APPLICATIONS OF ERGODIC THEORY TO THE INVESTIGATION OF MANIFOLDS OF NEGATIVE CURVATURE

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I. Notation and Definitions. Let M^n be an n-dimensional compact manifold of variable negative curvature, let N^n be a universal covering manifold of M^n and let π be its proper projection $M^n \xrightarrow{\pi} N^n$. If $m \in M^n$, we denote by $\pi^{-1}(m)$ the complete inverse image of point m under mapping π. If $n \in N^n$, we denote by $S_R(n)$ the area of a sphere of radius R with center at point n, and by $D_R(n)$ the volume of a ball of radius R with center at n. Let $n_1, n_2 \in N^n$. Then we denote by $a_{n_1, n_2}(R)$ the number of points of the set $\pi^{-1}(\pi(n_2))$ inside a closed circle of radius R with center at point n_1.

We denote by b(R) the number of closed geodesics on manifold M^n of length not greater than R.

We denote by W^{2n-1} the space of unit tangent vectors of M^n, i.e., the set of pairs of the form $w = (m, \xi)$, where $m \in M^n$ and ξ is the tangent vector of unit length to M^n at point m.

The notation $a_1(R) \sim a_2(R)$ signifies everywhere that $\lim\limits_{R\to\infty} \frac{a_1(R)}{a_2(R)} = 1$.

II. THEOREM 1. There exist continuous positive functions $c_1(m)$, $c_2(m_1, m_2)$ $(m, m_1, m_2 \in M^n)$ and a constant d, such that for any $m, m_1, m_2 \in M^n$:

1) $S_R(n) \sim c_1(m)e^{dR}$ for any point $n \in \pi^{-1}(m)$;

2) for any $n_1 \in \pi^{-1}(m_1)$ and $n_2 \in \pi^{-1}(m_2)$, it is true that $a_{n_1,n_2}(R) \sim c_2(m_1, m_2)e^{dR}$.

Remark 1. It follows at once from 1) that $D_R(n) \sim \frac{c_1(m)}{d} e^{dR}$.

Remark 2. If the curvature of manifold M^n in any two-dimensional direction at any point is between the constants $-K_1^2$ and $-K_2^2$, then it may be asserted that $(n-1)K_2 < d < (n-1)K_1$.

In the case when the curvature of M^n is constant and equal to $-K^2$, the following holds:

THEOREM 2. The constant d in Theorem 1 is equal to $(n-1)K$, the function $c_2(m_1, m_2)$ is constant, and

$$c_2(m_1, m_2) = \frac{S(n-1)}{2^{n-1}(n-1)K\,V(M^n)},$$

where $S(n-1)$ is the area of the unit sphere in $(n-1)$-dimensional Euclidean space and $V(M^n)$ is the volume of manifold M^n.

THEOREM 3. There exist constants c and d such that $b(R) \sim \frac{ce^{dR}}{R}$.

Remark 3. The constant d in Theorem 3 is the same as in Theorem 1.

THEOREM 4. If the curvature of manifold M^n is constant and equal to $-K^2$, then $b(R) \sim \frac{e^{R(n-1)K}}{R(n-1)K}$.

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Remark 4. It is apparent from Theorem 4 that the asymptotic behavior of the function b(R) depends only on the dimensions of M^n and not on the topology of M^n. It is possible that the topological properties of M^n have an effect on the higher asymptotic terms.

Remark 5. Some results have been obtained in [3] on the behavior of a function in (R).

III. The geodesic flux T^t on the manifold W^{2n-1} is a C-flux (sometimes called U-flux; see [1]). We denote by $\mathfrak{S}^k$ a foliation whose layers are expansible orispheres, and by $\mathfrak{S}^{l+1}$ a foliation whose layers are contractile leaves.

Let X be the set of all subsets of manifold W^{2n-1} satisfying the following conditions: a) if $x \in X$, there is a layer $\mathfrak{s}(x)$ of foliation $\mathfrak{S}^k$, such that $x \subset \mathfrak{s}(x)$; b) x as a subset of the manifold $\mathfrak{s}(x)$ is a compact open set (it must be stressed that $\mathfrak{s}(x)$ depends on x). If we replace $\mathfrak{S}^k$ by $\mathfrak{S}^{l+1}$, the set Y can be defined analogously.

Definition. We say that two $x_1, x_2 \in X$ ($y_1, y_2 \in Y$) are ε-equivalent if there is a homeomorphism $x_1 \times I \xrightarrow{h} W^n$ ($y_1 \times W^n$), where I is a unit segment, such that for any $w \in x_1 (w \in y_1)$: 1) if $h_w = h(w, I)$, then $h_w \in Y$ ($h_w \in X$) and h_w is a smooth curve whose length does not exceed ε; 2) $h(w, 0) = w$, $h(w, 1) \in x_2 (h(w, 1) \in y_2)$ and the mapping $\tilde{h}$ under which w is taken into h(w, 1) is a homeomorphism x_1 onto x_2 (y_1 onto y_2) (the topology on x_1, x_2, y_1, and y_2 is induced by the topology of the layers on which they exist).

THEOREM 5. There exist a constant c and a countably-additive measure μ_X (μ_Y) defined on the σ-algebra generated by subsets of X(Y), such that for any $x \in X$ ($y \in Y$) it is true that $\mu_X(x) < \infty$ ($\mu_Y(y) < \infty$),

$$\frac{\mu_X(T^t x)}{\mu_X(x)} = e^{ct} \quad \left(\frac{\mu_Y(T^t y)}{\mu_Y(y)} = e^{ct}\right)$$

and for any $\delta > 0$ there is an $\varepsilon > 0$ such that if $x_1, x_2 \in X$($y_1, y_2 \in Y$) are ε-equivalent, then

$$|\mu_X(x_1) - \mu_X(x_2)| < \delta\mu_X(x_2) \quad (|\mu_Y(y_1) - \mu_Y(y_2)| < \delta\mu_Y(y_2)).$$

Remark 6. Theorem 5 is used for the proof of Theorems 1 and 3.

IV. If M^n is a manifold of constant negative curvature −1, then M^n is isometric to the space L^n/Γ, where L^n is an n-dimensional Lobachevskii space and Γ is a discrete subgroup, containing no elements of finite order, of the group of motions of L^n. Let n = 2. Then the group of motions of space L^2 is isomorphic to the group $SL(2, R)/Z_2$, where SL(2, R) is the group of unimodular matrices of second order and Z_2 is a subgroup consisting of the matrices E and −E. Let $T_\Gamma(g)$ be the representation of group G generated by the homogeneous space G/Γ (see [2], p. 35).

The following theorem is a refinement of Theorem 2. We set $d(R) = \frac{\pi e^R}{V(M^n)}$.

THEOREM 6. Let $\alpha_{n_1,n_2} = \overline{\lim_{R\to\infty}} \frac{\ln(a_{n_1,n_2}(R) - d(R))}{R}$. Then:

1) for any subgroup Γ and for any n_1 and n_2 the quantity $\alpha_{n_1,n_2} < 1$;

2) if in the decomposition of representation T(g) into irreducible representations there are no representations of the complementary series, then for any n_1 and n_2

$$\alpha_{n_1,n_2} \leqslant \frac{1}{2};$$

3) if for any n_1 and n_2 the quantity $\alpha_{n_1,n_2} \leq \frac{1}{2}$, then in the decomposition of the representation into irreducible representations there are no representations of the complementary series.

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