ON THE QUESTION OF THE SOLVABILITY OF BISINGULAR AND POLYSINGULAR EQUATIONS

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Let C be the unit circle in the complex variable plane, $T = C \times C$, the operator S ($\in \mathrm{Hom}[L_2(T), L_2(T)]$)* has the form

$$(Sf)(t) = a_0(t) f(t) + \frac{1}{\pi i} \int_C \frac{a_1(t, \tau_1)}{\tau_1 - t_1} f(\tau_1, t_2)\, d\tau_1 + \frac{1}{\pi i} \int_C \frac{a_2(t, \tau_2)}{\tau_2 - t_2} f(t_1, \tau_2)\, d\tau_2$$
$$+ \left(\frac{1}{\pi i}\right)^2 \int_C \int_C \frac{a_{12}(t, \tau) f(\tau)\, d\tau_1 d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)}, \quad t = (t_1, t_2), \ \tau = (\tau_1, \tau_2), \ t \in T. \tag{1}$$

Concerning the kernels a_1, a_2, a_{12} we assume that they are expanded into absolutely-convergent Fourier series, i.e.,

$$a_1(t, \tau_1) = \sum_{i=1}^{+\infty} \overset{1}{\gamma}_i t^{\alpha_i} \tau_1^{\beta_i'}, \qquad a_2(t, \tau_2) = \sum_{i=1}^{+\infty} \overset{2}{\gamma}_i t^{\alpha_i} \tau_2^{\beta_i''}, \qquad a_{12}(t, \tau) = \sum_{i=1}^{+\infty} \overset{12}{\gamma}_i t^{\alpha_i} \tau^{\beta_i},$$

$$\alpha_i = (\alpha_i', \alpha_i''), \quad \beta_i = (\beta_i', \beta_i''), \ t^{\alpha_i} = t_1^{\alpha_i'} t_2^{\alpha_i''}, \quad \tau^{\beta_i} = \tau_1^{\beta_i'} \tau_2^{\beta_i''};$$

α_i', α_i'', β_i', β_i'' are integers and, moreover, $\sum_{i=1}^{+\infty} (|\overset{1}{\gamma}_i| + |\overset{2}{\gamma}_i| + |\overset{12}{\gamma}_i|) < +\infty$; the coefficient $a_0(t)$ is assumed continuous.

By A_0, A_i, B_i, B_i', B_i'' we denote the operators of multiplication by the functions $a_0(t)$, t^{α_i}, t^{β_i}, $\tau_1^{\beta_i'}$, $\tau_2^{\beta_i''}$, respectively, and by S_1, S_2 we denote the operators defined by the equalities $(S_1\varphi)(t) = \frac{1}{\pi i} \int_C \frac{\varphi(\tau_1, t_2)}{\tau_1 - t_1} d\tau_1$, $(S_2\varphi)(t) = \frac{1}{\pi i} \int_C \frac{\varphi(t_1, \tau_2)}{\tau_2 - t_2} d\tau_2$.

The operator S given by formula (1) is defined as the sum

$$S = A_0 + \sum_{i=1}^{+\infty} \overset{1}{\gamma}_i A_i S_1 B_i' + \sum_{i=1}^{+\infty} \overset{2}{\gamma}_i A_i S_2 B_i'' + \sum_{i=1}^{+\infty} \overset{12}{\gamma}_i A_i S_1 S_2 B_i.$$

It is not difficult to be convinced that such a definition of a bisingular operator coincides with generally accepted ones† when the latter have meaning, for example, when the kernels additionally satisfy a Hölder condition.

THEOREM. Let $\overset{1}{A}{}_\lambda^{\pm}$, $\overset{2}{A}{}_\lambda^{\pm}$ ($|\lambda| = 1$) be one-dimensional singular operators from $\mathrm{Hom}[L_2(C), L_2(C)]$, defined by the equalities

*$\mathrm{Hom}(B_1, B_2)$ is the space of linear operators acting from the Banach space B_1 into the Banach space B_2.
†A single integral is to be understood in the principal value (p.v.) sense, and a double integral as repeated p.v. integrals.

Rostov State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 5, No. 1, pp. 93-94, January-March, 1971. Original article submitted March 30, 1970.

$$(\overset{1}{A}{}_{\lambda}^{\pm}\varphi)(\theta)=[a_0(\theta,\lambda)\pm a_2(\theta,\lambda,\theta)]\varphi(\theta)+\frac{1}{\pi i}\int_C \frac{a_1(\theta,\lambda,\theta')\pm a_{12}(\theta,\lambda,\theta',\lambda)}{\theta-\theta'}\varphi(\theta')\,d\theta',$$

$$(\overset{2}{A}{}_{\lambda}^{\pm}\varphi)(\theta)=[a_0(\lambda,\theta)\pm a_1(\lambda,\theta,\theta)]\varphi(\theta)+\frac{1}{\pi i}\int_C \frac{a_2(\lambda,\theta,\theta')\pm a_{12}(\lambda,\theta,\lambda,\theta')}{\theta-\theta'}\varphi(\theta')\,d\theta'.$$

Then for operator S to be a Noetherian operator it is necessary and sufficient that the operators $\overset{1}{A}{}_{\lambda}^{\pm}$, $\overset{2}{A}{}_{\lambda}^{\pm}$ be invertible for all values of the parameter λ.

Proof. Let Z_2 be a discrete plane, i.e., the set of points on a plane with integral coordinates. By F we denote an operator from $\mathrm{Hom}[L_2(T), L_2(Z_2)]$* which associates with a function on a torus the double sequence of coefficients of its Fourier series; by $\tilde{Z}_2$ we denote the compactification of the discrete plane by an infinitely-distant sphere (see [1-3]) and we consider the operator $FSF^{-1}(\in \mathrm{Hom}[L_2(\tilde{Z}_2), L_2(\tilde{Z}_2)])$.

This operator is an operator of local type (see [1-5]).

Just as was done in [1-3] we carry out a local analysis of this operator at infinitely distant points. At finite points, by virtue of the discreteness of space Z_2, any operator from $\mathrm{Hom}[L_2(\tilde{Z}_2), L_2(\tilde{Z}_2)]$ is locally Noetherian.

We subdivide the set of infinitely distant points into eight parts: the four sets $\tilde{\Gamma}^{++}$, $\tilde{\Gamma}^{+-}$, $\tilde{\Gamma}^{-+}$, $\tilde{\Gamma}^{--}$, consisting of infinitely distant points corresponding to the rays issuing from the origin and being located in the open squares $E^{++}(x > 0, y > 0)$, $E^{+-}(x > 0, y < 0)$, $E^{-+}(x < 0, y > 0)$, $E^{--}(x < 0, y < 0)$, respectively, and the four sets consisting of one point $M_{+\infty,0}$, $M_{0,+\infty}$, $M_{-\infty,0}$, $M_{0,-\infty}$, corresponding to the rays $x > 0, y = 0$; $x = 0, y > 0$; $x < 0, y = 0$; $x = 0, y < 0$, respectively.

For each type of point the operator FSF^{-1} is locally equivalent to the simpler operator:

$$\pm\infty, 0)\ FSF^{-1}\overset{M_{\pm\infty,0}}{\sim}F\left(A_0\pm\sum_{i=1}^{+\infty}\overset{1}{\gamma}_i A_i B_i'+\sum_{i=1}^{+\infty}\overset{2}{\gamma}_i A_i S_2 B_i''\pm\sum_{i=1}^{+\infty}\overset{12}{\gamma}_i A_i S_2 B_i\right)F^{-1}; \tag{2}$$

$$0, \pm\infty)\ FSF^{-1}\overset{M_{0,\pm\infty}}{\sim}F\left(A_0+\sum_{i=1}^{+\infty}\overset{1}{\gamma}_i A_i S_1 B_i'\pm\sum_{i=1}^{+\infty}\overset{2}{\gamma}_i A_i B_i''\pm\sum_{i=1}^{+\infty}\overset{12}{\gamma}_i A_i S_1 B_i\right)F^{-1}; \tag{3}$$

$$\pm +)\ FSF^{-1}\overset{M}{\sim}F\left(A_0\pm\sum_{i=1}^{+\infty}\overset{1}{\gamma}_i A_i B_i'+\sum_{i=1}^{+\infty}\overset{2}{\gamma}_i A_i B_i''\pm\sum_{i=1}^{+\infty}\overset{12}{\gamma}_i A_i B_i\right)F^{-1}, M\in\tilde{\Gamma}^{\pm +}; \tag{4}$$

$$\pm -)\ FSF^{-1}\overset{M}{\sim}F\left(A_0\pm\sum_{i=1}^{+\infty}\overset{1}{\gamma}_i A_i B_i'-\sum_{i=1}^{+\infty}\overset{2}{\gamma}_i A_i B_i''\mp\sum_{i=1}^{+\infty}\overset{12}{\gamma}_i A_i B_i\right)F^{-1}, M\in\tilde{\Gamma}^{\pm -}. \tag{5}$$

We now note that the invertibility of the right hand sides of the equivalences (2)-(5) ensues from the theorem's hypotheses. Hence, on the basis of the results in [4, 5], it follows that the operator FSF^{-1} and, consequently, also the operator S, are Noetherian. The sufficiency of the conditions of the theorem is proved.

Necessity. If the operator S is a Noetherian operator, then the right hand sides of equivalences (2)-(5) are locally Noetherian at the corresponding points (see [4, 5]). Hence ensues the invertibility of these operators (see [1-3]).

Remark. From the proof it is clear that necessary and sufficient conditions for being Noetherian can be formulated also for polysingular equations, which would consist of the invertibility of the polysingular equations on a unity of lesser order.

The author thanks V. A. Kakichev for useful discussions on the work.

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*The measure of each point of Z_2 equals unity.

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