AN EXTREMAL PROBLEM OF THE THEORY OF POSITIVE HERMITIAN FUNCTIONS

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1. Let F be the union of all real, continuous, positive Hermitian functions $\varphi(t)$ such that $\varphi(0) = 1$, $\varphi(t) = 0$ for $|t| \geq 1$. We let $\Omega(t) = \sup\{|\varphi(t)| \mid \varphi \in F\}$. It is obvious that $\Omega(0) = 1$, $\Omega(t) = 0$ for $|t| \geq 1$. It follows from one of Cramer's results (see [1]) that $\Omega(t) \leq 1 - (t^2/8)$ for $|t| < 1$. The fundamental result of this paper is the following:

THEOREM 1. $\Omega(t) = \cos[\pi/(n+1)]$ for $|t| \in [1/n, 1/(n-1))$, $n - 2, 3, \ldots$.

We note that the equality $\Omega(t) = 1/2$ for $|t| \in [1/2, 1)$ is proved by A. I. Il'inskii* using other means.

We need the following lemma, which we state without proof.

LEMMA. The union F coincides with the union of all functions $\varphi(t)$, admitting the representation

$$\varphi(t) = \int g(x)\overline{g(x+t)}\,dx, \tag{1}$$

where $g \in L^2(-\infty, \infty)$, $\|g\| = 1$, and $g(x) = 0$ for $x \notin [0, 1]$.

It follows from the lemma that upon evaluation of $\Omega(t)$ one can obtain the value $\sup |\varphi(t)|$ only with respect to those functions $\varphi(t) \in F$ for which the function $g(x)$ in (1) is nonnegative.

We denote by $L^2(0, 1)$ the union of all real functions of $L^2(-\infty, \infty)$ which equal zero outside of the interval $(0, 1)$. In $L^2(0, 1)$ we define the operators A_t, $t \in (-1, 1)$, by setting $(A_t f)(x) = \chi_0(x) f(x+t)$, where $\chi_0(x)$ is the indicator function of the interval $(0, 1)$, i.e., $\chi_0(x) = 1$ for $x \in (0, 1)$ and $\chi_0(x) = 0$ for $x \notin (0, 1)$. The operators $T_t = (A_t + A_{-t})/2$ are self-adjoint. It is easy to see that the expression on the right side of (1) can for real functions $g(x) \in L^2(0, 1)$ be written as $(T_t g, g)$. Therefore, the equality $\|T_t\| = \Omega(t)$ is valid. We will show that $\|T_t\| = \cos[\pi/(n+1)]$ for $|t| \in [1/n, 1/(n-1))$, $n = 2, 3, \ldots$.

We first consider the case where $t = 1/n$, $n = 2, 3, \ldots$. Let $f \in L^2(0, 1)$. We define the function $f_n(x)$ by setting

$$f_n(x) = \sum_{k=1}^{n} \xi_k \chi\left(x - \frac{k-1}{n}\right), \tag{2}$$

where $\chi(x)$ is the indicator function of the interval $(0, 1/n)$, and $\xi_k = \left(n \int_{(k-1)/n}^{k/n} |f(x)|^2 dx\right)^{1/2}$. It follows from the definition of $f_n(x)$ that $\|f_n\| = \|f\|$. We will write $f(x)$ in the form

$$f(x) = \sum_{k=1}^{n} \xi_k \psi_k\left(x - \frac{k-1}{n}\right), \tag{3}$$

where $\psi_k(x) = 0$ for $x \notin (0, 1/n)$. From (2) and (3) we find

$$4\|T_t f\|^2 = \sum_{k=1}^{n} \int_0^{1/n} (\xi_{k-1}\psi_{k-1}(x) + \xi_{k+1}\psi_{k+1}(x))^2\,dx. \tag{4}$$

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$$4\|T_t f_n\|^2 = \sum_{k=1}^{n} \int_0^{1/n} (\xi_{k-1}\chi(x) + \xi_{k+1}\chi(x))^2\,dx, \tag{5}$$

where $\xi_0 = \xi_{n+1} = 0$. Since $\|\xi_k\psi_k\| = \|\xi_n\chi\|$, from (4), (5), and the Cauchy–Bunyakovski inequality we conclude that $\|T_t f_n\| \geq \|T_t f\|$.

From the argument used above it easily follows that the norm of the operator T_t coincides with the norm of the operator $\widetilde{T}_t$, where $\widetilde{T}_t$ is the restriction of T_t to the subspace of functions of the form (2). The operator $\widetilde{T}_t$ can be regarded as an operator acting on the space R^n of vectors $\{\xi_k\}$. Its matrix has the form

$$\|a_{jk}\|_{j,k=1}^{n} = \|(\delta_{j+1,k} + \delta_{j-1,k})/2\|_{j,k=1}^{n},$$

where δ_{jk} is the Kronecker delta. The norm of the operator T_t coincides with the maximum root of the characteristic polynomial $P_n(x) = \det\|a_{jk} + x\delta_{jk}\|_{j,k=1}^{n}$. It is easily seen that $P_n = xP_{n-1} - (P_{n-2}/4)$, $P_1 = x$, $P_2 = x^2 - (1/4)$. Therefore, $P_n = 2^{-n}U_n$, where U_n is a Chebyshev polynomial of the second kind (see [2]), and consequently $\|\widetilde{T}_{1/n}\| = \cos[\pi/(n+1)]$.

We now consider the case $|t| \in [1/n, 1/(n-1))$, $n = 2, 3, \ldots$. We note that $\Omega(\vartheta t) \geq \Omega(t)$ for $\theta \in [-1, 1]$, since $\Omega(\theta t) = \sup\{|\varphi(\theta t)| : \varphi(t) \in F\} \geqslant \sup\{|\varphi(\theta t)| : \varphi(\theta t) \in F\} = \Omega(t)$. Therefore

$$\|T_t\| \leqslant \|T_{1\,n}\| = \|\widetilde{T}_{1\,n}\| = \cos[\pi/(n+1)]. \tag{6}$$

To obtain the bound on the norm of T_t we apply the operator T_t to the function

$$f_t(x) = \sum_{k=1}^{n} \chi(x - t(k-1))\,\xi_k,$$

where $\chi(x)$ is the indicator function of the interval $(0, 1 - t(n-1))$ and $\{\xi_k\}_{k=1}^{n}$ is the eigenvector of the operator $\widetilde{T}_{1/n}$ corresponding to the maximum eigenvalue. It is obvious that

$$\|T_t\| \geqslant \|T_t f_t\|\,\|f_t\| = \|\widetilde{T}_{1/n}\| = \cos[\pi/(n+1)]. \tag{7}$$

The statement of Theorem 1 follows from the bounds (6) and (7).

2. Let A be an open subset of R^n such that for all $x \in A$ and all λ, $|\lambda| \leq 1$, we have $\lambda x \in A$. We denote by F_A the set of all continuous, positive Hermitian functions $\varphi(t)$ in R^n such that $\varphi(0) = 1$ and $\varphi(t) = 0$ for $t \notin A$. Let $\Omega(t, A) = \sup\{|\varphi(t)| : \varphi \in F_A\}$, and let $p(t)$ be the Minkovski functional of the set $A(p(t) = \inf\{|\lambda| : \lambda^{-1}t \in A\})$. It is easy to see that $\Omega(t, A) = 0$ for $p(t) \geq 1$ and $\Omega(t, A) = 1$ for $p(t) = 0$.

<u>THEOREM 2.</u> $\Omega(t, A) = \cos[\pi/(n+1)]$ for $1/n \leq p(t) < 1/(n-1)$, $n = 2, 3, \ldots$.

<u>Proof.</u> Let e be a unit vector in R^n such that $p(e) \neq 0$ and let $\varphi \in F_A$. The bound

$$\sup\{|\varphi(\lambda e)| : \varphi \in F_A\} \leqslant \sup\{|\gamma(\lambda p(e))| : \gamma \in F\} = \Omega(\lambda p(e)), \tag{8}$$

is obvious, where $\Omega(t)$ and F are defined as in Sec. 1.

Now let $\lambda \in \{\lambda : 1/n \leq p(\lambda e) < 1/(n-1)\}$; let the function $\alpha(t) \in F_A$ be such that $\alpha(t) = 0$ for $|t| \geq \varepsilon$; and let $\beta(t) \in F$. Since the sequence $\{\beta(kp(\lambda e))\}$, $k = 0, \pm 1, \pm 2, \ldots$, is positive Hermitian, the function

$$\psi_\lambda(t) = \sum_k \beta(kp(\lambda e))\,\alpha(t - k\lambda e)$$

for sufficiently small $\varepsilon > 0$ belongs to the set F_A and the equality $\psi_\lambda(\lambda e) = \beta(p(\lambda e))$ is valid. From this we find that

$$\sup\{|\varphi(\lambda e)| : \varphi \in F_A\} \geqslant \sup\{|\beta(\lambda p(e))| : \beta \in F\} = \Omega(\lambda p(e)). \tag{9}$$

Theorem 2 follows from inequalities (8) and (9).

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<u>Remark.</u> After the paper was submitted for publication, the author became aware of the article of O. Szasz'a ["Über harmonische Functionen and L-Form," Math. Z., 1, 149-162 (1918)] in which a result similar to Theorem 1 is obtained.

LITERATURE CITED

1. H. Cramer, Random Variables and Probability Distributions, Cambridge Univ. Press (1970).
2. G. Szegö, Orthogonal Polynomials, Am. Math. Soc. (1967).