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FRACTIONAL POWERS OF OPERATORS AND HAMILTONIAN SYSTEMS

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In [1] it was discovered that the nonlinear Korteweg–de Vries (KdV) equation admits in some sense of an exact integration procedure. The principal bases of this procedure were clarified in [2]: It was connected with the problem of seeking differential operators of any order for the Sturm–Liouville operator $L = -d^2/dx^2 + u(x)$, which would commute in the maximally possible way with L. Usually this is called the construction of Lax's L,A-pairs.* In [3, 4] it was shown that the KdV equation is a Hamiltonian system having a complete collection of first integrals in involutions. Already in [2] it was noticed that instead of a second-order operator L we can choose higher-order operators and look for those which pair with them. An algorithm for this was proposed in [5]. This was done in the modern way in [6] on the basis of a development of the technique in [7]. As a result, a system of equations was constructed generalizing the KdV equation and the complete integrability was proved of the corresponding stationary time-independent equations (also see survey [8]). As far as we know, the Hamiltonian mechanics of these systems, analogous to that for the case of second-order operators L, has not been constructed anywhere.

In the present article we shall show that by a sequential application of the technique suggested in the authors' previous articles [9, 10] we can construct a theory of generalized systems of the KdV type, including the Hamiltonian structure.

1. Ring of Polynomials of $u_k(x)$, $u_k'(x)$, ... By A we denote the ring of polynomials of several functions $u_k(x)$ and their derivatives of any order. The algebra and the variational calculus in such a ring of one function were presented in detail in [9]. Here we list briefly the information needed for the case of any finite number of functions. Differentiations or "vector fields" $\sum b_{k,i}\frac{\partial}{\partial u_k^{(i)}}$, $b_{k,i}\in A$ act in the ring. The collection of differentiations is named TA. There is one preferred differentiation $\frac{d}{dx}=\sum u_k^{(i+1)}\frac{\partial}{\partial u_k^{(i)}}$. Let dA/dx be the set of elements of A, representable in the form df/dx, $f\in A$. We set $\tilde{A} = A/(dA/dx)$. The elements of $\tilde{A}$ are called functionals [if we examine some boundary conditions on $u_k(x)$, making it possible to talk about the integrals $\int f\,dx$, e.g., the condition of dying out at ±∞ or of periodicity, etc., then a one-to-one correspondence exists between the equivalence

*In what follows we shall call them P,L-pairs since, firstly, the letter A will be firmly occupied, and, secondly, it honors P. Lax.

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classes with respect to dA/dx and the integrals]. The mapping which associates with each $f \in A$ its class $\tilde{f}$ in $\tilde{A}$ is denoted $\int dx$: $\tilde{f} = \int f\,dx$. Here the integral is defined purely algebraically without any convergence conditions. Let us determine the operators of the variational derivatives $\frac{\delta}{\delta u_k} = \sum\left(-\frac{d}{dx}\right)^i \frac{\partial}{\partial u_k^{(i)}}$. It can be proved that for $f \in dA/dx$, it is necessary and sufficient that all $\delta f/\delta u_k = 0$. For this reason $\delta/\delta u_k$ can be transferred from A to $\tilde{A}$ (but they take values as before in A). Obviously, $\frac{\delta f}{\delta u_k} = \frac{\delta}{\delta u_k}\int f\,dx$. Differential operators commuting with d/dx have the form $\sum b_k^{(i)} \frac{\partial}{\partial u_k^{(i)}}$, where $b_k^{(i)} = \left(\frac{d}{dx}\right)^i b_k$. These operators can be taken as acting in $\tilde{A}$. If we apply the obvious formula for integration by parts to $\int f'g\,dx = -\int fg'dx$, then we obtain

$$\left(\sum b_k^{(i)} \frac{\partial}{\partial u_k^{(i)}}\right)\tilde{f} = \int \sum b_k^{(i)} \frac{\partial f}{\partial u_k^{(i)}}\,dx = \sum \int b_k \left(-\frac{d}{dx}\right)^i \frac{\partial f}{\partial u_k^{(i)}}\,dx = \sum_k b_k \frac{\delta}{\delta u_k} f\,dx.$$

Later on we examine the module of the differential forms $TA^* = \{\sum a_{ij\ldots}^{rs\ldots}\delta u_i^{(r)} \wedge \delta u_j^{(s)} \wedge \ldots\}$ over A. In it acts the operator δ, i.e., the exterior differential, and d/dx (acts both on the coefficients as well as on $\delta u_i^{(r)}$). We set $\widetilde{TA}^* = TA^* \Big/ \frac{d}{dx} TA^*$. The operators δ and d/dx commute; therefore, δ can be examined in $\widetilde{TA}^*$ as well; here $\delta \int \omega\,dx = \int \delta\omega\,dx$.

There holds

<u>THEOREM 1.</u> If $f \in A$, then δf can be uniquely represented in the form $\sum R_k \delta u_k + \frac{d}{dx}\omega$, where $R_k \in A$, and ω is some 1-form. Here the coefficients R_k equal $\delta f/\delta u_k$. Another formulation: $\delta \int f\,dx$ can be uniquely written as $\int \sum R_k \delta u_k dx$, and here $R_k = \frac{\delta}{\delta u_k}\int f\,dx$.

<u>2. Differential Operator L and Its Resolvent.</u> We examine the operator

$$L\left(-i\frac{d}{dx}\right) = \sum_{k=0}^{n} u_k(x)\left(-i\frac{d}{dx}\right)^k; \quad u_n \equiv 1, \quad u_{n-1} \equiv 0. \tag{1}$$

The coefficients $u_k(x)$ are arbitrary functions.* The symbol of this operator is $L(\xi) = \sum_k u_k \xi^k$. By $\circ$ we denote the operation of multiplication of symbols:

$$\sigma_1 \circ \sigma_2 = \sum_\nu \frac{1}{\nu!}\left(\frac{\partial}{\partial \xi}\right)^\nu \sigma_1 \cdot \left(-i\frac{d}{dx}\right)^\nu \sigma_2.$$

Let b be the symbol inverse to $L(\xi) - z$:

$$b \circ (L(\xi) - z) = (L(\xi) - z) \circ b = 1 \tag{2}$$

(i.e., the symbol of the resolvent). We seek b in the form†

$$b(\xi, x; z) = \sum_{l,m} B_{l,m}(-1)^{\frac{l+m}{n}} \xi^m (\xi^n - z)^{-1-\frac{l+m}{n}}.$$

Only those nonnegative l and m for which $(l + m)/n$ is an integer are present in the sum. Equation (2) yields the recurrence relations

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*Almost all the results — the construction of P,L-pairs, the Hamiltonian formalism — are preserved when the u_k are matrices. Here for simplicity we restrict ourselves to the scalar case, but we hope to return to the more general case in another article wherein we shall apply another technique which is more natural for the matrix case.

†Such expansions were analyzed in [11] for a second-order operator and in [12] for any elliptic pseudodifferential operator. For our purposes the technique of symbols is especially convenient since it enables us to carry out purely local analyses without using boundary conditions or spectra.

$$B_{0,0}=1,\qquad B_{l,m}=0,\quad \text{if}\quad l<0\quad \text{or}\quad m<0,$$

$$B_{l+n,\,m}=\sum_{k=0}^{n}\sum_{\nu=0}^{k}{}' \binom{k}{\nu} u_k \left(-i\frac{d}{dx}\right)^{\nu} B_{l+(k-\nu),\,m-(k-\nu)} \tag{4}$$

(the prime on the summation sign denotes that the values $k = n$ and $\nu = 0$ are omitted). From the recurrence formula it is easy to get that the $B_{l,m}$ are polynomials of u_k and their derivatives.

3. Fractional Power. For $|\xi| \geqslant 1/2$ we examine

$$\tilde{a}(\xi, x; s)=\frac{1}{2\pi i}\int_{\Gamma} z^s b(\xi, x; z)\,dz. \tag{5}$$

The contour Γ is shown in Fig. 1. The integral converges for Re $s < -1$. By z^s we mean $|z|^s e^{s \arg z}$, where $-\pi/2 \leqslant \arg z \leqslant 3\pi/2$. Let $\chi(\xi)$ be a smooth function, $\chi(\xi) = 1$ when $|\xi| \geq 1$ and $\chi(\xi) = 0$ when $|\xi| \leq 1/2$. We introduce the symbol

$$a(\xi, x; s)=\chi(\xi)\,\tilde{a}(\xi, x; s) \tag{6}$$

[generally speaking, the class of functions of ξ and x, distinguished in the finite range of ξ, is named the symbol; when speaking of a symbol $a(\xi, x; s)$ we shall have in mind a written concrete representative of an equivalence class. As a matter of fact, those important characteristics which we shall need do not depend upon the choice of the representative of the class, for instance, upon the smoothing function $\chi(\xi)$]. It is not difficult to prove the formulas

$$\frac{b(\xi, x; z_1)-b(\xi, x; z_2)}{z_1-z_2}=b(\xi, x; z_1)\circ b(\xi, x; z_2) \tag{7}$$

(the functional equation of the resolvent) and

$$a(\xi, x; s_1)\circ a(\xi, x; s_2)=a(\xi, x; s_1+s_2). \tag{8}$$

From Eq. (7) it follows that

$$\frac{\partial b(\xi, x; z)}{\partial z}=b(\xi, x; z)\circ b(\xi, x; z). \tag{9}$$

Substituting expansion (3) into Eq. (5), we have

$$a(\xi, x; s)=\frac{1}{2\pi i}\int_{\Gamma}\chi(\xi)\sum_{l,m}B_{l,m}(-1)^{\frac{l+m}{n}}\xi^m(\xi^n-z)^{-1-\frac{l+m}{n}}z^s\,dz=\sum_{l,m}B_{l,m}(\xi^n)^{s-\frac{l+m}{n}}\xi^m\cdot\binom{s}{\frac{l+m}{n}}.$$

Introducing the notation

$$A_l(s)=\sum_m B_{l,m}\binom{s}{\frac{l+m}{n}}, \tag{10}$$

we obtain

$$a(\xi, x; s)=\chi(\xi)\sum_{l=0}^{\infty}A_l(s)\,(\xi^n)^s\,\xi^{-l}, \tag{11}$$

where, according to the choice of branch, $\xi^n = 0$ for $\xi^n > 0$ and arg $\xi^n = \pi$ for $\xi^n < 0$.

4. Diagonal of the Kernel. If $\sigma(\xi, x)$ is some symbol or, more precisely, a representative of the class of the symbol, we set

$$\overline{\sigma}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma(\xi, x)\,d\xi,$$

if this integral converges. $\overline{\sigma}(x)$ depends upon the choice of the representative of the class. [$\overline{\sigma}(x)$ is the diagonal of the kernel of the operator which can be constructed from the symbol;

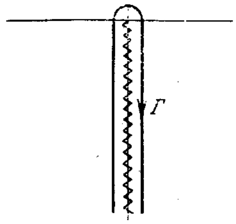

Fig. 1

such an operator is defined ambiguously by the symbol, i.e., to within a smoothing operator.] We consider

$$\bar{a}_k(x;s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\xi^k a(\xi,x;s)\,d\xi,$$

where k is any integer. Later on we shall be interested in the analytic continuation of this function onto the whole plane of the complex variable s, or, more precisely, in the residues of this function. It is easy to see that they are determined by the symbol and do not depend upon the choice of the smoothing function $\chi(\xi)$. They coincide with the residues of the integral

$$J(s)=\frac{1}{2\pi}\int_{|\xi|\geqslant 1}\sum_{l}A_l(s)(\xi^n)^s\xi^{-l+k}\,d\xi.$$

We have

$$J(s)=-\frac{1}{\pi}\sum_{l}A_l(s)\frac{1}{2(ns-l+k+1)}[(-1)^{l+k}+1]$$

for even n and

$$J(s)=-\frac{1}{\pi}\sum_{l}A_l(s)\frac{1}{2(ns-l+k+1)}[e^{\pi i[s+l+k]}+1]$$

for odd n. The residue at $s=(l-k-1)/n$ equals

$$-\frac{1}{2\pi n}A_l\left(\frac{l-k-1}{n}\right)\varepsilon_{l,k},$$

where $\varepsilon_{l,k}=(-1)^{l+k}+1$ for even n and $\varepsilon_{l,k}=e^{\pi i((l-k-1/n)+l+k)}+1$ for odd n.

5. Asymptotics of the Diagonal of the Resolvent's Kernel. We rewrite Eq. (5) as

$$\bar{a}(\xi,x;s)=-\frac{1}{2\pi}\left(e^{\frac{3}{2}\pi is}-e^{-\frac{\pi}{2}is}\right)\int_0^{\infty}z^s b(\xi,x;-iz)\,dz=-i\frac{e^{\frac{\pi}{2}is}\sin\pi s}{\pi}\int_0^{\infty}z^s b(\xi,x;-iz)\,dz.$$

The integral converges for $-1<\operatorname{Re}s<-1/2$. Having set $b_k=\xi^k b$, we have

$$\bar{a}_k(x;s)=-\frac{i}{\pi}e^{\frac{\pi}{2}is}\sin\pi s\int_0^{\infty}z^s\,\overline{\chi(\xi)b_k(\xi,x;-iz)}\,dz.$$

By the Mellin inversion formula (see [13]) we obtain

$$\bar{b}_k(x;-iz)=\frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}z^{-s-1}e^{-\frac{\pi}{2}is}\frac{\pi\bar{a}_k(x;s)}{\sin\pi s}\,ds.$$

Hence follows the asymptotic behavior

$$\bar{b}_k(x; -iz) = \frac{i}{2n}\sum_l A_l\left(\frac{l-k+1}{n}\right)\frac{e^{-\frac{\pi}{2}i\frac{l-k-1}{n}}}{\sin\frac{\pi}{n}(l-k-1)} z^{\frac{-l+k+1}{n}-1}\varepsilon_{l,k}. \tag{12}$$

6. Theorem on Variational Derivatives.

THEOREM 2.

$$\frac{\delta}{\delta u_k}\bar{b}(x; z) = -\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\partial}{\partial z}\left(-i\frac{d}{dx}\right)^{\nu}\bar{b}_{k-\nu}(x; z). \tag{13}$$

To prove Theorem 2 we compute the differential δb and we write it as $R_k \delta u_k + d/dx(\ )$; then $\delta b/\delta u_k = R_k$ (see Paragraph 1). We apply operator δ to $b \circ [L(\xi) - z] = 1$:

$$\delta b \circ [L(\xi) - z] + \sum_k b \circ \delta u_k \circ \xi^k = 0.$$

We multiply from the right by b and we apply the operator $\bar{\ }$:

$$\delta\bar{b} = -\sum_k \overline{b \circ \delta u_k \circ \xi^k \circ b}.$$

For any σ_1 and σ_2 there holds $\overline{\sigma_1 \circ \sigma_2} = \overline{\sigma_2 \circ \sigma_1} + \frac{d}{dx}(\ )$, which follows easily from the definition of the multiplication $\sigma_1 \circ \sigma_2$. We have

$$\delta\bar{b} = -\sum_k \overline{\delta u_k \circ \xi^k \circ b \circ b} + \frac{d}{dx}(\ ).$$

Now Eq. (9) yields $\frac{\delta\bar{b}}{\delta u_k} = -\overline{\xi^k \circ \frac{\partial b}{\partial z}}$. It remains to compute the product of symbols:

$$\frac{\delta\bar{b}}{\delta u_k} = -\sum_\nu \frac{1}{\nu!}\overline{\left[\left(\frac{\partial}{\partial\xi}\right)^{\nu}\xi^k\right]\cdot\left(-i\frac{d}{dx}\right)^{\nu}\frac{\partial b}{\partial z}} = -\sum_{\nu=0}^{\infty}\binom{k}{\nu}\overline{\xi^{k-\nu}\left(-i\frac{d}{dx}\right)^{\nu}\frac{\partial b}{\partial z}} = -\sum_{\nu=0}^{\infty}\binom{k}{\nu}\left(-i\frac{d}{dx}\right)^{\nu}\frac{\partial\bar{b}_{k-\nu}}{\partial z}.$$

QED.

COROLLARY 1.

$$\frac{\delta}{\delta u_k}A_l\left(\frac{l-1}{n}\right) = \frac{l-1}{n}\sum_{\nu=0}^{\infty}\binom{k}{\nu}\left(-i\frac{d}{dx}\right)^{\nu}A_{l-n+k-\nu}\left(\frac{l-n-1}{n}\right). \tag{14}$$

The proof of Corollary 1 can be obtained without difficulty from the connection of $\bar{b}_k$ and A_l [see (12)].

7. Lax's P,L-Pairs. Let us consider Eq. (10) with s = N/n, where N is an integer not divisible by n. We indicate by

$$P_1(\xi) = \sum_{l=0}^{N} A_l\left(\frac{N}{n}\right)(\xi^n)^{N/n}\xi^{-l}$$

the part of the symbol of a nonnegative power of ξ. Let

$$P(\xi, x) = \sum_{l=0}^{N} A_l\left(\frac{N}{n}\right)\xi^{N-l}.$$

Then, when n is odd $P_1 \equiv P$, and when n is even $P_1 \equiv (\operatorname{sign}\xi)^N P$. $P(\xi, x)$ is the symbol of the differential operator

$$P\left(-i\frac{d}{dx}, x\right) = \sum_{l=0}^{N} A_l\left(\frac{N}{n}\right)\left(-i\frac{d}{dx}\right)^{N-l}. \tag{15}$$

We compute the commutator [P(ξ, x), L(ξ)], using the fact that $a(\xi, x; s)$ commutes with L(ξ) [since L(ξ) = $a(\xi, x; 1)$ and in accord with Eq. (8)]. We have

$$[P_1(\xi, x), L(\xi)] = \left[L(\xi), \sum_{l=N+1}^{\infty} A_l\left(\frac{N}{n}\right)\xi^{N-l}\right]\cdot\varepsilon =$$

$$= \sum_{\nu=0}^{\infty}\sum_{k=0}^{n}\sum_{l=N+1}^{\infty}(-i)^{\nu}\left[\binom{k}{\nu}u_k\xi^{k-\nu}A_l^{(\nu)}\left(\frac{N}{n}\right)\xi^{N-l} - \binom{N-l}{\nu}A_l\left(\frac{N}{n}\right)\xi^{N-l-\nu}u_k^{(\nu)}\xi^k\right]\cdot\varepsilon,$$

where ε = 1 for odd n and ε = (sign $\xi)^N$ for even n. Further,

$$[P(\xi, x), L(\xi)] = \sum_{l=N+1}^{\infty}\sum_{\nu=0}^{\infty}\sum_{k=0}^{n}(-i)^{\nu}\left[\binom{k}{\nu}A_l^{(\nu)}\left(\frac{N}{n}\right)u_k - \binom{N-l}{\nu}A_l\left(\frac{N}{n}\right)u_k^{(\nu)}\right]\xi^{N-l+k-\nu}. \tag{16}$$

There are no negative powers of ξ in the left-hand side of Eq. (16); therefore, they are mutually annulled in the right-hand side; the maximal power of ξ is n − 2.

Thus, for every integer N not divisible by n, we have found a differential operator P(−id/dx, x) such that the commutator [P, L] is a differential operator of order n − 2. This is a direct generalization of Lax's theorem for n = 2.

8. Systems of KdV Type. Let the functions u_r depend upon one more variable t. We write the operator equation

$$\frac{d}{dt}L(\xi) = [P, L]. \tag{17}$$

This is equivalent to a system for the functions u_r

$$\frac{du_r}{dt} = \sum_{\substack{N-l+k-\nu=r\\(l>N)}}(-i)^{\nu}\left[\binom{k}{\nu}A_l^{(\nu)}\left(\frac{N}{n}\right)u_k - \binom{N-l}{\nu}A_l\left(\frac{N}{n}\right)u_k^{(\nu)}\right] \tag{18}$$

$$(r = 0, 1, \ldots, n-2).$$

We call this system a system of KdV type.

THEOREM 3. System (18) can be written as

$$\frac{d\mathbf{u}}{dt} = l\frac{\delta}{\delta u}A_{N+n+1}\left(\frac{N+n}{n}\right)\cdot\frac{n}{N+n}, \tag{19}$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_{n-2}), \frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_0}, \ldots, \frac{\delta}{\delta u_{n-2}}\right)$, l is a matrix consisting of the differential operators

$$l_{rs} = \sum_{\gamma=0}^{n-1-r-s}\left[\binom{\gamma+r}{r}u_{r+s+\gamma+1}\left(-i\frac{d}{dx}\right)^{\gamma} - \binom{\gamma+s}{s}\left(i\frac{d}{dx}\right)^{\gamma}u_{r+s+\gamma+1}\right]. \tag{20}$$

Proof. We can invert Eq. (14):

$$A_{l-n+k}\left(\frac{l-n-1}{n}\right) = \frac{n}{l-1}\sum_{\nu=0}^{k}\binom{k}{\nu}\left(i\frac{d}{dx}\right)^{\nu}\frac{\delta}{\delta u_{k-\nu}}A_l\left(\frac{l-1}{n}\right). \tag{21}$$

We substitute these expressions into the right-hand side of Eq. (18). The first one of the two summands yields

$$\frac{n}{N+n}\sum_{\mu,\nu,s}(-i)^{\nu}i^{\mu}\binom{r+s+\mu+\nu+1}{\nu}\binom{s+\mu}{s}u_{r+s+\mu+\nu+1}\left(\frac{d}{dx}\right)^{\mu+\nu}\frac{\delta}{\delta u_s}A_{N+n+1}\left(\frac{N+n}{n}\right).$$

We set μ + ν = γ and we make use of the identity

$$\sum_{\nu=0}^{\gamma}(-1)^{\nu}\binom{r+s+\gamma+1}{\nu}\binom{s+\gamma-\nu}{s} = (-1)^{\gamma}\binom{\gamma+r}{r}, \tag{22}$$

easily provable by induction over s. We have

$$\frac{n}{N+n}\sum_{s=0}^{n-2}\sum_{\gamma=0}^{n-1-r-s}\binom{\gamma+r}{r}u_{r+s+\gamma+1}\left(-i\frac{d}{dx}\right)^{\gamma}\frac{\delta}{\delta u_s}A_{N+n+1}\left(\frac{N+n}{n}\right).$$

The second summand in the right-hand side of Eq. (18) yields

$$-\frac{n}{N+n}\sum_{\mu,\nu,s}(-i)^{\nu}\binom{-1-\mu-s}{\nu}i^{\mu}\binom{s+\mu}{\mu}u^{(\nu)}_{r+s+\mu+\nu+1}\left(\frac{d}{dx}\right)^{\mu}\frac{\delta}{\delta u_s}A_{N+n+1}\left(\frac{N+n}{n}\right).$$

Setting $\mu+\nu=\gamma$, we have

$$-\frac{n}{N+n}\sum_{s=0}^{n-2}\sum_{\gamma=0}^{n-1-r-s}\binom{\gamma+s}{s}i^{\gamma}\sum_{\nu=0}^{\gamma}\binom{\gamma}{\nu}u^{(\nu)}_{r+s+\gamma+1}\left(\frac{d}{dx}\right)^{\gamma-\nu}\frac{\delta}{\delta u_s}A_{N+n+1}\left(\frac{N+n}{n}\right)=$$

$$=-\frac{n}{N+1}\sum_{s=0}^{n-2}\sum_{\gamma=0}^{n-1-r-s}\binom{\gamma+s}{s}\left(i\frac{d}{dx}\right)^{\gamma}u_{r+s+\gamma+1}\frac{\delta}{\delta u_s}A_{N+n+1}\left(\frac{N+n}{n}\right).$$

The sum of the two terms yields what we require.

9. First Integrals.

THEOREM 4. For any p, $\int A_p\left(\frac{p-1}{n}\right)dx$ is a first integral of Eq. (19).

(We recall, see Paragraph 1, that the integral is defined purely algebraically as an equivalence class with respect to dA/dx; i.e., we need not speak about any convergence for the integral. If we restrict ourselves to the class of functions for which $\int f\,dx$ exists in the analytic sense and $\int\frac{d}{dx}f\,dx=0$, then the integral in Theorem 1 can be understood in such a sense.)

Proof of Theorem 4. Let u satisfy system (19). We differentiate the equations $b\circ[L(\xi)-z]=1$ with respect to t:

$$b_t\circ[L(\xi)-z]+b\circ L_t(\xi)=0.$$

But $L_t=P\circ L-L\circ P$ [since this is equivalent to (19)]. We substitute this into the equation and we apply the operation $\bar{\ }$:

$$\bar b_t+\overline{b\circ P\circ L\circ b}-\overline{b\circ L\circ P\circ b}=0.$$

Taking into account that $b\circ L=L\circ b=1+zb$, we obtain $\bar b_t+\overline{b\circ P}-\overline{P\circ b}=0$. We remember that $\overline{\sigma_1\circ\sigma_2}=\overline{\sigma_2\circ\sigma_1}+d/dx(\ )$. Hence, $\bar b_t=d/dx(\ )$. All the coefficients in the expansion $\bar b_t$ in powers of z turn out to be expressions of derivative type, $(\bar b_p)_t=d/dx(\ )$. It remains to note that $\bar b_p$ is proportional to $A_p[(p-1)/n]$.

THEOREM 5. For any p and q (p − 1 and q − 1 are not divisible by n) there exists $J_{q,p}$, a polynomial of u_r and their derivatives, such that

$$\sum_{r,s=0}^{n-2}\left[l_{rs}\left(\frac{\delta}{\delta u_s}A_q\left(\frac{q-1}{n}\right)\right)\right]\cdot\frac{\delta}{\delta u_r}A_p\left(\frac{p-1}{n}\right)=\frac{d}{dx}J_{q,p}. \qquad (23)$$

This is an immediate corollary of the preceding theorem:

$$0=\frac{d}{dt}\int A_p\left(\frac{p-1}{n}\right)dx=\int\sum\frac{\partial}{\partial u_r^{(k)}}A_p\left(\frac{p-1}{n}\right)\cdot(u_r^{(k)})_t\,dx=$$

$$=\int\sum\left(-i\frac{d}{dx}\right)^k\frac{\partial}{\partial u_r^{(k)}}A_p\left(\frac{p-1}{n}\right)\cdot(u_r)_t\,dx=\int\sum\frac{\delta}{\delta u_r}A_p\left(\frac{p-1}{n}\right)\cdot l_{rs}\frac{\delta}{\delta u_s}A_{N+n+1}\left(\frac{N+n}{n}\right)dx.$$

Setting q = N + n + 1 and allowing for the arbitrariness of N, we obtain the theorem's assertion.

10. Space A^{n-1}. Lattices of a Lie Algebra.

The space A^{n-1} consists of the collections $f=(f_0,\ldots,f_{n-2})$, where $f_k\in A$. In this space we now introduce a lattice of a Lie algebra. Namely, with each $\mathbf{f}$ we associate a differential operator $\partial_{\mathbf{f}} = \sum f_k^{(i)} \frac{\partial}{\partial u_k^{(i)}}$ commuting with d/dx. The space A^{n-1} turns out to be in a one-to-one correspondence with the space of such differential operators. The Lie algebra lattice, existing in the latter and introducible by the usual commutation $[\partial_{\mathbf{f}}, \partial_{\mathbf{g}}] = \partial_{\mathbf{f}}\partial_{\mathbf{g}} - \partial_{\mathbf{g}}\partial_{\mathbf{f}}$, is carried over to A^{n-1}. Namely,

$$[\mathbf{f}, \mathbf{g}]_0 = \partial_{\mathbf{f}}\mathbf{g} - \partial_{\mathbf{g}}\mathbf{f} \tag{24}$$

($\partial_{\mathbf{f}}$ and $\partial_{\mathbf{g}}$ act componentwise on $\mathbf{g}$ and $\mathbf{f}$). We have marked the commutator with a subscript 0 since we shall be introducing another commutator right away.

THEOREM 6. The matrix-valued differential operator l maps A^{n-1} onto a Lie subalgebra relative to the commutator $[\ ]_0$, i.e., for any $\mathbf{f}$ and $\mathbf{g}$ there exists an element $\mathbf{h}$ such that

$$[l\mathbf{f}, l\mathbf{g}]_0 = l\mathbf{h}. \tag{25}$$

Here $\mathbf{h}$ is given by the formula

$$\mathbf{h} = \partial_{l\mathbf{g}}\mathbf{g} - \partial_{l\mathbf{g}}\mathbf{f} + \mathbf{h}_1, \tag{26}$$

where

$$(h_1)_k = \sum_{\substack{r,\,s=0\\ r+s<k}}^{n-2} \left[\binom{\gamma+r}{r}(-i)^{\gamma} f_s^{(\gamma)} g_r - \binom{\gamma+s}{s}(-i)^{\gamma} f_s g_r^{(\gamma)}\right],$$

$$\gamma = k - r - s - 1. \tag{27}$$

Theorem 6 can be proved by direct computation, using formulas of type (22). We cannot present these calculations here in view of their awkwardness.

If we exclude constants from ring A and, correspondingly, from A^{n-1}, then, as is easy to verify, the mapping l is a monomorphism. Then $\mathbf{h}$, constructed from $\mathbf{f}$ and $\mathbf{g}$, is a commutator induced from $[\ ]_0$ by mapping l. We shall denote it by $[\ ]_1$, i.e., $[l\mathbf{f}, l\mathbf{g}]_0 = l[\mathbf{f}, \mathbf{g}]_1$.

11. Poisson Brackets. Let $\tilde{F}, \tilde{G} \in \tilde{A}$ be two functionals. The following function:

$$\{\tilde{F}, \tilde{G}\} = -\int \sum_{r,\,s} \left[l_{rs}\left(\frac{\delta \tilde{G}}{\delta u_s}\right)\right] \cdot \frac{\delta \tilde{F}}{\delta u_r}\,dx. \tag{28}$$

is called the Poisson bracket of these functionals. From Eq. (20) for l_{rs} it is obvious that $l_{rs}^* = -l_{sr}$ (the asterisk denotes the formally adjoint differential operator). Hence follows the skew-symmetry of the Poisson bracket. The Jacobi identity is not obvious; it will follow from the next theorem below. We note that when n = 2 the vectors are turned into scalars and $l = -2i\,d/dx$; we arrive at the Gardner–Zakharov–Faddeev brackets.

THEOREM 7. The variational gradient operation $\boldsymbol{\delta}/\delta u$, mapping $\tilde{A}$ into A^{n-1}, leads the Poisson bracket $\{\ \}$ into the commutator $[\ ]_1$:

$$\frac{\delta}{\delta u}\{\tilde{F}, \tilde{G}\} = \left[\frac{\delta}{\delta u}\tilde{F}, \frac{\delta}{\delta u}\tilde{G}\right]_1. \tag{29}$$

We prove this theorem at once by the method applied in [14] to prove this same fact but in a somewhat different situation.

LEMMA 1. Let $\tilde{R} \in \tilde{A}$ be an arbitrary functional,

$$m_{rs} = \sum_k \frac{\partial}{\partial u_s^{(k)}}\left(\frac{\delta \tilde{R}}{\delta u_r}\right) \cdot \left(\frac{d}{dx}\right)^k.$$

Then the matrix-valued differential operator m is formally self-adjoint, m* = m, or

$$\sum_k \left(-\frac{d}{dx}\right)^k \frac{\partial}{\partial u_s^{(k)}}\left(\frac{\delta \tilde{R}}{\delta u_r}\right) = \sum_k \frac{\partial}{\partial u_r^{(k)}}\left(\frac{\delta \tilde{R}}{\delta u_s}\right)\left(\frac{d}{dx}\right)^k. \tag{30}$$

To prove the equality of the two operators it is sufficient to prove that they act alike on the vector-valued form $\delta\mathbf{u} = (\delta u_0, \ldots, \delta u_{n-2})$ (since the forms $\delta u_k^{(i)}$ are linearly independent over A). Let us verify this action for the left- and right-hand sides of the equality to be proved. We have

$$\sum_{r,k}\left(-\frac{d}{dx}\right)^k \frac{\partial}{\partial u_s^{(k)}}\left(\frac{\delta \tilde{R}}{\delta u_r}\right)\delta u_r = \sum_k \left(-\frac{d}{dx}\right)^k \frac{\partial}{\partial u_s^{(k)}} \sum_r \frac{\delta \tilde{R}}{\delta u_r}\delta u_r = \frac{\delta}{\delta u_s}\delta \tilde{R},$$

$$\sum_{r,k}\frac{\partial}{\partial u_r^{(k)}}\left(\frac{\delta \tilde{R}}{\delta u_s}\right)\left(\frac{d}{dx}\right)^k \delta u_r = \sum_{r,k}\frac{\partial}{\partial u_r^{(k)}}\left(\frac{\delta \tilde{R}}{\partial u_s}\right)\delta u_r^{(k)} = \delta\frac{\delta \tilde{R}}{\delta u_s} = \frac{\delta}{\delta u_s}\delta \tilde{R}.$$

Lemma 1 has been proved.

We go on to prove Theorem 7. We compute $\delta\{\tilde{F}, \tilde{G}\}$ and we represent it in the form $\int \sum R_k \delta u_k dx$, then (see Paragraph 1) $R_k = \frac{\delta}{\delta u_k}\{\tilde{F}, \tilde{G}\}$. We have

$$\delta\{\tilde{F}, \tilde{G}\} = -\int \delta \sum_{r,s}\left[l_{rs}\left(\frac{\delta \tilde{G}}{\delta u_s}\right)\right]\cdot \frac{\delta \tilde{F}}{\delta u_r}dx =$$

$$= -\int \sum_{r,s}\left[l_{rs}\left(\delta\frac{\delta \tilde{G}}{\delta u_s}\right)\right]\cdot\frac{\delta \tilde{F}}{\delta u_r}dx - \int\sum_{r,s}\left[l_{rs}\left(\frac{\delta \tilde{G}}{\delta u_s}\right)\right]\delta\frac{\delta \tilde{F}}{\delta u_r}dx - \int\sum_{r,s}\left[(\delta l_{rs})\left(\frac{\delta \tilde{G}}{\delta u_s}\right)\right]\cdot\frac{\delta \tilde{F}}{\delta u_r}dx.$$

The first two terms equal

$$+\int\sum_{r,s,k,i}\left[\frac{\partial}{\partial u_k^{(i)}}\left(\frac{\delta \tilde{G}}{\delta u_s}\right)\left(\frac{d}{dx}\right)^i \delta u_k\right]\cdot\left[l_{sr}\left(\frac{\delta \tilde{F}}{\delta u_r}\right)\right]dx - \int\sum_{r,s,k,i}\left[\frac{\partial}{\partial u_k^{(i)}}\left(\frac{\delta \tilde{F}}{\delta u_r}\right)\left(\frac{d}{dx}\right)^i \delta u_k\right]\cdot\left[l_{rs}\left(\frac{\delta \tilde{G}}{\delta u_s}\right)\right]dx.$$

By Lemma 1 this expression equals

$$\int\sum_{r,s,k,i}\left[l_{sr}\left(\frac{\delta \tilde{F}}{\delta u_r}\right)\right]^{(i)}\frac{\partial}{\partial u_s^{(i)}}\left(\frac{\delta \tilde{G}}{\delta u_k}\right)\delta u_k dx - \dots$$

(the dots denote terms differing by a commutation of $\tilde{F}$ and $\tilde{G}$). We obtain

$$\int\sum_k \partial_{l\frac{\delta F}{\delta u}}\frac{\delta \tilde{G}}{\delta u_k}\delta u_k dx - \dots$$

The third term equals

$$-\int\sum_{r,s}\sum_{\gamma=0}^{n-1-r-s}\left[\binom{\gamma+r}{r}\delta u_{r+s+\gamma+1}\left(-i\frac{d}{dx}\right)^\gamma - \binom{\gamma+s}{s}\left(i\frac{d}{dx}\right)^\gamma \delta u_{r+s+\gamma+1}\right]\left(\frac{\delta \tilde{G}}{\delta u_s}\right)\frac{\delta \tilde{F}}{\delta u_r}dx =$$

$$= -\int\sum_{r,s=0}^{n-2}\left[\binom{\gamma+r}{r}(-i)^\gamma\left(\frac{\delta \tilde{G}}{\delta u_s}\right)^{(\gamma)}\frac{\delta \tilde{F}}{\delta u_r} - \binom{\gamma+s}{s}\frac{\delta \tilde{G}}{\delta u_s}\left(\frac{\delta \tilde{F}}{\delta u_r}\right)^{(\gamma)}\right]\delta u_k dx.$$

Collecting all the terms and keeping Eqs. (26) and (27) in mind, we obtain the required equality

$$\delta\{\tilde{F}, \tilde{G}\} = \sum_k \int\left(\left[\frac{\delta}{\delta u}\tilde{F}, \frac{\delta}{\delta u}\tilde{G}\right]_{1,k}\delta u_k dx.\right.$$

<u>COROLLARY 2.</u> If the Poisson bracket of two functionals equals zero, then the differential operators $\partial_{l\frac{\delta \tilde{F}}{\delta u}}$, $\partial_{l\frac{\delta \tilde{G}}{\delta u}}$ commute.

<u>COROLLARY 3.</u> For any p and q the operators $\partial_{l\frac{\delta}{\delta u}\mathcal{L}}, \partial_{l\frac{\delta}{\delta u}\mathcal{M}}$, where $\mathcal{L} = A_p\left(\frac{p-1}{n}\right)$, $\mathcal{M} = A_q\left(\frac{q-1}{n}\right)$, commute.

<u>Remark 1.</u> We can examine an equation, somewhat more general than (19),

$$\frac{d\mathbf{u}}{dt} = l\frac{\delta}{\delta u}\mathcal{L}, \tag{31}$$

where the Hamiltonian $\mathcal{L}$ is an arbitrary linear combination $\sum_{p=1}^{p_0} c_p A_p\left(\frac{n-1}{n}\right)$. Then, as before, for every q we can find an element J_q such that for $\mathcal{H}_q = A_q\left(\frac{q-1}{n}\right)$

$$\sum_{r,\, s=0}^{n-2} \left[l_{r,\, s} \left(\frac{\delta}{\delta u_s}, \mathcal{H}_q \right) \right] \cdot \frac{\delta}{\delta u_r} \mathcal{L} = \frac{d}{dx} J_q$$

and the differential operators $\partial_{l\,\delta;\delta u}\,\mathcal{L}$, $\partial_{l\,\delta|\delta u}\,\mathcal{H}_q$ commute.

12. Stationary Equations. Now let u be independent of t and satisfy the stationary equation corresponding to Eq. (31)

$$\frac{\delta}{\delta u_r} \mathcal{L} = 0 \qquad (r = 0, \ldots, n-2). \tag{33}$$

Equation (32) shows that the quantities J_q are first integrals of a stationary system. Our next problem is to show that a stationary system can be represented in Hamiltonian form and to compute the Poisson brackets of the first integrals indicated. The Poisson brackets turn out to be equal to zero; i.e., the first integrals turn out to be in involutions.

By $I_{\mathcal{L}}$ we denote an ideal in ring A, generated by the left-hand sides of Eqs. (32) and all their derivatives with respect to x. In other words, to this ideal belong the polynomials of u_i, u_i', . . ., which vanish by virtue of system (33). We set $A_{\mathcal{L}} = A/I_{\mathcal{L}}$.

We now restrict somewhat the generality of the analysis by introducing additional requirements whose meaning reduces to the possibility of solving Eqs. (32) relative to the highest derivatives. In ring A there is the following graduation: The number $n - i + k$ is called the weight of factor $u_i^{(k)}$, while the sum of the weights of the factors is called the sum of the monomial. [This graduation arises naturally from the very origin of the u_i as the coefficients of operator (1); besides, this is not important just now.] The collection of terms of the highest weight is called the leading part of $\mathcal{L}$. It is easy to see that all terms of the polynomials $A_p(s)$ have one and the same weight p; therefore, the leading part of $\mathcal{L}$ is $A_{p_0}[(p_0 - 1)/n]$. In each variational derivative $\delta/\delta u_r \mathcal{L}$ we pick out the linear part of highest weight

$$\frac{\delta}{\delta u_0} \mathcal{L} = \sum_{j=0}^{n-2} k_{0j} u_j^{(p_0-2n+j)} + \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\delta}{\delta u_i} \mathcal{L} = \sum_{j=0}^{n-2} k_{ij} u_j^{(p_0-2n+i+j)} + \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

The terms not written out either are of lesser weight or are nonlinear. In this and other cases they contain derivatives of the functions u_j, of orders lower than in the terms written out. It is easy to see that $k_{ij} = (-1)^{p_0-2n+i+j} k_{ji}$. From now on we shall examine only those Lagrangians for which

$$\Delta_0 \equiv k_{00} \neq 0, \quad \Delta_1 \equiv \begin{vmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{vmatrix} \neq 0, \ldots, \Delta_{n-2} \equiv \begin{vmatrix} k_{00} & \cdots & k_{0\,n-2} \\ \vdots & & \vdots \\ k_{n-2\,0} & \cdots & k_{n-2\,n-2} \end{vmatrix} \neq 0. \tag{34}$$

Hence, it follows already that p_0 must be even and that $p_0 \geqslant 2n$. We set $p_0 - 2n = 2\mu$. We shall analyze only this case (for systems of KdV type this signifies that $n + N + 1$ is even).

LEMMA 2. As independent generators in ring A we can take the system

$$\begin{aligned} &\{u_i^{(s)}\} \quad (i = 0, \ldots, n-2;\ s \leqslant 2\mu + 2i - 1), \text{(a)} \\ &\left\{\left(\frac{d}{dx}\right)^r \frac{\delta \mathcal{L}}{\delta u_i}\right\} \quad (i = 0, \ldots, n-2;\ r = 0, 1, 2, \ldots). \text{(b)} \end{aligned} \tag{35}$$

The functions (35a) are called the principal derivatives. We prove Lemma 2 by induction over the weight of the derivatives $u_i^{(s)}$, which we must express in terms of the quantities (35). Suppose that this has been done for the weight $2\mu + n + m$. We write the system

$$\left(\frac{d}{dx}\right)^m \frac{\delta \mathcal{L}}{\delta u_0} = \sum_{j=0}^{m} k_{0j} u_j^{(2\mu+m+j)} + \sum_{j=m+1}^{n-2} k_{0j} u_j^{(2\mu+m+j)} + \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{\delta \mathcal{L}}{\delta u_m} = \sum_{j=0}^{m} k_{mj} u_j^{(2\mu+m+j)} + \sum_{j=m+1}^{n-2} k_{mj} u_j^{(2\mu+m+j)} + \dots,$$

whence we express the derivatives $u_0^{(2\mu+m)}, \dots, u_m^{(2\mu+2m)}$, since the system's determinant is nonzero.

LEMMA 3. If

$$\sum_{r,\, i} a_i^r \left(\frac{d}{dx}\right)^r \frac{\delta \mathcal{L}}{\delta u_i} = 0, \tag{36}$$

where $a_i^r \in A$, then $a_i^r \in I_{\mathcal{L}}$.

Proof. We express a_i^r in terms of the generators (35). If in these expressions there were terms containing only the generators (35a) and not (35b), then the left-hand side of (36) would contain terms linear with respect to (35b), which would not mutually annihilate anything, and that contradicts the independence of the generators.

Definition 1. A vector field (differentiation) ξ is called tangent if it contains the ideal $I_{\mathcal{L}}$: $\xi I_{\mathcal{L}} \subset I_{\mathcal{L}}$. Two tangent fields are equivalent if $(\xi - \eta) A \subset I_{\mathcal{L}}$. The set of equivalence classes is called $TA_{\mathcal{L}}$.

LEMMA 3. A tangent field $\xi \in TA_{\mathcal{L}}$ is uniquely defined by the principal coordinates ξ_i^s $(s \leqslant 2\mu + 2i - 1)$, which can be taken arbitrarily.

The proof is carried out similarly to the proof of Lemma 2. The nonprincipal coordinates are determined successively from the systems

$$\xi \left(\frac{\delta \mathcal{L}}{\delta u_0}\right)^{(m)} = \sum_{j=0}^{m} k_{0j} \xi_j^{2\mu+j+m} + \sum_{j=m+1}^{n-2} k_{0j} \xi_j^{2\mu+j+m} + \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\xi \frac{\delta \mathcal{L}}{\delta u_0} = \sum_{j=0}^{m} k_{mj} \xi_j^{2\mu+j+m} + \sum_{j=m+1}^{n-2} k_{mj} \xi_j^{2\mu+j+m} + \dots$$

The left-hand sides belong to the ideal. The coordinates are uniquely determined in $TA_{\mathcal{L}}$

13. Characteristics of the Ideal's Elements. We construct the mapping $I_{\mathcal{L}} \Big/ \frac{d}{dx} I_{\mathcal{L}} \to A_{\mathcal{L}}^{n-1}$, which we shall call a characteristic of the element $I_{\mathcal{L}} \Big/ \frac{d}{dx} I_{\mathcal{L}}$. Let $f \in I_{\mathcal{L}}$ be any representative of the class. As an element of the ideal it can be written (not uniquely) as

$$f = \sum_{i,\, s} a_i^s \left(\frac{\delta \mathcal{L}}{\delta u_i}\right)^{(s)} \qquad (a_i^s \in A).$$

With it we associate $a_i = \left(\sum_s \left(-\frac{d}{dx}\right)^s a_i^s\right)_{\mathcal{L}}$ (the notation $(\ )_{\mathcal{L}}$ signifies the natural projection $A \to A_{\mathcal{L}}$). It is necessary to show that this is indeed a single-valued mapping of the class $I_{\mathcal{L}} \Big/ \frac{d}{dx} I_{\mathcal{L}}$ onto the class $A_{\mathcal{L}}^{n-1}$. Let $f = \sum_{i,\, s} b_i^s \left(\frac{\delta \mathcal{L}}{\delta u_i}\right)^{(s)}$ be another way of writing the same f in terms of the generators of the ideal. By Lemma 3, 2, $a_i^s - b_i^s \in I_{\mathcal{L}}$, and, therefore, a_i =

b_i. Now let g be another representative of this same class, $g=\sum_{i,s} b_i^s \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(s)}$. Then $f-g \in \frac{d}{dx} I_{\mathscr{L}}$, i.e.,

$$\sum_{i,s} (a_i^s - b_i^s) \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(s)} = \frac{d}{dx} \sum c_i^s \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(s)},$$

whence

$$a_i^s - b_i^s - (c_i^s)' - c_i^{s-1} \in I_{\mathscr{L}}, \quad \sum_s \left(-\frac{d}{dx}\right)^s a_i^s - \sum_s \left(-\frac{d}{dx}\right)^s b_i^s \in I_{\mathscr{L}}.$$

Definition 2. $F \in A$ is called a first integral if $\frac{dF}{dx} \in I_{\mathscr{L}}$. Its class $(F)_{\mathscr{L}}$ also will be called a first integral.

Obviously, the class of dF/dx in $I_{\mathscr{L}} \Big/ \frac{d}{dx} I_{\mathscr{L}}$ is uniquely determined by class $(F)_{\mathscr{L}}$.

Definition 3. The characteristic of dF/dx in $I_{\mathscr{L}} \Big/ \frac{d}{dx} I_{\mathscr{L}}$ is called the characteristic of the first integral.

14. Hamiltonian Lattice. As we know (Paragraph 1), a certain form $\Omega^{(1)}$ exists such that

$$\delta \mathscr{L} = \sum_k \frac{\delta \mathscr{L}}{\delta u_k} \delta u_k + \frac{d}{dx} \Omega^{(1)}.$$

Let $\Omega^{(2)} = \delta \Omega^{(1)}$, i.e.,

$$\frac{d}{dx} \Omega^{(2)} = -\sum_k \delta \frac{\delta \mathscr{L}}{\delta u_k} \wedge \delta u_k. \tag{37}$$

We shall treat $\Omega^{(2)}$ as a form over $TA_{\mathscr{L}}$ with values in $A_{\mathscr{L}}$. As the differential of $\Omega^{(1)}$, this form is closed.

THEOREM 8. $\Omega^{(2)}$ is a nondegenerate form in $TA_{\mathscr{L}}$.

Proof. In $d\Omega^{(2)}/dx$ we pick out the terms highest with respect to the total weight of the differentials

$$\frac{d}{dx} \Omega^{(2)} = -\sum k_{ij} \delta u_j^{(2\mu+i+j)} \wedge \delta u_i + \dots$$

Hence we find that

$$\Omega^{(2)} = -\sum \tilde{k}_{ij} (\delta u_j^{(2\mu+i+j-1)} \wedge \delta u_i - \delta u_j^{(2\mu+i+j-2)} \wedge \delta u_i' + \dots$$
$$\dots + (-1)^{i+j-1} \delta u_j \wedge \delta u_i^{(2\mu+i+j-1)}) + \dots,$$

where

$$\tilde{k}_{ij} = \begin{cases} k_{ij}, & i \neq j, \\ k_{ij}/2, & i = j, \end{cases}$$

while the dots denote terms which contain $\delta u_j^{(s)} \wedge \delta u_i^{(r)}$ with $s + r < 2\mu + i + j - 1$. Later on it is necessary to express $\Omega^{(2)}$ in terms of the differentials of the principal derivatives [and of the differentials of variables (35b), but the latter yield a form zero in $TA_{\mathscr{L}}$].

We do not present the simple but lengthy calculations, but write out at once the resulting formula. If we introduce the bordered determinant

$$\Delta_{m;i,l}=\begin{vmatrix} k_{00} & \dots & k_{0m} & k_{0l}\\ \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots\\ k_{m0} & \dots & k_{mm} & k_{ml}\\ k_{i0} & \dots & k_{im} & k_{il}\end{vmatrix}, \qquad \frac{\Delta_{-1;\,i,\,l}}{\Delta_{-1}}=k_{il},$$

then

$$\Omega^{(2)}=\Omega_*^{(2)}+\Omega_{\mathscr{L}}^{(2)},$$

where $\Omega_{\mathscr{L}}^{(2)}=\sum a\cdot\delta\left(\frac{\delta L}{\delta u_i}\right)^{(r)}\wedge\delta q_j$, while the q_j are any variables.

$$\Omega_*^{(2)}=\sum_{i\geqslant l}\sum_{r=i-l}^{i-1}(-1)^{r-1}\frac{\Delta_{i-1-r;\,i,\,l}}{\Delta_{i-1-r}}\delta u_l^{(s)}\wedge\delta u_i^{(r)}+\sum_{i>l}\sum_{s=0}^{l-1}(-1)^s\frac{\Delta_{l-1-s;\,l,\,i}}{\Delta_{l-1-s}}\delta u_l^{(s)}\wedge\delta u_i^{(r)}+$$
$$+\sum_{i\geqslant l}\sum_{r=i}^{2\mu+i-1}(-1)^{r-1}\bar{k}_{il}\delta u_l^{(s)}\wedge\delta u_i^{(r)}+\dots\qquad(r+s=2\mu+i+l-1). \tag{38}$$

The nondegeneracy of the form signifies that whatever be the 1-form ω, we can find a vector ξ such that $\omega=-i(\xi)\,\Omega^{(2)}$ in $TA_{\mathscr{L}}$. Obviously, we can take it right away that ω contains only the differentials of the principal variables, $\omega=\sum_{s\leqslant 2\mu+2i-1}\omega_i^s\delta u_i^{(s)}$, and as $\Omega^{(2)}$ we can take $\Omega_*^{(2)}$.

We obtain the sequence of systems

$$\left\{\omega_{n-2}^{2\mu+2n-5}=-\frac{\Delta_{n-3;\,n-2,\,n-2}}{\Delta_{n-3}}\xi_{n-2}^0,\right.$$

$$\left\{\begin{aligned}\omega_{n-3}^{2\mu+2n-7}&=-\frac{\Delta_{n-4;\,n-3,\,n-3}}{\Delta_{n-4}}\xi_{n-3}^0+\frac{\Delta_{n-4;\,n-2,\,n-3}}{\Delta_{n-4}}\xi_{n-2}^1+\dots,\\ \omega_{n-2}^{2\mu+2n-6}&=-\frac{\Delta_{n-4;\,n-3,\,n-2}}{\Delta_{n-4}}\xi_{n-3}^0+\frac{\Delta_{n-4;\,n-2,\,n-2}}{\Delta_{n-4}}\xi_{n-2}^1+\dots,\end{aligned}\right.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\left\{\begin{aligned}\omega_0^{(2\mu-1)}&=-\frac{\Delta_{-1;\,0,\,0}}{\Delta_{-1}}\xi_0^0+\dots+(-1)^{n-1}\frac{\Delta_{-1;\,n-2,\,0}}{\Delta_{-1}}\xi_{n-2}^{n-2}+\dots\\ &\dots\dots\dots\dots\dots\dots\\ \omega_{n-2}^{(2\mu+n-3)}&=-\frac{\Delta_{-1;\,0,n-2}}{\Delta_{-1}}\xi_0^0+\dots+(-1)^{n-1}\frac{\Delta_{-1;\,n-2,\,n-2}}{\Delta_{-1}}\xi_{n-2}^{n-2}+\dots\end{aligned}\right.$$

Besides the ones written down there are more terms containing the coordinates of ξ, determined from the preceding systems. The determinants of all these systems are nonzero by Sylvester's theorem, $\det(\Delta_{n-3-m;\,i,k})=\Delta_{n-3-m}^m\cdot\Delta_{n-2}$. These are still not all the systems needed. We need to finish writing several more systems for the determination of the missing coordinates. They all have one and the same matrix, just as in the last of the systems written out. Thus, all the coordinates ξ_i^s $(s\leqslant 2\mu+2i-1)$ are determined in succession.

Remark 2. As we saw from the proof, if as ω and $\Omega^{(2)}$ we take forms containing only the differentials of the fundamental variables, then the equation $\omega=-i(\xi)\,\Omega^{(2)}$ can be solved exactly, i.e., in A and not just in $A_{\mathscr{L}}$.

15. Construction of the Vector Field Corresponding to a First Integral. After the symplectic form $\Omega^{(2)}$ has been constructed from a given $\mathscr{L}$, we can develop the usual concepts of Hamiltonian mechanics (we refer to [9] for details). $\mathscr{L}$ is a Langrangian, d/dx is a vector field corresponding to the equation, ξ_F are the Hamiltonian vector fields corresponding to $F\in A_{\mathscr{L}}$, i.e., $\delta F=-i(\xi_F)\,\Omega^{(2)}$. If $F,G\in A_{\mathscr{L}}$, then their Poisson brackets are $\xi_F G=\xi_G F$. The vector field corresponding to the Poisson brackets of F and G is the commutator of the vector fields ξ_F and ξ_G; therefore, the Poisson brackets equal zero if and only if the corresponding vector fields commute. If $F\in A_{\mathscr{L}}$ is a first integral, i.e., dF/dx = 0 in $A_{\mathscr{L}}$, then field ξ_F commutes with vector field d/dx.

When speaking about forms we shall distinguish three kinds of equalities:

1) the identity equality $\omega_1 = \omega_2$, i.e., the coincidence of all the coefficients in A; 2) equality over $A_{\mathscr{E}}$, $\omega_1 = \omega_2 (A_{\mathscr{E}})$, when the values of the forms, treated as elements of $A_{\mathscr{E}}$, coincide on the tangent vector fields; 3) equivalence, $\omega_1 \simeq \omega_2$, when the values of the forms, treated as elements of $A_{\mathscr{E}}$, coincide on all the vector fields (i.e., the coefficients for the forms differ by the elements of ideal $I_{\mathscr{E}}$).

LEMMA 5. If $\delta F \simeq 0$, then $F \in I_{\mathscr{E}}$ and its characteristic equals zero.

Proof. We shall use variables (35). The relation $\delta F \simeq 0$ signifies that $\xi F \in I_{\mathscr{E}}$ for every vector field ξ. Let $F = F_1 + F_2$, where F_1 depends only on variables (35a) and $F_2 \in I_{\mathscr{E}}$. If as ξ we take the partial derivatives with respect to the variables (35a), then we obtain $\partial F_1 / \partial u_i^{(s)} = 0, F_1 = 0$. Hence $F = \sum a_i^r \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)}$. But $\partial F / \partial \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} \in I_{\mathscr{E}}$. Then all $a_i^r \in I_{\mathscr{E}}$, and the characteristic equals zero.

THEOREM 9. If F is a first integral and $\{f_i\}$ is its characteristic, then the vector field corresponding to this first integral is

$$\xi_F = -\sum_{i,s} f_i^{(s)} \frac{\partial}{\partial u_i^{(s)}} = -\partial_{\bar{f}}. \tag{39}$$

Proof. Let $\Omega^{(2)}$ be defined by the exact equality

$$\frac{d\Omega^{(2)}}{dx} = -\sum \delta \frac{\delta \mathscr{L}}{\delta u_i} \wedge \delta u_i.$$

As before we represent $\Omega^{(2)}$ as $\Omega_*^{(2)} + \Omega_{\mathscr{E}}^{(2)}$, where $\Omega_*^{(2)}$ depends only on the fundamental variables (35a) and $\Omega_{\mathscr{E}}^{(2)}$ is a form of type $\sum a \delta \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} \wedge \delta q_j$, where the q_j are any coordinates. F can be reckoned as depending only on the fundamental variables. By Remark 2 to Theorem 8 in Paragraph 15 there exists a vector field ξ such that the exact equality $\delta F = -i(\xi)\,\Omega_*^{(2)}$ holds. Then

$$\delta F = -i(\xi)\,\Omega^{(2)} + i(\xi)\Omega_{\mathscr{E}}^{(2)}.$$

We apply d/dx to both sides. We note that $\frac{d}{dx} i(\xi)\,\Omega^{(2)} = i(\xi)\frac{d}{dx}\Omega^{(2)} + i\left(\left[\frac{d}{dx}, \xi\right]\right)\Omega^{(2)}$. But d/dx and ξ commute in $A_{\mathscr{E}}$, i.e., in the vector field [d/dx, ξ] all coordinates belong to $I_{\mathscr{E}}$. Its convolution with any form yields a form all of whose coefficients belong to $I_{\mathscr{E}}$, i.e., is equivalent to zero. We have

$$\delta \frac{dF}{dx} \simeq -i(\xi)\frac{d\Omega^{(2)}}{dx} + \frac{d}{dx} i(\xi)\,\Omega_{\mathscr{E}}^{(2)}$$

or

$$\delta \sum f_i^r \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} \simeq i(\xi) \sum \delta \frac{\delta \mathscr{L}}{\delta u_i} \wedge \delta u_i + \frac{d}{dx} i(\xi) \sum a \delta \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} \wedge \delta q_j.$$

Taking into account that $\xi \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} \in I_{\mathscr{E}}$, since $\xi \in TA_{\mathscr{E}}$, and discarding the forms equivalent to zero, we obtain

$$\delta \sum f_i^r \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} \simeq -\sum \xi_i^0 \delta \frac{\delta \mathscr{L}}{\delta u_i} - \frac{d}{dx} \sum (\xi q_j)\, a \delta \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} \simeq -\delta \left[\sum \xi_i^0 \frac{\delta \mathscr{L}}{\delta u_i} + \frac{d}{dx} \sum (\xi q_j)\, a \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)}\right].$$

i.e.,

$$\delta \left[\sum f_i^r \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)} + \sum \xi_i^0 \frac{\delta \mathscr{L}}{\delta u_i} - \frac{d}{dx} \sum (\xi q_j)\, a \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)}\right] \simeq 0.$$

By Lemma 5 the expression within the brackets has a zero characteristic. But the last term is an arbitrary element of the ideal and so its characteristic is zero. Therefore, the characteristic of $\sum f_i^{(r)} \left(\frac{\delta \mathscr{L}}{\delta u_i}\right)^{(r)}$ equals the characteristic of $-\xi_i^0 \frac{\delta \mathscr{L}}{\delta u_i}$, i.e.,

$$\xi_i^0 = -f_i = -\sum_r \left(-\frac{d}{dx}\right)^r f_i^{(r)}.$$

Now Eq. (39) follows from the fact that ξ_F commutes in $A_{\mathscr{L}}$ with d/dx; hence, $\xi_i^r = \left(\frac{d}{dx}\right)^r \xi_i^0$.

The preceding analysis was of a more or less general nature. We now turn to the case $\mathscr{L} = \sum_{p=1}^{p_0} c_p A_p \left(\frac{n-1}{n}\right)$. Equation (32) shows that the quantities J_q are the first integrals of system (33), with characteristics $f_r = \sum_{s=0}^{n-2} l_{rs}\left(\frac{\delta}{\delta u_s} \cdot \mathscr{M}_q\right)$ or $\mathfrak{f} = l \frac{\delta}{\delta u} \cdot \mathscr{M}_q$.

<u>THEOREM 10.</u> The first integrals J_q of system (33) are in involutions among themselves for any q.

<u>Proof.</u> By what was said at the end of Paragraph 10 the operators $\partial_{l \frac{\delta}{\delta u} \mathscr{M}_{q_1}}$ and $\partial_{l \frac{\delta}{\delta u} \mathscr{M}_{q_2}}$ commute.

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