RECONSTRUCTION OF A DIFFERENTIAL EQUATION FROM THE SPECTRUM

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We consider the following boundary value problem on the interval $0 \le x \le \pi$:

$$(-1)^m y^{(2m)} + q(x) y = \lambda y, \tag{1}$$

$$y^{(2\nu)}(0) = y^{(2\nu)}(\pi) = 0, \quad \nu = 0, 1, \ldots, m-1, \tag{2}$$

with $m \ge 1$ and $q(x) \in L_2(0, \pi)$. Denote by $\{\lambda_n\}_1^\infty$ the sequence of eigenvalues of problem (1), (2). We pose the problem of determining the function q(x) from a given sequence $\{\lambda_n\}_1^\infty$ of eigenvalues. We shall assume that

$$q(x) = q(\pi - x). \tag{3}$$

Restriction (3) is natural, since replacing q(x) by $q(\pi - x)$ does not change the spectrum. It is not assumed that the function q(x) is real. Without loss of generality, we assume that

$$\int_0^\pi q(x)\, dx = 0. \tag{4}$$

Denote by $L_2'(0, \pi)$ the class of functions in $L_2(0, \pi)$, satisfying conditions (3) and (4). Uniqueness and existence of the solution to the inverse problem have been proved (see [1-4]) for the case when m = 1 under condition (3) and the natural assumption regarding the sequence $\{\lambda_n\}_1^\infty$. In this paper, for $m \ge 2$ it is assumed that the sequence $\{\lambda_n\}_1^\infty$ (in general, a complex sequence) deviates slightly from the sequence $\{n^{2m}\}_1^\infty$ of eigenvalues of problem (1), (2) with $q(x) \equiv 0$, and we prove existence and uniqueness of the corresponding function q(x) belonging to some ball $\|q\|_2 < \rho'$ in the space $L_2'(0, \pi)$. It is shown that outside this ball, in the general case, there can exist other functions q(x) generating the same spectrum.

The aim of this note is the following assertion, which we formulate for $m \ge 2$.

THEOREM. Let the sequence of numbers $\{\lambda_n\}_1^\infty$ be given, satisfying the condition

$$\left(\sum_{n=1}^\infty |\lambda_n - n^{2m}|^2\right)^{1/2} < \frac{1}{5}(2^{2m} - 1).$$

Then in the ball $\|q\|_2 < \sqrt{\frac{2}{\pi}}\left[(2^{2m}-1) - \left(\sum_{n=1}^\infty |\lambda_n - n^{2m}|^2\right)^{1/2}\right]$ of the space $L_2'(0, \pi)$ there exists a unique function q(x) such that $\{\lambda_n\}_1^\infty$ is the sequence of eigenvalues of problem (1), (2) with this function. Moreover, the function q(x) can be found by the method of successive approximations from the following equation:

$$q(x) = f(x) + 2\sum_{j=2}^\infty \underbrace{\int_0^\pi \cdots \int_0^\pi}_{j} \Phi(x, t_1, \ldots, t_j)\, q(t_1) \ldots q(t_j)\, dt_1 \ldots dt_j, \tag{5}$$

where

$$f(x) = -2\sum_{n=1}^\infty (\lambda_n - n^{2m}) \cos 2nx,$$
$$\Phi(x, t_1, \ldots, t_j) = \frac{2}{\pi}\sum_{n=1}^\infty [\sin nt_1 G_n(t_1, t_2) \ldots G_n(t_{j-1}, t_j) \sin nt_j] \cos 2nx, \tag{6}$$

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and $G_n(x,t)=\frac{2}{\pi}\sum_k \frac{\sin kx \sin kt}{\lambda_n - k^{2m}}$ (summation is carried out over k such that $k \neq n$ and $k+n$ is even).

1. For the series on the right-hand side of (5) to converge, it is sufficient that $\sqrt{\frac{2}{\pi}}\|q\|_2+\left(\sum_{n=1}^{\infty}|\lambda_n-n^{2m}|^2\right)^{1/2}<3^{2m}-1$. Equation (5) can be solved for q(x) by the method of successive approximations under the condition

$$\left(\sum_{n=1}^{\infty}|\lambda_n-n^{2m}|^2\right)^{1/2}<\frac{3^{2m}-1}{10\sqrt{2\pi}}. \tag{7}$$

2. We show that the solution q(x) of Eq. (5) is a solution to the inverse problem. In fact, from (5) we obtain that

$$\frac{1}{2}\int_0^{\pi} q(x)\cos 2nx\,dx=-\frac{\pi}{2}(\lambda_n-n^{2m})+\sum_{j=2}^{\infty}\underbrace{\int_0^{\pi}\cdots\int_0^{\pi}}_{j}\sin nt_1 G_n(t_1,t_2)\ldots G_n(t_{j-1},t_j)\sin nt_j q(t_1)\ldots q(t_j)\,dt_1\ldots dt_j,\ n=1,2,3,\ldots.$$

Setting

$$y_n(x)=\sin nx+\sum_{j=1}^{\infty}\int_0^{\pi}\cdots\int_0^{\pi}G_n(x,t_1)G_n(t_1,t_2)\ldots G_n(t_{j-1},t_j)\sin nt_j q(t_1)\ldots q(t_j)\,dt_1\ldots dt_j,$$

we find by direct verification that $y_n(x)$ satisfies the integral equation

$$y_n(x)=\sin nx+\int_0^{\pi}G_n(x,t)q(t)y_n(t)\,dt, \tag{8}$$

with $y_n^{(2\nu)}(0)=y_n^{(2\nu)}(\pi)=0,\ \nu=0,1,\ldots,m-1$. From the formulas given for $y_n(x)$ it follows that

$$\int_0^{2\pi}\sin nx y_n(x)[q(x)-\lambda_n+n^{2m}]\,dx=0,\ n=1,2,3,\ldots. \tag{9}$$

Differentiating (8) 2m times and taking into account (9) and the formula for $G_n(x,t)$, we finally obtain

$$(-1)^m y_n^{(2m)}(x)+q(x)\,y_n(x)=\lambda_n y_n(x),\quad n=1,2,3,\ldots.$$

3. We show that in the general case the solution to the inverse problem may not be unique. For simplicity, let m = 5. We specify the number sequences $\{\lambda_n'\}_1^{\infty}$ and $\{\lambda_n''\}_1^{\infty}$ in the following manner: $\lambda_n'=n^{10}$ ($n=1,2,3,\ldots$), $\lambda_1''=2^{10}$, $\lambda_2''=1$, $\lambda_n''=n^{10}$ ($n=3,4,5,\ldots$). It is essential that both sequences satisfy inequality (7). We set up two functions f(x) [see (6)]: $f_1(x)\equiv 0$ and $f_2(x)=-2\,(2^{10}-1)(\cos 2x-\cos 4x)$. From Eq. (5) we obtain in the first case $q_1(x)\equiv 0$, while in the second case $q_2(x)\not\equiv 0$, since $f_2(x)\not\equiv 0$.

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