal{R}\varphi(w, v)$ is the sum of two summands of the form

$$\int \Phi_i(w^1 + tv^1, w^2 + tv^2, z^3, \ldots, z^n)\, \bar v^i\, dt \wedge d\bar t \wedge dz^3 \wedge \ldots \wedge dz^n \wedge d\bar z^3 \wedge \ldots \wedge d\bar z^n,$$

whence one verifies directly that each of the summands is annihilated by the operator $\frac{\partial^2}{\partial w^1 \partial v^2} - \frac{\partial^2}{\partial w^2 \partial v^1}$.

Let $\psi(w, v)$ be a holomorphic function on the manifold of frames (w, v). By analogy with the form $L\psi$ from Sec. 1 and [5], we introduce the holomorphic (2n − 1)-form

$$\varkappa\psi = \frac{(-1)^n}{(2\pi i)^n} \sum_{s_1, \ldots, s_{n-1}} \frac{\partial^{n-1}\psi(Z, v)}{\partial z^{s_1} \ldots \partial z^{s_{n-1}}}\, dv_1^{s_1} \wedge \ldots \wedge dv_{n-1}^{s_{n-1}} \wedge \omega^*(Z),$$

where the summation is over all collections $0 \leqslant s_j \leqslant n$, and $\omega^*(Z) = \sum_{k=0}^{n} (-1)^k z^k \bigwedge_{j \neq k} dz^j$.

LEMMA 2.3. Let $\psi(w, v)$ be defined for all (w, v), for which $\zeta(w, v) \in D^*$, and let ψ satisfy (ii) of Lemma 2.1 and (2.4). Then $\varkappa\psi$ drops to D in Z, and in v to the corresponding set of $(n-2)$-dimensional analytic planes $\zeta'(v)$. Moreover, the form $\varkappa\psi$ is closed; in particular, $\varkappa\mathscr{R}\varphi$ is a closed form.

Proof. That it is closed in Z follows from the fact that $\varkappa\psi$ drops to D in it and has maximal degree there. That it is closed in v is a direct consequence of (2.4).

Proposition 2.4. Let the holomorphic function $\psi(w, v)$ for $\zeta(w, v) \in D^*$ satisfy (ii), (iii) of Lemma 2.1 and (2.4). Let, further, $z \mapsto v(z) = (v_1(z), \ldots, v_{n-1}(z))$ be a smooth function on D such that $\zeta(z, v(z)) \in D^*$. Then $\varkappa\psi|_{v=v(z)}$ (the restriction of $\varkappa\psi$ to the graph of the map $z \mapsto v(z)$) be a $\bar\partial$-closed $(n, n-1)$-form on D and

$$\mathscr{R}(\varkappa\psi|_{v=v(z)}) = \psi(w, v), \qquad \zeta(w, v) \in D^*. \tag{2.5}$$

In other words, the Radon transformation of the form $\varkappa\psi|_{v=v(z)}$ coincides with ψ. In particular, $\mathscr{R}(\varkappa\mathscr{R}\varphi|_{v=v(z)}) = \mathscr{R}\varphi$.

Proof. Since the bundle $(Z, v) \mapsto (Z, \zeta(Z, v))$ has contractible fiber, we get (using Michael's theorem [10]), that the functions $z \mapsto v(z)$ on D exist, and it suffices to verify (2.5) for one section v(z). Using Proposition 1.2, one can fix $\eta \in D^*$ and consider v(z) only on $D \setminus \eta$, provided only that $\varkappa\psi|_{v=v(z)}$ lies in the image of the map ρ_η: $H^{n-1}(D, \Omega^n) \to H^{n-1}(D \setminus \eta, \Omega^n)$. Let $\zeta(1, 0, \ldots, 0) \in D^*$, i.e., the hyperplane $\eta = \{z^0 = 0\}$ be contained in D. We introduce inhomogeneous coordinates $z^0 = 1$. Let $\xi \in \mathbf{C}^{n+1} \setminus \{0\}$. We set

$$\begin{aligned} v_1(\xi) &= (0, \xi_n, 0, \ldots, 0, -\xi_1), \\ v_2(\xi) &= (0, 0, \xi_n, \ldots, 0, -\xi_2), \\ &\ldots\ldots\ldots\ldots\ldots \\ v_{n-1}(\xi) &= (0, 0, 0, \ldots, \xi_n, -\xi_{n-1}). \end{aligned}$$

It is clear that $\zeta(Z, v(\xi)) = \zeta(\xi)$, if $\langle \xi, Z\rangle = 0$, and $\zeta'(v(\xi))$ is distinguished in ζ by the condition $z^0 = 0$. For $\zeta(\xi) \in D^*$, $\langle \xi, z\rangle = 0$ we set $\tilde\psi(Z, \xi) = \psi(Z, v(\xi))$. Then

$$L\tilde\psi(Z, \xi) = \varkappa\psi(Z, v(\xi)).$$

To prove this equation, it is convenient, using homogeneity, to assume that $\xi_n = 1$. Then in $\varkappa\psi$ (taking account of the special form of $v(\xi)$) there remains only one summand

$$c\frac{\partial^{n-1}}{(\partial z^n)^{n-1}}\psi(Z, v)\, dv_1^n \wedge \ldots \wedge dv_{n-1}^n \wedge dz^1 \wedge \ldots \wedge dz^n,$$

whence, substituting $v_j^n = -\xi_j$ and considering that here $\dfrac{\partial}{\partial\xi_0} = \dfrac{\partial}{\partial z^n}$ we get that the equation is proved.

As a result, for any section $z \mapsto \xi(z)$ over $D \setminus \eta$, from Theorem 1 we can construct a map $z \mapsto (Z, v(\xi(z)))$ for which (2.5) is satisfied. The proof is concluded.

Remark. From Proposition 2.4 and its proof, it is evident that it admits a significantly wider choice of sections over D, and by the same token, a wider class of inversion formulas for the Radon transformation than in Theorem I. The inversion formulas of Theorem I correspond to the cases when in Proposition 2.4, v(z) is chosen so that $\zeta'(v(z))$ lies in a fixed hyperplane $\eta \in D^*$, i.e., they are connected with some fixed affinization of projective space.

3. Radon Transformation of (n, q)-Forms in (n − q − 1)-Linearly Concave Domains in CP^n

Let D be an $(n-q-1)$-linearly concave domain in $\mathbf{C}P^n$; $D^* \subset G_{n+1, q+1}$ be the set of q-dimensional analytic planes contained in D; $F(D^*)$ be the set of flags (ζ, ζ'), $\zeta \in D^*$, $\zeta' \subset \zeta$ being a $(q-1)$-dimensional analytic plane. We introduce homogeneous coordinates in CP^n and just as for $q = n-1$, we consider two parametrizations of the space of flags, by means of frames in the original or conjugate space:

a) $\zeta(w, v_1, \ldots, v_q) = \zeta(w, v) = \{Z = t^0 w + t^1 v_1 + \ldots + t^q v_q\}$, $\zeta'(v) = \{Z = t^1 v_1 + \ldots + t^q v_q\}$;

b) $\zeta(\xi_1, \ldots, \xi_{n-q}) = \zeta(\xi) = \{Z\colon \langle \xi_1, Z\rangle = 0, \ldots, \langle \xi_{n-q}, Z\rangle = 0\}$, $\zeta'(\xi, \eta) = \{Z\colon z \in \zeta(\xi), \langle \eta, Z\rangle = 0\}$.

For given flags of $F(D^*)$ it is convenient to require in addition that any $n-q$ of the hyperplanes $\langle \xi_1, Z\rangle = 0, \ldots, \langle \xi_{n-q}, Z\rangle = 0$, $\langle \eta, Z\rangle = 0$ should intersect in an element of D^* (i.e., in a q-dimensional plane lying in D).

Accordingly we give in two variants formulas for the Radon transformation.

Definition. Let φ be a $\bar\partial$-closed (n, q)-form on D; we set

$$\mathscr{R}\varphi(\xi_1,\ldots,\xi_{n-q};\eta)=\int\limits_{z\in\zeta(\xi)}\langle\xi_1,dZ\rangle\wedge\ldots\wedge\langle\xi_{n-q},dZ\rangle\rfloor\varphi\cdot\langle\eta,Z\rangle^{n-q}, \tag{3.1}$$

if $\zeta(\xi)\in D^*$, and

$$\mathscr{R}\varphi(w,v_1,\ldots,v_q)=\frac{1}{[w,v,u]^{n-q}}\int\limits_{z\in\zeta(w,v)}[w,v,dZ]\rfloor\varphi\cdot[Z,v,u]^{n-q}, \tag{3.2}$$

where $\zeta(w,v)\in D^*$, $u=(u_1,\ldots,u_{n-q})$ is an arbitrary frame of n − q vectors supplementary to the frame w, v, $[w,v,dZ]=[w,v_1,\ldots,v_q,dZ,\ldots,dZ]$ is the determinant of the matrix in which the column dZ is repeated n − q times, while in the formation of the determinant the exterior differentials are multiplied.

LEMMA 3.1. (i) (3.2) is independent of the choice of u;

(ii) $\mathscr{R}\varphi(\mu\xi,\lambda\eta)=(\det\mu)^{-1}\lambda^{n-q}\mathscr{R}\varphi(\xi,\eta)$, where $\mu\in GL(n-q;\mathbf{C})$, $\lambda\in\mathbf{C}$;

(iii) $\mathscr{R}\varphi(\lambda w,\nu v)=\lambda^{q-n-1}(\det\nu)^{-1}\mathscr{R}\varphi(w,v)$, where $\lambda\in\mathbf{C}$, $\nu\in GL(q,\mathbf{C})$.

Thus, in both variants of the definition, $\mathscr{R}\varphi$ drops to a section of a one-dimensional bundle over F(D*).

(iv) The bundle sections $\mathscr{R}\varphi(\xi,\eta)$ and $\mathscr{R}\varphi(w,v)$ over F(D*) are compatible, i.e., $\mathscr{R}\varphi(\xi,\eta)=c(\xi,\eta,w,v)\,\mathscr{R}\varphi(w,v)$, where c is independent of φ, if $\zeta(\xi)=\zeta(w,v)$, $\zeta'(\xi,\eta)=\zeta'(v)$.

The proof does not differ from the proof of Lemma 2.1, excluding (iv). To prove (iv) we can, for fixed flag (ζ,ζ'), choose appropriate bases in $\mathbf{C}^{n+1}$, $(\mathbf{C}^{n+1})^*$ and frames, without changing (ζ,ζ'). If one assumes $\langle\eta,Z\rangle=z^0$, $\langle\xi_j,Z\rangle=z^j$, i.e., that ζ on $\{z_0=1\}$ has the form $\{z^{q+1}=\ldots=z^n=0\}$, then

$$\mathscr{R}\varphi(\xi,\eta)=\int dz^{q+1}\wedge\ldots\wedge dz^n\rfloor\varphi.$$

Analogously, if $v_i=\delta_i, u_i=\delta_{q+i}$, $w^0=v^0=1$, then $[Z,v,u]=[w,v,u]=1$, $[w,v,dZ]=c\,dz^{q+1}\wedge\ldots\wedge dz^n$ and $\mathscr{R}\varphi(\xi,\eta)=\mathscr{R}\varphi(w,v)$.

We note that for $\xi_1=\ldots=\xi_{n-1}=\eta$, the function $\mathscr{R}\varphi(\xi,\eta)$ has a zero of order n − q; in (3.2), this zero is extinguished by the factor 1/[w, v, u].

We denote by $\varphi_{(\xi_2,\ldots,\xi_{n-q};\eta)}$ the restriction of the form $\langle\xi_2,dZ\rangle\wedge\ldots\wedge\langle\xi_{n-q},dZ\rangle\rfloor\varphi\langle\eta,Z\rangle^{n-q-1}$ to the subspace $\{Z:\langle\xi_2,Z\rangle=0,\ldots,\langle\xi_{n-q},Z\rangle=0\}$. One verifies directly that this operation preserves $\bar\partial$-closedness. Comparing the definitions, we get the following assertion.

LEMMA 3.2. We have

$$\mathscr{R}\varphi(\xi_1,\xi_2,\ldots,\xi_{n-q};\eta)=\mathscr{R}_{q+1}\varphi_{(\xi_2,\ldots,\xi_{n-q};\eta)}(\tilde\xi_1,\tilde\eta), \tag{3.3}$$

where on the right-hand side is the Radon transformation in $\mathbf{C}P^{q+1}$ and $\tilde\xi_1,\tilde\eta$ are the images of ξ_1 and η in the quotient space by $\{\lambda_2\xi_2+\ldots+\lambda_{n-q}\xi_{n-q}\}$.

Lemma 3.2 allows one to reduce a series of questions about the Radon transformation in (n − q − 1)-linearly convex domains for any q to the case q = n − 1 considered in Secs. 1 and 2. In particular, one has

COROLLARY. The functions $\mathscr{R}\varphi(\xi,\eta)$, $\mathscr{R}\varphi(w,v)$ drop to holomorphic sections of one-dimensional bundles over F(D*).

Just as in Sec. 2, one proves

Proposition 3.3. The function $\mathscr{R}\varphi(w,v)$ satisfies the system of differential equations

$$\frac{\partial^2\mathscr{R}\varphi}{\partial w^j\,\partial v_i^k}-\frac{\partial^2\mathscr{R}\varphi}{\partial w^k\,\partial v_i^j}=0,\qquad 1\leqslant i\leqslant q,\ 0\leqslant j,k\leqslant n. \tag{3.4}$$

By analogy with Sec. 2, for a holomorphic function $\psi(w, v)$ one introduces the (n + q)-form

$$\varkappa\psi = \frac{(-1)^q}{(2\pi i)^{q-1}} \sum_{s_1, \ldots, s_q} \frac{\partial^q \psi(Z, v)}{\partial z^{s_1} \ldots \partial z^{s_q}} dv_1^{s_1} \wedge \ldots \wedge dv_q^{s_q} \wedge \omega^*(Z). \tag{3.5}$$

LEMMA 3.4. Let $\psi(Z, v)$ be defined and holomorphic for all (Z, v) for which $\zeta(Z, v) \in D^*$. Let, further, ψ satisfy (iii) of Lemma 3.1 and (3.4). Then the form $\varkappa\psi$ drops to D in Z, and in v to the corresponding set of $(q-1)$-dimensional planes $\zeta'(v)$. Moreover, the form $\varkappa\psi$ is closed. In particular, $\varkappa\mathcal{R}\varphi$ is a closed form.

The proof is no different from the case $q = n - 1$ of Sec. 2.

THEOREM II. Let D be an $(n-q-1)$-linearly concave domain in CP^n, such that for any point $z \in D$ the set of q-dimensional analytic planes in D passing through z is contractible. Let, further, $\psi(w, v)$ be a holomorphic function on the set of those frames (w, v) such that $\zeta(w, v) \in D^*$, satisfying the hypotheses of Lemma 3.4. If $z \mapsto v(z)$ is any smooth map into the set of frames $\{v_1, \ldots, v_q\}$, for which $\zeta(z, v(z)) \in D^*$, then $\varkappa\psi|_{v=v(z)}$, the restriction of $\varkappa\psi$ to the graph of $v(z)$, is a $\bar\partial$-closed form on D, while

$$\mathcal{R}(\varkappa\psi|_{v=v(z)}) = \psi. \tag{3.6}$$

Proof. By virtue of the conditions imposed on the bundle, a section $v(z)$ with the properties necessary exists, and the restrictions of $\varkappa\psi$ to different sections are cohomologous. Thus, it suffices to prove (3.6) for one section $v(z)$. Further, if we want to prove (3.6) for some fixed pair (w, v), one can restrict the section $v(z)$ to any subdomain $D_1 \subset D$, containing the plane $\zeta(w, v)$. The section over D_1 can be deformed in its own right, providing only that $\zeta(z, v(z)) \in D^*$, without worrying about whether the section obtained over D_1 extends to a section over all of D. Thus, let the pair (w, v) be such that $\zeta(w, v) \in D^*$. Without loss of generality, one can assume that $w^0 = 1$, $v_i^0 = 0$, $1 \leqslant i \leqslant q$, $v_i^j = 0$, $1 \leqslant i \leqslant q$, $j > q$. Then, passing to inhomogeneous coordinates $(z^0 = 1)$, it will be assumed that $z = (z', z'')$, $z' = (z^1, \ldots, z^q)$, $z'' = (z^{q+1}, \ldots, z^n)$, i.e., our plane is defined by the conditions $z'' = w''$, $v'' = 0$, $z' \in \zeta(w', v')$. We include it in the domain $D_1 = D' \times D''$, where D'' is a small neighborhood of w'', and D' is a linearly concave domain in the subspace $C_{z'}^{q+1}$. If φ is an (n, q)-form on D_1, then by Lemma 3.2, its Radon transformation coincides with the Radon transformation of the form $\varphi_{w''} = \omega(z'') \rfloor \varphi$ for fixed $w'' \in D''$, i.e. (we note that $v'' = 0$), for $\zeta(w, v) \in D_1^*$, $\mathcal{R}\varphi(w', v') = \mathcal{R}\varphi_{w''}(w', v')$. One verifies directly that $\varkappa\psi(z, v') = \varkappa_{(z', v')}\psi(z', z'', v')$, where on the right, the operator $\varkappa$ is applied for fixed z''. As a consequence, if we consider over D_1 a section $z \mapsto v(z)$, such that $\zeta(z, v(z)) \in D_1^*$, then the validity of (3.6) will be a consequence of the analogous assertion for $q = n - 1$ (see Proposition 2.4). Whence, as remarked above, it follows that the assertion is proved.

4. A Criterion for $\bar\partial$-Exactness of Closed (n, q)-Forms in the Domain $\{z \in C^n : 1 + |z^1|^2 + \ldots + |z^{n-q-1}|^2 < |z^{n-q}|^2 + \ldots + |z^n|^2 < M^2\}$

In this section we will adapt the arguments of [7] (see in [7] the section devoted to the local solution of J. Kohn's equation) to derive a criterion needed later for the $\bar\partial$-exactness of closed (n, q)-forms in a domain of the form

$$D_q^M = \{w \in C^n : 1 + |w^1|^2 + \ldots + |w^{n-q-1}|^2 < |w^{n-q}|^2 + \ldots + |w^n|^2 < M^2\},$$

where $w = (w^1, w^2, \ldots, w^n)$, $q = 1, 2, \ldots, n-1$, $1 < M < \infty$.

We set $w' = (w^1, \ldots, w^{n-q-1})$, $|w'|^2 = |w^1|^2 + \ldots + |w^{n-q-1}|^2$, $dw' = dw^1 \wedge \ldots \wedge dw^{n-q-1}$; $w'' = (w^{n-q}, \ldots, w^n)$. By $D_{q,z'}^M$ we denote the section of the domain D_q^M of the form

$$D_{q,z'}^M = D_q^M \cap \{w : w' = z'\}.$$

THEOREM III. In order that the (n, q)-form f, which is continuous and closed on $\bar D_q^M$, be $\bar\partial$-exact in the domain D_q^M, it suffices that for all w', the $(q + 1, q)$-form $dz' \rfloor f(z)$ be $\bar\partial$-exact on the section $D_{q,w'}^M$ of the domain D_q^M.

In the formulation of Lemma 4.1, the following notation is used (see [7]):

$$\Phi(w, z) = \sum_{j=1}^n P_j(w, z)(w^j - z^j),$$

$$\Phi_0(w, z) = \sum_{j=1}^n P_j^0(w, z)(w^j - z^j),$$

$$P(w,z)=(P_1,P_2,\dots,P_n)=(\bar w^1,\dots,\bar w^{n-q-1},-\bar z^{n-q},\dots,-\bar z^n),$$
$$P^0(w,z)=(P_1^0,P_2^0,\dots,P_n^0)=(0,\dots,0,\bar w^{n-q},\dots,\bar w^n),$$
$$\sigma_0=\{w\in\partial D_q^M:\ |w''|^2<M^2\},$$
$$\sigma_1=\{w\in\partial D_q^M:\ 1+|w'|^2<|w''|^2\},\quad \sigma_{01}=\partial\sigma_0,\quad \sigma_{10}=\partial\sigma_1.$$

We orient the space C^n with the variable w so that the form $(-i)^n\omega(\bar w)\wedge\omega(w)$ is positive. We equip the manifold ∂D_q^M with the orientation induced by the orientation of the domain D_q^M. W have $\sigma_{10}=-\sigma_{01}$.

LEMMA 4.1. For any $w\in\bar\sigma_1$ and $z\in D_q^M$, we have

$$\Phi(w,z)\neq 0. \tag{4.1}$$

For any $w\in\sigma_{01}$ and $z\in D_q^M$ we have

$$\Phi_0(w,z)\neq 0. \tag{4.2}$$

It is also convenient for later formulations to introduce the following differential forms. In the space C^{3n} of the variables (ξ, w, z) on the analytic surface

$$\left\{(\xi,w,z):\ \sum_{k=1}^n \xi_k(w^k-z^k)=1\right\} \tag{4.3}$$

we consider the holomorphic form

$$\omega'(\xi)\wedge\omega(w)\wedge\omega(z). \tag{4.4}$$

If $\xi=\xi(w,z,\lambda)$ is a function of w, z, and the parameter λ, then in view of the closedness of the form (4.4) on the manifold (4.3), we have the equation

$$d_\lambda\omega'(\xi)+\bar\partial_w\omega'(\xi)+\bar\partial_z\omega'(\xi)=0. \tag{4.5}$$

We represent the form (4.4) in the form

$$\sum_{q=0}^{n-1}\omega_q'(\xi)\wedge\omega(w)\wedge\omega(z), \tag{4.6}$$

where $\omega_q'(\xi)$ is a form of order q with respect to $d\bar z$ and correspondingly of order $(n-q-1)$ with respect to $d\bar w$ and $d\lambda$. From (4.5) and (4.6) follows the following relation:

$$d_\lambda\omega_q'(\xi)+\bar\partial_w\omega_q'(\xi)=-\bar\partial_z\omega_{q-1}'(\xi),\qquad q=1,2,\dots,n-1. \tag{4.7}$$

For any continuous (n, q)-form f on $\bar D_q^M$, by virtue of Lemma 4.2, the following (0, q)-form is well-defined:

$$K_qf(z)=\int_{w\in\sigma_1} f(w)\wedge\omega_q'\left(\frac{P(w,z)}{\Phi(w,z)}\right)+\int_{\substack{w\in\sigma_{10}\\ \lambda\in[0,1]}} f(w)\,\omega_q'\left((1-\lambda)\frac{P(w,z)}{\Phi(w,z)}+\lambda\frac{P^0(w,z)}{\Phi_0(w,z)}\right),$$

where $z\in D_q^M$.

Now we take a step toward the proof of Theorem III.

LEMMA 4.2. Let f be a continuous $\bar\partial$-closed form on $\bar D_q^M$ such that the (q + 1, q)-form $dw'\rfloor f(w)$ is $\bar\partial$-exact on any section $D_{q,z'}^M$ of the domain D_q^M. Then the (0, q)-form K_qf is $\bar\partial$-exact in the domain D_q^M.

Proof. Since for any $z\in D_q^M$ the form $dw'\rfloor f(w)$ is exact on the section $D_{q,z'}^M$, and the function $[\bar z^{n-q}(w^{n-q}-z^{n-q})+\dots+\bar z^n(w^n-z^n)]^{-q-1}$ is holomorphic in the variable $w\in\sigma_1\cap D_{q,z'}^M$, by means of integration by parts over the closed manifold $\sigma_1\cap D_{q,z'}^M$, we get the equation

$$If(z)=\int_{\{w\in\sigma_1:\ w'=z'\}}\frac{dw'\rfloor f(w)}{[\bar z^{n-q}(w^{n-q}-z^{n-q})+\dots+\bar z^n(w^n-z^n)]^{q+1}}=0 \tag{4.8}$$

for any $z\in D_q^M$.

Using Martinelli's formula (see [7]), one verifies directly, on the other hand, that one has the equation

$$If(z) = \lim_{\varepsilon\to 0} \int_{\{w\in\sigma_1:\ |w'-z'|=\varepsilon\}} \frac{dw' \rfloor f(w) \wedge \left\{ \sum_{k=1}^{n-q-1} (-1)^k (\bar{w}^k - \bar{z}^k) \bigwedge_{j\neq k} d\bar{w}^j \right\} \wedge dw'}{\Phi^{q+1} [|w'-z'|^2 + \ldots + |w^{n-q-1} - z^{n-q-1}|^2]^{n-q-1}}. \tag{4.9}$$

Combining (4.8) and (4.9), we have

$$\lim_{\varepsilon\to 0} \int_{\{w\in\sigma_1:\ |w'-z'|=\varepsilon\}} \frac{f(w) \wedge \sum_{k=1}^{n-q-1} (-1)^k (\bar{w}^k - \bar{z}^k) \bigwedge_{j\neq k} d\bar{w}^j}{\Phi^{q+1} |w'-z'|^{2(n-q-1)}} = 0. \tag{4.10}$$

We set, further, $\Phi_1 = |w'-z'|^2$, $P^1 = (\bar{w}^1 - \bar{z}^1, \ldots, \bar{w}^{n-q-1} - \bar{z}^{n-q-1}, 0, \ldots, 0)$. Direct verification shows that (4.10) itself implies

$$I_q f(z) = \lim_{\varepsilon\to 0} \int_{\substack{\{w\in\sigma_1:\ |w'-z'|=\varepsilon\} \\ \lambda\in[0,1]}} f(w) \wedge \omega_q' \left((1-\lambda)\frac{P}{\Phi} + \lambda \frac{P^1}{\Phi_1} \right) = 0, \tag{4.11}$$

where $I_q f(z)$ is a form of type (0, q) in the domain D_q^M. Now we transform the left side of (4.11) by means of Stokes' formula. We have, taking (4.7) into account,

$$I_q f(z) = -\lim_{\varepsilon\to 0} \int_{\{w\in\sigma_1:\ |w'-z'|\geqslant\varepsilon\}} f(w) \wedge \omega_q' \left(\frac{P^1(w,z)}{\Phi_1(w,z)} \right) + \lim_{\varepsilon\to 0} \int_{\{w\in\sigma_1:\ |w'-z'|\geqslant\varepsilon\}} f(w) \wedge \omega_q' \left(\frac{P(w,z)}{\Phi(w,z)} \right) +$$

$$+ \int_{\substack{w\in\sigma_{10} \\ \lambda\in[0,1]}} f(w) \wedge \omega_q' \left((1-\lambda)\frac{P}{\Phi} + \lambda\frac{P^1}{\Phi_1} \right) - \bar{\partial}_z \int_{\substack{w\in\sigma_1 \\ \lambda\in[0,1]}} f(w) \wedge \omega_{q-1}' \left((1-\lambda)\frac{P}{\Phi} + \lambda\frac{P^1}{\Phi_1} \right). \tag{4.12}$$

Since the first summand on the right-hand side of (4.12) is equal to zero (the vector-function P^1/Φ_1 has null components!), it follows from (4.11) and (4.12) that a (0, q)-form of the form

$$\tilde{K}_q f = \int_{w\in\sigma_1} f(w) \wedge \omega_l' \left(\frac{P(w,z)}{\Phi(w,z)} \right) + \int_{\substack{w\in\sigma_{10} \\ \lambda\in[0,1]}} f \wedge \omega_q' \left((1-\lambda)\frac{P}{\Phi} + \lambda\frac{P^1}{\Phi_1} \right) \tag{4.13}$$

is $\bar{\partial}$-exact in D_q^M. Now we set

$$\Phi_2 = \bar{w}^1 (w^1 - z^1) + \ldots + \bar{w}^{n-q-1} (w^{n-q-1} - z^{n-q-1}),$$
$$P^2 = (\bar{w}^1, \ldots, \bar{w}^{n-q-1}, 0, \ldots, 0).$$

The second summand in (4.13) can be transformed with the aid of Stokes' formula as follows:

$$\int_{\substack{w\in\sigma_{10} \\ \lambda\in[0,1]}} f \wedge \omega_q' \left((1-\lambda)\frac{P}{\Phi} + \lambda\frac{P^1}{\Phi_1} \right) =$$

$$= \int_{\substack{w\in\sigma_{10} \\ \lambda\in[0,1]}} f \wedge \omega_q' \left((1-\lambda)\frac{P}{\Phi} + \lambda\frac{P^2}{\Phi_2} \right) + \int_{\substack{w\in\sigma_{10} \\ \lambda\in[0,1]}} f \wedge \omega_q' \left((1-\lambda)\frac{P^2}{\Phi_2} + \lambda\frac{P^1}{\Phi_1} \right) -$$

$$- \bar{\partial}_z \int_{\substack{w\in\sigma_{10} \\ \lambda_1+\lambda_2\leqslant 1 \\ \lambda_1,\lambda_2\geqslant 0}} f \wedge \omega_{q-1}' \left((1-\lambda_1-\lambda_2)\frac{P}{\Phi} + \lambda_1\frac{P^1}{\Phi_1} + \lambda_2\frac{P^2}{\Phi_2} \right). \tag{4.14}$$

Noting that the second summand on the right-hand side of (4.14) is equal to zero (the vector-function $(1-\lambda)\frac{P^2}{\Phi_2} + \lambda\frac{P^1}{\Phi_1}$ has null components!), we extract from (4.14) and the exactness of the form $\tilde{K}_q f$ that a (0, q)-form of the form

$$\widetilde{\widetilde{K}}_q f = \int\limits_{w\in\sigma_1} f\wedge\omega_q'\left(\frac{P}{\Phi}\right) + \int\limits_{\substack{w\in\sigma_{10}\\ \lambda\in[0,1]}} f\wedge\omega_q'\left((1-\lambda)\frac{P}{\Phi}+\lambda\frac{P^2}{\Phi_2}\right) \tag{4.15}$$

is $\bar\partial$-exact in the domain D_q^M.

Finally, the second summand in (4.15) can be transformed to the form

$$\int\limits_{\substack{w\in\sigma_{10}\\ \lambda\in[0,1]}} f\wedge\omega_q'\left((1-\lambda)\frac{P}{\Phi}+\lambda\frac{P^2}{\Phi_2}\right) =$$

$$= \int\limits_{\substack{w\in\sigma_{10}\\ \lambda\in[0,1]}} f\wedge\omega_q'\left((1-\lambda)\frac{P}{\Phi}+\lambda\frac{P^0}{\Phi_0}\right) + \int\limits_{\substack{w\in\sigma_{10}\\ \lambda\in[0,1]}} f\wedge\omega_q'\left((1-\lambda)\frac{P^0}{\Phi_0}+\lambda\frac{P^1}{\Phi_1}\right) -$$

$$-\bar\partial_z \int\limits_{\substack{w\in\sigma_{10}\\ \lambda_1+\lambda_2\leqslant 1\\ \lambda_1,\lambda_2\geqslant 0}} f\wedge\omega_{q-1}'(1-\lambda_1-\lambda_2)\frac{P}{\Phi}+\lambda_1\frac{P^1}{\Phi_1}+\lambda_2\frac{P^0}{\Phi_0}\Big). \tag{4.16}$$

Noting here that the second summand on the right-hand side of (4.16) is equal to zero because of the holomorphicity in z of the vector-function $(1-\lambda)\frac{P^0}{\Phi_0}+\lambda\frac{P^1}{\Phi_1}$, we deduce from (4.15) and (4.16) that the (0, q)-form $K_q f$ is $\bar\partial$-exact in the domain D_q^M. Lemma 4.2 is proved.

Another essential step in proving Theorem III is the following variant of the integral representation of (n, q)-forms in piecewise smooth domains (see [7, 14]).

LEMMA 4.3. Any continuous, $\bar\partial$-closed (n, q)-form f on $\bar D_q^M$ is cohomologous in the space $H^q(D_q^M, \Omega^n)$ to an (n, q)-form of the form

$$(-1)^q\frac{(n-1)!}{(2\pi i)^n}\int\limits_{w\in\sigma_1} f(w)\wedge\omega_q'\left(\frac{P(w,z)}{\Phi(w,z)}\right)\wedge\omega(z) + \int\limits_{\substack{w\in\sigma_{10}\\ \lambda\in[0,1]}} f\wedge\omega_q'\left((1-\lambda)\frac{P}{\Phi}+\lambda\frac{P^0}{\Phi_0}\right)\wedge\omega(z),$$

where $z\in D_q^M$.

The assertion of Theorem III follows directly from Lemmas 4.2 and 4.3.

5. Description of (n, q)-Forms f, for which the Radon Transformation $\Re f$ is Equal to Zero

Following [2], a compactum K on an n-dimensional complex manifold Ω will be called (n − q − 1)-pseudoconvex with respect to Ω, if for any two neighborhoods Ω_0, Ω_1 $(\Omega_0 \subset\subset \Omega_1)$ of the compactum K, one can find in the domain $\Omega_1\setminus\bar\Omega_0$ an (n − q − 1)-psuedoconvex function ρ, such that $\rho|_{\partial\Omega_0}=0$ and $\rho|_{\partial\Omega_1}=1$.

We recall (see [1]) that a smooth function ρ is called (n − q − 1)-pseudoconvex on the manifold Ω, if for any z (in local coordinates) the restriction of the quadratic form $\sum\limits_{1\leqslant j,k\leqslant n}\frac{\partial^2\rho(z)}{\partial z^j\partial\bar z^k}w^j\bar w^k$ to the plane $\sum\limits_{j=1}^{n}\frac{\partial\rho}{\partial z^j}w^j=0$ has at least q positive eigenvalues.

THEOREM IV. Let D be an (n − q − 1)-linearly concave domain in CP^n, such that for some q-plane $\zeta(\xi)$, the compactum $CP^n\setminus D$ is (n − q − 1)-pseudoconvex with respect to the domain $CP^n\setminus\zeta$. If under these conditions for the (n, q)-form φ, the Radon transformation $\mathscr{R}\varphi(\xi,\eta)$ (for some fixed η) is equal to zero in a neighborhood of the point ξ, then the form φ is $\bar\partial$-exact in the domain D.

We fix in CP^n homogeneous coordinates $Z = (z^0, \ldots, z^n)$ and we consider a q-linearly concave domain D_q of the following special form:

$$D_q = \{z\in CP^n: |z^n|^2+\ldots+|z^{n-q}|^2-|z^{n-q-1}|^2-\ldots-|z^1|^2-|z^0|^2>0\}.$$

Let, further, the hyperplane η in CP^n have the following special form: $\eta=\{z\in CP^n: z^0=0\}$.

LEMMA 5.1. Suppose for a continuous and $\bar\partial$-closed in (n, q)-form φ in $\bar D_q$ the Radon transform $\mathscr{R}\varphi(\xi,\eta)=0$ for any q-plane $\zeta=\zeta(\xi)\subset D_q$. Then for any M < ∞ the form φ is $\bar\partial$-exact in a domain of the form

$$D_q^M = \{z\in D_q: |z^n|^2+\ldots+|z^{n-q}|^2<M^2|z^0|^2\}.$$

Proof. We consider in D_q a q-plane $\zeta = \zeta(\xi)$, having the form

$$\zeta = \zeta(\xi) = \{z \in D_q: z^1 = \ldots = z^{n-q-1} = \langle \xi, Z \rangle = 0\},$$

where $\xi = \xi_{n-q} \in \mathbf{C}^{n+1} \setminus \{0\}$.

By virtue of (3.1) and Lemma 3.1, the equation $\mathcal{R}\varphi(\xi, \eta) = 0$ for the ξ and η chosen here be written in the form

$$\int\limits_{\{z:\, z^1 = \ldots = z^{n-q-1} = \langle \xi, Z \rangle = 0\}} (dz' \wedge \langle \xi, dZ \rangle) \rfloor \varphi (z^0)^{n-q} = 0. \tag{5.1}$$

(5.1) means that the (q + 1, q)-form $dz' \rfloor \varphi \cdot (z^0)^{n-q-1}$ in the section of the domain D_q of the (q + 1)-plane $z^1 = \ldots = z^{n-q-1} = 0$ has Radon transform equal to zero.

By virtue of Proposition 1.3, in this same section the Fantapié indicatrix $\mathcal{F}\varphi$, is equal to zero, i.e.,

$$\int\limits_{z \in \partial D_q \cap \{z:\, z^1 = \ldots = z^{n-q-1} = 0\}} \frac{dz' \rfloor \varphi \cdot (z^0)^{n-q}}{\langle \xi, Z \rangle} = 0 \tag{5.2}$$

for all ξ such that the q-plane $\zeta(\xi) \subset D_q$. Differentiating (5.2) q times with respect to the parameter ξ_0, we get

$$\int\limits_{z \in \partial D_q \cap \{z:\, z' = 0\}} \frac{dz' \rfloor \varphi \cdot (z^0)^{n}}{\langle \xi, Z \rangle^{q+1}} = 0, \quad \zeta(\xi) \subset D_q. \tag{5.3}$$

For each point $z \in D_q$ we substitute into (5.3) a vector $\xi(z)$ of the form

$$\xi(z) = (-|z|^2, \bar{z}^1, \ldots, \bar{z}^{n-q-1}, \bar{z}^{n-q}, \ldots, \bar{z}^n).$$

Then in inhomogeneous coordinates ($z^0 = 1$, $w^0 = 1$), (5.3) assumes the form

$$\int\limits_{w \in \partial D_q \cap \{w:\, w' = 0\}} \frac{dw' \rfloor \varphi}{[\bar{z}^{n-q}(w^{n-q} - z^{n-q}) + \ldots + \bar{z}^n(w^n - z^n)]^{q+1}} = 0. \tag{5.4}$$

(5.4) means, in the terminology of [7], that in the section $D_q \cap \{z' = 0\}$, the Martineau indicatrix $K^*(dz' \rfloor \varphi)$ of the form $dz' \rfloor \varphi$ is equal to zero. By Theorem 1.4 of [7], the form $dz' \rfloor \varphi$ is $\bar{\partial}$-exact in the section $D_q \cap \{z' = 0\}$. Whence, and from Theorem IV it follows that the form φ is for any M, $\bar{\partial}$-exact in the domain D_q^M. The lemma is proved.

LEMMA 5.2. Let φ be a continuous and $\bar{\partial}$-closed (n, q)-form in a neighborhood of the compactum $\bar{D}_q$ such that for sufficiently large M, the form φ is $\bar{\partial}$-exact in a neighborhood of the compactum D_q^M. Then the form φ is $\bar{\partial}$-exact in the domain D_q.

Proof. Since the form φ is $\bar{\partial}$-closed in a neighborhood of the compactum $\bar{D}_q$, for some $\delta > 0$ the form φ is $\bar{\partial}$-closed in the domain of the form

$$D_{q,\delta} = \{z \in \mathbf{C}P^n: (1 + 2\delta^2)|z''|^2 - |z'|^2 - |z^0|^2 > 0\},$$

where $z' = (z^1, \ldots, z^{n-q-1})$, $z'' = (z^{n-q}, \ldots, z^n)$.

We introduce into consideration the domain of the form

$$D_{q+1} = \{z \in \mathbf{C}P^n: |z''|^2 - |z'|^2 + |z^0|^2 > 0\}.$$

Let $M \geqslant 1/\delta$. Then we have the inclusion

$$D_{q+1} \setminus D_q^M \subset D_{q,\delta}. \tag{5.5}$$

If $z \in D_{q+1} \setminus D_q^M$, then $|z^0|^2 < M^{-2}|z''|^2 \leqslant \delta^2 |z''|^2$ and $|z''|^2 - |z'|^2 + 2|z^0|^2 - |z^0|^2 > 0$. Whence, we have $(1 + 2\delta^2)|z''|^2 - |z'|^2 - |z^0|^2 > 0$, i.e., $z \in D_{q,\delta}$. Now suppose in the domain $D_{q,\delta} \cap \{z: |z''|^2 < M^2 |z^0|^2\}$ we have

$$\varphi = \bar{\partial} g, \tag{5.6}$$

where g is some (n, q − 1)-form.

We denote by $\chi(z)$ a smooth function in the domain D_{q+1} such that

$$\chi(z)=\begin{cases}1 & \text{for} \quad z\in D_q\bigcup(D_{q+1}\setminus D^M_{q+1}),\\ 0 & \text{for} \quad z\in\partial D_{q,\delta}\bigcap D_{q+1},\end{cases} \tag{5.7}$$

where $D^M_{q+1}=\{z\in D_{q+1}:|z''|^2<M^2|z^0|^2\}$. Now we consider in the domain D_{q+1} a form of the form

$$\tilde{\varphi}(z)=\begin{cases}\varphi, & \text{if} \quad z\in D_{q+1}\setminus D^M_{q+1},\\ \bar{\partial}\chi g, & \text{if} \quad z\in D^{2M}_{q+1}\bigcap D_{q,\delta},\\ 0, & \text{if} \quad z\in D_{q+1}\setminus D_{q,\delta}.\end{cases}$$

By virtue of (5.5)-(5.7) the form $\tilde{\varphi}$ is well defined and $\bar{\partial}$-closed in the domain D_{q+1} and coincides with the form φ on D_q.

We recall (see [2]) that the domain $CP^n\setminus\bar{D}_{q+1}$ is $(n-q-2)$-complete. Hence by the theorem of Andreotti–Grauert [1], we have

$$H^k(CP^n\setminus\bar{D}_{q+1},\Omega^0)=0 \text{ for } k\geqslant n-q-1.$$

By virtue of the duality theorem (see [1]), we have from this

$$H^{q+1}_{CP^n\setminus D_{q+1}}(CP^n,\Omega^n)=0, \tag{5.8}$$

where $H^{q+1}_{CP^n\setminus D_{q+1}}(CP^n,\Omega^n)$ is the $(q+1)$-dimensional cohomology of the space CP^n with coefficients in Ω^n and with supports in $CP^n\setminus D_{q+1}$. In order to derive from (5.8) the equation $H^q(D_{q+1},\Omega^n)=0$, we consider the following exact sequence of spaces and natural maps between them (see [6]):

$$\to H^q(CP^n,\Omega^n)\to H^q(D,\Omega^n)\to H^{q+1}_{CP^n\setminus D}(CP^n,\Omega^n)\to H^{q+1}(CP^n,\Omega^n)\to\dots\,. \tag{5.9}$$

From (5.8), (5.9), and the known equation $H^q(CP^n,\Omega^n)=0$ for $q\neq n$, we get the assertion

$$H^q(D_{q+1},\Omega^n)=0. \tag{5.10}$$

By virtue of (5.10), the form $\tilde{\varphi}$ is $\bar{\partial}$-exact in the domain D_{q+1}. By virtue of the equation $\tilde{\varphi}=\varphi$ for $z\in D_q$ we get the $\bar{\partial}$-exactness in the domain $D_q\subset D_{q+1}$ of the form φ.

LEMMA 5.3. Let D be a domain in CP^n such that for some q-plane $\eta\subset D$ the compactum $CP^n\setminus D$ is $(n-q-1)$-pseudoconvex with respect to the domain $CP^n\setminus\eta$. Then if the $\bar{\partial}$-closed (n, q)-form φ in D is $\bar{\partial}$-exact in a neighborhood of the q-plane η, then φ is $\bar{\partial}$-exact in D.

Proof. By virtue of the relative cohomology theory (see [6]), we have the following exact sequence of spaces and natural maps between them:

$$\to H^q(CP^n,\Omega^n)\to H^q(D,\Omega^n)\to H^{q+1}_{CP^n\setminus D}(CP^n,\Omega^n)\to H^{q+1}(CP^n,\Omega^n)\to\dots\,. \tag{5.11}$$

On the other hand, by virtue of duality theory (see [1]), one has a canonical isomorphism

$$H^{q+1}_{CP^n\setminus D}(CP^n,\Omega^n)\simeq H^{n-q-1}(CP^n\setminus D,\Omega^0). \tag{5.12}$$

From (5.10)-(5.12) it follows that to prove the $\bar{\partial}$-exactness of the form φ in D it suffices to verify that φ generates the null functional on the space $H^{n-q-1}(CP^n\setminus D,\Omega^0)$.

In other words, it suffices to prove that for a fundamental system of neighborhoods $\{\omega_\nu\}$ of the compactum $CP^n\setminus D$ we have

$$\int_{\partial\omega_\nu}\psi\wedge\varphi=0, \tag{5.13}$$

where ψ is an arbitrary $\bar{\partial}$-closed $(0, n-q-1)$-form in $\bar{\omega}_\nu$.

Let D_q be a neighborhood of the q-plane $\eta\in D$, such that $\varphi=\bar{\partial}g$ on ω_0.

By virtue of the $(n-q-1)$-**pseudoconvexity** of the compactum $CP^n\setminus D$ with respect $CP^n\setminus\eta$, one can use the approximation theorem of Andreotti–Grauert [1], which gives a sequence of $\bar{\partial}$-closed $(0, n-q-1)$-forms $\{\psi_k\}$ in the domain $CP^n\setminus D_q$, which approximates the form ψ uniformly on $\bar{\omega}_\nu$. Whence

$$\lim_{k\to\infty}\int_{\partial\omega_\nu}\psi_k\wedge\varphi=\int_{\partial\omega_\nu}\psi\wedge\varphi. \tag{5.14}$$

On the other hand, by Stokes' formula, we have

$$\int_{\partial\tilde{\omega}_V} \psi_k \wedge \varphi = -\int_{\partial D_q} \psi_k \wedge \varphi = -\int_{\partial D_q} \psi_k \wedge \bar{\partial} g = (-1)^{n-q-1} \int_{\partial D_q} \bar{\partial}\psi_k \wedge g = 0. \tag{5.15}$$

From (5.14) and (5.15) follows (5.13). The lemma is proved.

The assertion of Theorem IV follows from Lemmas 5.1-5.3.

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