COHOMOLOGIES OF BRAID GROUPS

F. V. Vainshtein UDC 513.836

In this paper we present the results of computing the cohomologies of groups B(n) of n-thread braids. The study of braid group cohomologies was initiated by Arnol'd [1]. Up to the present time we knew the ring of braid group cohomologies with coefficients in Z_2 (see [3]) and the cohomology ring of the group $B(\infty) = \varinjlim B(n)$ (see [2]).

Our method is close to Fuks' approach [3]. At first, using triangulation we compute the cohomologies with coefficients in Z_p, $p > 2$, and then we compute the cohomologies of the Bokshtein complex. The information obtained, together with the results in [3] and the formula for the universal coefficients, enables us to compute the ring of integral cohomologies of B(n).

1. Let $\tilde{G}_n$ be a subspace of space C^n, consisting of points with pairwise distinct coordinates, G_n be the factor space of $\tilde{G}_n$ with respect to the substitution group S(n), and G_n^* be a one-point compactification of G_n. It is well known (see [1]) that $G_n = K(B(n), 1)$. By definition $H^*(B(n); \mathbf{Z}) = H^*(G_n; \mathbf{Z})$. The Poincaré isomorphism $\tilde{H}^*(G_n; \mathbf{Z}) \approx \tilde{H}_*(G_n^*; \mathbf{Z})$ reduces the computation of the cohomologies of B(n) to the computation of the homologies of G_n^*.

Let us consider the triangulation of G_n^*. It is made up from one 0-dimensional cell $\infty \in G_n^* \setminus G_n$ and the (n + q)-dimensional cells $e(m_1, \ldots, m_q)$, where $q, m_i > 0$, $\sum_{i=1}^{q} m_i = n$. By $e(m_1, \ldots, m_q)$ we have denoted a subset of space G_n, consisting of points $\{z_1, \ldots, z_n\} \in G_n$ such that the points $z_1, \ldots, z_n$ of plane C lie on q pairwise distinct vertical lines and, moreover, exactly m_i points from them lie on the i-th line, counting from the left. The characteristic maps have been described in [3]. The boundary of the chain $e(m_1, \ldots, m_q)$ is expressed by the formula

$$\partial e(m_1, \ldots, m_q) = \sum_{i=0}^{q-1} (-1)^i P_{m_i - m_{i+1}}^{m_i} e(m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_q),$$

where $P_m^k = 0$, if $m - k \equiv k \equiv 1 \bmod 2$, and $P_m^k = C_{[m/2]}^{[k/2]}$ in the remaining cases ($[\alpha]$ is the integer part of number α). By $C_{.}(G_n^*; \mathbf{Z})$ we denote the cell complex corresponding to the triangulation described.

LEMMA. The complex $C_{.}(G_n^*; \mathbf{Z})$ can be expanded into a direct sum of subcomplexes $C_{.}(G_n^*; \mathbf{Z}) = C_0(G_n^*; \mathbf{Z}) \oplus K_0(n) \oplus K_1(n) \oplus \ldots \oplus K_n(n)$, where $K_i(n)$ is the subgroup of $C_{.}(G_n^*; \mathbf{Z})$, generated by by the cells $e(m_1, \ldots, m_q)$ for which exactly i of the numbers $m_1, \ldots, m_q$ are odd.

The next statement shows that it is sufficient to compute the homologies of complexes K_0.

Proposition 1. The canonic isomorphism $H_q(K_i(n)) \approx H_{q-i}(K_0(n-i))$ holds for any $q \geqslant 0$.

COROLLARY 1. For any $l \geqslant 1$ and $q \geqslant 0$ the natural inclusion $B(n) \subset B(n+l)$ induces the epimorphisms $H^q(B(n+l); \mathbf{Z}) \to H^q(B(n); \mathbf{Z})$ which are isomorphisms when $n \equiv 0 \bmod 2$ and $l = 1$.

COROLLARY 2. **(Stabilization).** The canoninic isomorphisms $H^q(B(n); \mathbf{Z}) \approx H^q(B(n+1); \mathbf{Z})$ hold for $q < [q/2]$.

The statement on isomorphism in Corollaries 1 and 2 was first proved by Arnol'd (see [1]).

2. Let $\mathfrak{L}_p(k) = K_0(2k) \otimes Z_p$ and $\mathfrak{L}_p = \bigoplus_k \mathfrak{L}_p(k)$. We define a linear operator Φ in the Z_p-space $\mathfrak{L}_p$. Let $(m_1, \ldots, m_r)$ be an integral vector with positive coefficients. By $\mathfrak{S}(r)$ we denote the linear space generated over Z_p by the substitution group S(r). We set

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$$\varphi_{m_1,\ldots,m_r}(i_1,\ldots,i_r)=\sum_{q=1}^{r-1}\delta(m_{i_{q+1}},i_{q+1},i_q)\,C_{m_{i_q}+m_{i_{q+1}}}^{m_{i_q}}(i_1,\ldots,i_{q-1},i_{q+1},i_q,i_{q+2},\ldots,i_r),$$

where $\delta(m, i, j) = 0$ if m is not a power of p or if $i < j$ and $\delta(m, i, j) = 1$ otherwise. We set $E(m_1,\ldots,m_r)=e(2m_1,\ldots,2m_r)\otimes 1$ and we define the linear map $\Psi_{m_1,\ldots,m_r}\colon \mathfrak{F}(r)\to\mathfrak{L}_p$ by the formula

$$\Psi_{m_1,\ldots,m_r}(i_1,\ldots,i_r)=E(m_{i_1},\ldots,m_{i_r}).$$

We now set

$$\Phi E(m_1,\ldots,m_r)=\Psi_{m_1,\ldots,m_r}\left(\sum_{i=0}^{\infty}(-\varphi_{m_1,\ldots m_r})^i\,(1,2,\ldots,r)\right).$$

Let $\mathfrak{A}$ be the set of all finite unordered collections of nonnegative integers. For $R\in\mathfrak{A}$ we denote the set of ordered collections representing R by S(R). For $R=\{r_1,\ldots,r_s\}\in\mathfrak{A}$ we set $N_p(R)=p(p^{r_1}+\ldots+p^{r_s})$, $\lambda(R)=s$, and

$$E(m_1,\ldots,m_{k_1},\{R\},m_{k_1+2s+1},\ldots,m_{k_2})=\sum_{(i_1,\ldots,i_s)\in S(R)}E(m_1,\ldots,m_{k_1},p^{r_{i_1}},(p-1)p^{r_{i_1}},\ldots,p^{r_{i_s}},(p-1)p^{r_{i_s}},m_{k_1+2s+1},\ldots,m_{k_2}).$$

Further, let $\mathfrak{B}$ be the set of all finite collections, ordered strictly by growth, of nonnegative integers and let $L=(l_1,\ldots,l_t)\in\mathfrak{B}$. We set $N_p(L)=p^{l_1}+\ldots+p^{l_t}$, $\lambda(L)=t$ and

$$E(m_1,\ldots,m_{k_1},(L),m_{k_1+t+1},\ldots,m_{k_2})=E(m_1,\ldots,m_{k},p^{l_1},\ldots,p^{l_t},m_{k_1+t+1},\ldots,m_{k_2}).$$

Proposition 2. a) Each homology class of complex $\mathfrak{L}_p$ is uniquely represented by a linear combination of cycles of the form

$$\Phi E(\{R\},(L)), \tag{1}$$

where $R\in\mathfrak{A}, L\in\mathfrak{B}$ (the collections R and L can be empty), and each chain of form (1) is a cycle. Thus, the homology space of complex $\mathfrak{L}_p$ is canonically isomorphic with the subspace generated over Z_p by cycles of form (1).

b) The space $H_q(K_0(2k);\ \mathbf{Z}_p)=H_q(\mathfrak{L}_p(k))$ is canonically isomorphic with the space generated by cycles of form (1) for which $N_p(R)+N_p(L)=k$ and $2\lambda(R)+\lambda(L)=q$.

We obtain the next statement as a corollary of Proposition 2.

THEOREM 1. The rank of group $H^q(B(n);\ \mathbf{Z}_p)$ equals the number $N_{1c,q}+N_{2,q}+\ldots+N_{[n/2],q}$, where $N_{k,q}$ is the number of ways in which the number k can be represented as a sum of $2k - q$ powers of number p (Among these powers there can be coinciding ones, while representations differing only by the order of the summands are reckoned like.).

3. The ring structure of the cohomologies of interest to us can be described by the following theorem.

THEOREM 2. The ring $H^*(B(\infty);\mathbf{Z}_p)$ is a tensor product of a polynomial algebra with the generators x_i, $\dim x_i=2p^{i+1}-2$, $i=0,1,\ldots$, and an exterior algebra with the generators $\dim y_j=2p^j-1$, $j=0,1,\ldots$. The kernel of isomorphism $H^*(B(\infty);\mathbf{Z}_p)\to H^*(B(n);\ \mathbf{Z}_p)$ is generated by the monomials $x_{r_1}\cdot\ldots\cdot x_{r_s}\cdot y_{l_1}\cdot\ldots\cdot y_{l_t}$ with $2(p^{r_1+1}+\ldots+p^{r_s+1}+p^{l_1}+\ldots+p^{l_t})>n$.

Let p be a prime and β_p be the Bokshtein homomorphism corresponding to the exact sequence $0\to\mathbf{Z}_p\to\mathbf{Z}_{p^2}\to\mathbf{Z}_p\to 0$.

THEOREM 3. The following isomorphisms hold:

$$H^0(B(n);\mathbf{Z})=H^1(B(n);\mathbf{Z})=\mathbf{Z},$$
$$H^q(B(n);\mathbf{Z})=\bigoplus_p\beta_p H^{q-1}(B(n);\mathbf{Z}_p)\ \text{for}\ q\geqslant 2,$$

where the sum extends onto all primes. The action of the Bokshtein homomorphism is described by the following formulas: $\beta_p x_i=y_{i+1}$, $\beta_p y_j=0$. In particular, p^2-torsion is absent in the braid group cohomologies.

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INVARIANT ALGEBRAS OF FUNCTIONS ON LIE GROUPS

V. M. Gichev

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1. In this paper we examine algebras, invariant relative to left and right shifts, of continuous functions on Lie groups. We reveal the connection of such algebras with the cones, invariant relative to adjoined representation, in the Lie algebras of these groups.

In the following G denotes a unimodular linear Lie group. We shall say that A is an invariant algebra on G if the following conditions are fulfilled:

1) A is a closed and separating point of subalgebra $C_0(G)$;

2) A is invariant relative to left and right shifts of G;

3) A contains an approximate identity element;

4) functions integrable relative to the Haar measure σ of group G are dense in A.

By $\mathcal{M}_A$ we denote the space of maximal ideals of A, with its usual topology.

II. Let $\varphi, \psi \in \mathcal{M}_A$. For arbitrary $f \in A$ we set $\varphi\psi(f) = \int f(gh)d\mu(g)d\nu(h)$, where μ and ν are representative measures for φ and ψ, respectively. From 3) it follows that $\varphi\psi \neq 0$ for $\varphi, \psi \in \mathcal{M}_A$.

THEOREM 1. The multiplication defined above introduces in $\mathcal{M}_A$ the structure of a locally compact topological semigroup; here the multiplication on G coincides with group multiplication and the group of invertible elements of $\mathcal{M}_A$ equals G.

III. We note that $A \cap L_1[G, \sigma] \subset L_2[G, \sigma]$. Let $A_0 = A \cap L_2[G, \sigma]$ and H be the closure of A_0 in $L_2[G, \sigma]$. With each $\varphi \in \mathcal{M}_A$ we associate an operator R_φ in $H: R_\varphi = \int R_g d\mu(g)$, where μ is the representative measure for φ, $R_g f(h) = f(hg)$. All R_φ, $\varphi \in \mathcal{M}_A$, commute with all left shifts, $\|R_\varphi\| \leqslant 1$, the mapping $\varphi \to R_\varphi$ is continuous in the strong operator topology, R_φ are multiplicative on A_0, $R_\varphi R_\psi = R_{\varphi\psi}, \varphi, \psi \in \mathcal{M}_A$.

A continuous homomorphism γ of the semigroup of nonnegative numbers with addition in $\mathcal{M}_A$, such that γ(0) = e, where e is the identity element of G, is called a one-parameter semigroup in $\mathcal{M}_A$.

Proposition 1. Let $\{T_t\}$ be a strongly continuous semigroup of operators in H, all T_t, $t \geqslant 0$, commute with all left shifts of G; let X be its infinitesimal generating operator. In order that a one-parameter semigroup γ exists in $\mathcal{M}_A$, such that $T_t = R_{\gamma(t)}, t \geqslant 0$, it is necessary and sufficient that the set $A_1 = \{f \in A_0 \cap \mathcal{D}(X) \mid Xf \in A_0\}$ be an algebra and that $f_1, f_2 \in A_1$ be fulfilled for any $X(f_1 f_2) = (Xf_1) f_2 + f_1 (Xf_2)$.

We denote involution by *: $f^*(g) = \overline{f(g^{-1})}$; from 2) and 4) it follows that A* = A; by the same token symmetry is introduced in $\mathcal{M}_A$: $\varphi^*(f) = \overline{\varphi(f^*)}$. By computation we obtain that $R_{\varphi^*} = R_\varphi^*$, where R_φ^* is the operator adjoint to R_φ in H. By an application of Stone's theorem (see [1, p. 472]), from Proposition 1 we obtain the following statement.

Let γ be a symmetric one-parameter semigroup in $\mathcal{M}_A$, and X be the infinitesimal generating operator of semigroup $R_{\gamma(t)}$. Then iX is the generating operator of the one-parameter group of right shifts of G.

IV. Analogously to the way in which the existence of a one-parameter subgroup in the NSS-group was proved in [2], we can prove the following statement.

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