SERRE DUALITY FOR PROJECTIVE SUPERMANIFOLDS

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The goal of the present note is to generalize the Serre duality theorem to projective supermanifolds. The dualizing sheaf turns out to be the shear of volume forms Ber X. The proof goes basically parallel to the classical one, but the characteristic singularities of supermathematics appear in it.

If $\tilde{X}$ is a smooth projective manifold over the algebraically closed field $\mathcal{K}$, and $\tilde{\mathcal{F}}$ is a coherent sheaf on it, then Serre's theorem asserts that there is a canonical isomorphism of finite-dimensional vector spaces: $\mathrm{Ext}^i(\tilde{\mathcal{F}}, \Omega^n\tilde{X}) = H^{n-i}(\tilde{X}, \tilde{\mathcal{F}})^*$, where $0 \leqslant i \leqslant n$, $n = \dim \tilde{X}$, $\Omega^n\tilde{X}$ is the canonical sheaf, and * denotes the dual vector space. In particular, it follows from this that if $\tilde{\mathcal{F}}$ is locally free, then $H^i(\tilde{X}, \tilde{\mathcal{F}}) = H^{n-i}(\tilde{X}, \tilde{\mathcal{F}}^* \otimes \Omega^n\tilde{X})^*$ (cf., e.g., [3]).

Now let X be a smooth supermanifold of dimension n|m over $\mathcal{K}$, $\mathcal{K} = \bar{\mathcal{K}}$ and let $\mathcal{O}_X$ be its structure sheaf (notation $\mathbf{X} = (X, \mathcal{O}_X)$). If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then by $\mathcal{F}_{red}$ we shall denote the sheaf of $\mathcal{O}_{red} = \mathcal{O}_X/\mathcal{N}\mathcal{O}_X$-modules $\mathcal{F}/\mathcal{N}\mathcal{F}$, where $\mathcal{N} \subset \mathcal{O}_X$ is the ideal of nilpotents. In the category of supermanifolds there is defined the functor Gr, which carries X into the supermanifold $Gr\mathbf{X} = (X, \Lambda^{\cdot}(\mathcal{N}/\mathcal{N}^2))$, where $\Lambda^{\cdot}(\mathcal{N}/\mathcal{N}^2)$ is the Grassman algebra of the sheaf of $\mathcal{O}_{red}$-modules $\mathcal{N}/\mathcal{N}^2$. We call X decomposable if there exists an isomorphism between X and GrX. If a decomposition of X is given (i.e., an isomorphism between X and GrX), then any $\mathcal{O}_X$-module becomes an $\mathcal{O}_{red}$-module.

In the category of $\mathcal{O}_X$-modules there is also defined the intrinsic functor $\mathcal{H}om$: $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ object representing the functor $\mathcal{C}, \mathcal{H}om(\mathcal{C} \otimes \mathcal{A}, \mathcal{B})$. We denote $\Gamma(\mathbf{X}, \mathcal{H}om)$ by <u>Hom</u>, and the corresponding derived functors by <u>$\mathcal{E}xt^i$</u> and <u>Ext^i</u>.

We denote the functor which changes parity by Π.

If $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a sheaf of $\mathcal{O}_X$-modules, then the cohomology groups of X with coefficients in $\mathcal{F} - H^i(\mathbf{X}, \mathcal{F})$ are $\mathcal{K}$-superspaces.

For any locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank p|q there is defined an invertible $\mathcal{O}_X$-module Ber $\mathcal{F}$ of parity $1/2(1-(-1)^{p+q})$ [1, 2].

We denote by $\Omega^k X$ the differential forms of degree k. Integral forms are defined by the equation $\Sigma_{n-m-k}\mathbf{X} = (\Omega^k\mathbf{X})^* \otimes \mathrm{Ber}\,\mathbf{X}$ [2] (by Ber X we denote $(\mathrm{Ber}\,\Omega^1\mathbf{X})^* = \Sigma_{n-m}\mathbf{X})$).

$\mathbf{P}^{N|M} = \mathbf{P}$ denotes the projective space of dimension N|M over $\mathcal{K}$. $\mathbf{P}^{N|M}$ is canonically decomposable: $\mathbf{P}^{N|M} = (\mathbf{P}^N, \Lambda^{\cdot}(M\mathcal{O}_{\mathbf{P}_N}(-1)))$.

<u>LEMMA 1.</u> Let X be decomposable, and $\mathbf{X}_{red}$ be projective. Then Ber X is a dualizing sheaf, i.e., there is given an even trace morphism $t: H^n(\mathbf{X}, \mathrm{Ber}\,\mathbf{X}) \to \mathcal{K}$ such that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ the pairing $t \circ \alpha$, where $\alpha: \underline{\mathrm{Hom}}_{\mathbf{X}}(\mathcal{F}, \mathrm{Ber}\,\mathbf{X}) \times H^n(\mathbf{X}, \mathcal{F}) \to H^n(\mathbf{X}, \mathrm{Ber}\,\mathbf{X})$, is nondegenerate.

<u>Proof.</u> We denote $\mathcal{N}/\mathcal{N}^2$ by E. It is easy to verify that there exists an isomorphism

$$\mathrm{Ber}\,\mathbf{X} \simeq \Lambda^{\cdot}(E^*) \underset{\mathcal{O}_{red}}{\otimes} \Omega^n(\mathbf{X}_{red}). \tag{1}$$

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Thus, the trace morphism t is induced by the map Ber X $\to \Omega^n(X_{red})$, and the nondegeneracy of the (even) pairing $t \circ \alpha$ follows from the equation $\underline{\mathrm{Hom}}_X(\mathscr{F}, \mathrm{Ber}\,X) = \underline{\mathrm{Hom}}_{X_{red}}(\mathscr{F}, \Omega^n(X_{red}))$ (which follows from (1)) and from Serre duality for the coherent $\mathscr{O}_{red}$-module $\mathscr{F}$).

THEOREM 1. Let X be a closed subsupermanifold in P of codimension $r\,|\,\rho$. Then $\underline{Ext}^r_P(\mathscr{O}_X, \mathrm{Ber}\,P)$ is a dualizing invertible sheaf of parity $1/2(1-(-1)^{n+m})$.

The proof of this theorem follows from the one one has in the purely even case (cf., e.g., [3]); it is based on the duality for P.

It follows from Theorem 1 that the $\mathscr{O}_X$-module $\underline{Ext}^r_P(\mathscr{O}_X, \mathrm{Ber}\,P)$ does not depend on the projective imbedding (up to isomorphism). We denote it by ω_X. Just as in the classical case one has:

COROLLARY 1. For a coherent $\mathscr{O}_X$-module $\mathscr{F}$ one has

$$\underline{\mathrm{Ext}}^i(\mathscr{F}, \omega_X) = H^{n-i}(X, \mathscr{F})^*, \quad \forall i \geqslant 0.$$

Remark. It is new, compared with the even case, that even for a smooth supermanifold X the equation $H^n(X, \omega_X) = \mathscr{K}$ does not necessarily hold; example — $X = (P^n, \Lambda^\cdot(m\mathscr{O}(1)))$, $m, n \geqslant 1$.

THEOREM 2. Let $X \subsetneq Y$ be a closed imbedding of supermanifolds of codimension $r|\rho$. Then $\underline{Ext}^i_Y(\mathscr{O}_X, \mathrm{Ber}\,Y) = 0$ for $i \neq r$, and $\underline{Ext}^r_Y(\mathscr{O}_X, \mathrm{Ber}\,Y) = \mathrm{Ber}\,X$.

COROLLARY 2. For a projective supermanifold X the dualizing sheaf ω_X is isomorphic with Ber X.

The proof of Theorem 2 is based on properties of the Koszul complex and its dual. Let A be a supercommutative ring. We call a collection of homogeneous elements $(Y^1, \ldots, Y^R)$ regular, if $\forall i$, $1 \leqslant i \leqslant R$, one has $\mathrm{Ann}_{A/(Y^1,\ldots,Y^{i-1})} Y^i = (Y^i - (-1)^{\bar{Y}^i} Y^i)$. From a regular collection of elements of A $(y^1, \ldots, y^r, \eta^1, \ldots, \eta^\rho)$, $\bar{y} = 0$, $\bar{\eta} = 1$, generating ideal J, one can canonically construct a Koszul complex $K_{\{y,\eta\}}$ and its dual $\bar{K}_{\{y,\eta\}} = \underline{\mathrm{Hom}}(K_{\{y,\eta\}}, A)$. The members of the Koszul complex are the homogeneous components of the symmetric algebra $S(\Pi L)$, where L is a free A-module of rank $r|\rho$ with basis $l_1, \ldots, l_{r+\rho}$, $\bar{l}_i = 0$, $i \leqslant r$ and $\bar{l}_i = 1$, $i \geqslant r+1$; its differential is the operator $Y^i \frac{\partial}{\partial \pi l^i}$ (the sum over i is meant).

Proposition:

a) $H_0(K_{\{y,\eta\}}) = A/J$; $H_i(K_{\{y,\eta\}}) = 0$ for $i > 0$.

b) $H^r(\bar{K}_{\{y,\eta\}}) = \mathrm{Ber}\,(J/J^2)^*$; $H^i(\bar{K}_{\{y,\eta\}}) = 0$ for $i \neq r$.

Point a) shows that the Koszul complex constructed from a regular sequence of local equations of X in Y is a locally free resolution of the sheaf $\mathscr{O}_X$. From point b) we get $\underline{Ext}^r_Y(\mathscr{O}_X\,\mathrm{Ber}\,Y) = \mathrm{Ber}\,(\mathscr{J}/\mathscr{J}^2)^* \otimes_{\mathscr{O}_Y} \mathrm{Ber}\,Y$, where $\mathscr{J}$ is the ideal of X in $\mathscr{O}_Y$. It remains to prove the formula $\mathrm{Ber}\,(\mathscr{J}/\mathscr{J}^2)^* \otimes_{\mathscr{O}_Y} \mathrm{Ber}\,Y = \mathrm{Ber}\,X$. This is the adjoint formula in the supercase and it can be verified by direct calculation in local coordinates.

The following corollary follows directly from Theorems 1 and 2.

COROLLARY 3. For projective X one has a canonical isomorphism: $H^i(X, \Omega^k X) = H^{n-i}(X, \Sigma_{n-m-k} X)^*$.

Our proposition allows us to give the following invariant characterization of a Berezin module.

COROLLARY 4. Let A be a supercommutative ring, M be a free A-module with basis $e_1, \ldots, e_N$; $e^1, \ldots, e^N$ be the dual basis of M*, $\pi e^1, \ldots, \pi e^N$ be a basis of ΠM*. We denote by

$\tilde{A}$ the symmetric algebra $S(M^*)$, $\tilde{M} := \tilde{A} \otimes_A M$. In $S_{\tilde{A}}(\widetilde{\Pi M^*})$ there is a canonical "homological" differentiation D; in coordinates it has the form $D = Y^i \frac{\partial}{\partial \pi e^i}$ (here Y^i are elements of $\tilde{A}$, equal to e^i from M^*). By S^* we denote the dual complex to $S_{\tilde{A}}(\widetilde{\Pi M^*})$. Then $H(S^*) = \mathrm{Ber}\, M$.

This assertion carries over completely to locally free sheaves of modules.

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REPRESENTATIONS OF THE LIE SUPERALGEBRAS $\mathfrak{gl}$ (n, m) AND Q(n) ON THE SPACE OF TENSORS

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In what follows the ground field is always **C**. It is known (cf. [1]) that all irreducible finite-dimensional representations of simple Lie algebras of the series A_n can be obtained by decomposing tensor powers of the identity representation. In the present paper, following the method of H. Weyl, we study the expansion of tensor powers of the identity representation of the Lie superalgebras $\mathfrak{gl}$ (n, m) and Q(n). As a corollary we get a formula for the characters of the irreducible finite-dimensional representations of the superalgebra Q(n), appearing in the tensor algebra of the identity representation. Moreover, the results of the present paper clarify the use of Young diagrams for describing subrepresentations of the Lie superalgebra GL(n, m) in the tensor algebra as is done in [7, 8].

Let A be a free commutative superalgebra, generated by a family of generators $\{x_i\}_{i \in I}$. For $x = (x_1, \ldots, x_k)$ we define an element p(x) by the rule $p(x) = (p(x_1), \ldots, p(x_k))$, where $p(x)_i$ is the parity of x_i. If S_k is the symmetric group of order k and $\sigma \in S_k$, then $c(p(x), \sigma)$ is determined from the relation $c(p(x), \sigma)\, x_1 \ldots x_k = x_{\sigma(1)} \ldots x_{\sigma(k)}$. It is easy to verify that c is a cocycle, i.e., $c(p(x), \sigma\tau) = c(\sigma^{-1}p(x), \tau)\, c(p(x), \sigma)$. Let V be the identity representation of $\mathfrak{g} = \mathfrak{gl}(n, m)$ and W be its k-th tensor power, ρ_k be the corresponding representation of $\mathfrak{g}$ and $U(\mathfrak{g})$ (the universal enveloping algebra of the superalgebra $\mathfrak{g}$). The group S_k acts on W according to the rule $\pi(\sigma)(v_1 \otimes \ldots \otimes v_k) = c(\sigma^{-1}(p(v)), \sigma)\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. The decomposition of the module W into irreducible $\mathfrak{g}$ submodules is based on the following result, which also proves its complete reducibility.

THEOREM 1. $\pi(S_k)^! = \rho_k(U(\mathfrak{g}))$ (! is the notation for the commutant).

To describe the decomposition of the module W we introduce the following objects (cf. [2]): μ is a partition of k_1, ν is a partition of k_2, where $k = k_1 + k_2$, λ is a partition of k. By S^λ we denote the irreducible S_k-module corresponding to λ, and by V^λ the irreducible

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