HOMOGENEOUS COMPACT ALMOST-CONTACT MANIFOLDS

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1. By an almost-contact structure on a (2n + 1)-dimensional manifold M is meant a reduction of the bundle of bases of the manifold M to the subgroup $U(n) \times 1$ (see [1]).

Let π: $M \to B$ be a principal T-bundle (T is a one-dimensional torus); an almost-contact structure on M associated with this bundle will mean a pair (η, J), where η is a connection on the bundle π, and J is an almost-contact structure on B. A triple (η, J, g), where (J, g) is an almost-hermitian metric on B, will be called an almost-contact metric structure associated with π. Under certain restrictions any almost-contact structure is uniquely associated with some principal T-bundle.

We shall say that the connection η is normal relative to J, if $\Omega(JX, JY) = \Omega(X, Y)$, where $\pi^*\Omega = d\eta$. An almost-contact structure (η, J) will be called normal, if J is integrable, and η is normal relative to J. We shall call an almost-contact metric structure quasisasaki, if it is normal and the hermitian form F of the metric (J, g) is closed, and sasaki, if it is normal and $d\eta = \pi^*F$.

One can show that for associated structures our definitions coincide with the standard ones, which one can find, e.g., in [2].

2. We shall call an almost-contact structure on M homogeneous, if one can find a Lie group of automorphisms of the corresponding $(U(n) \times 1)$-subbundle of the bundle of bases, transitive on M.

Let $K_1 \subset K$ be subgroups of the group G; we shall call the subgroup K_1 rigidly imbedded in the subgroup K, if K_1 is a normal subgroup of K and the factor-group K/K_1 is abelian. We shall call a subgroup $K \subset G$ regular, if it can be rigidly imbedded in a subgroup of maximal rank. The number, equal to rank G − rank K, will be called the defect of the regular subgroup K.

THEOREM 1. If a regular homogeneous space admits an invariant almost-contact structure, then the homogeneous base of the principal T-bundle defined by this structure admits an invariant almost-complex structure.

Homogeneous compact almost-complex spaces of positive Euler−Poincaré characteristic, i.e., factors by subgroups of maximal rank, have been studied in sufficient detail. A summary account can be found in [3]. Homogeneous complex spaces of semisimple Lie groups are regular and are described in [4]. The classification of regular almost-complex homogeneous spaces is completed by the following theorem.

THEOREM 2. The homogeneous space $E_8/3A_2$ is the only one, up to conjugacy, which is an almost-complex noncomplex regular homogeneous space of zero characteristic of a simple compact Lie group.

In the class of regular homogeneous spaces is contained all homogeneous spaces of semisimple compact Lie groups, which admit normal, quasisasaki and sasaki almost-contact structures.

THEOREM 3. In order that a homogeneous space of a semisimple compact Lie group admit a normal almost-contact structure, it is necessary and sufficient that the isotropy subgroup be rigidly imbedded with odd defect in the centralizer of some torus.

COROLLARY. Regular homogeneous almost-contact spaces of simple compact Lie groups, which do not admit a normal almost-contact structure, up to conjugacy, are exhausted by the following: $E_7/3A_2$,

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$E_8/A_5 + A_2$, $E_8/3A_2 + A_1$, $E_8/3A_2 + T$.

Quasisasaki and sasaki structures are analogs of Kähler structures and Hodge structures in the theory of almost-hermitian structures. Homogeneous compact Kähler manifolds and Hodge manifolds coincide and are exhausted by factors by centralizers of tori. In the almost-contact case the situation is essentially different.

Let $\Psi = \{\psi_1, \ldots, \psi_l\}$ be a system of simple roots of the semisimple compact Lie group G and let $\Psi_K = \{\psi_{k+1}, \ldots, \psi_l\}$ be the subsystem of simple roots corresponding to some centralizer of a torus K. Any differential of a character is a purely imaginary linear form $\eta: C(\mathfrak{K}) \to \sqrt{-1}\mathbf{R}$, where $C(\mathfrak{K})$ is the center of the algebra $\mathfrak{K}$, since on $\mathfrak{K}'$ any differential is identically equal to zero. Any differential can be written in the following form: $\eta = \sum_{i=1}^{k} a_i\bar{\omega}_i$, where $a_i \in \mathbf{Z}$, $\{\bar{\omega}_i\}_{1 \leqslant i \leqslant l}$ is the basis dual to $\{\psi_i^{\vee}\}_{1 \leqslant i \leqslant l}$.

THEOREM 4. For a homogeneous space of a semisimple compact Lie group to admit an invariant quasisasaki structure, it is necessary and sufficient that the isotropy subgroup be rigidly imbedded in the centralizer of some torus with defect one, i.e., be the kernel of some character of this centralizer.

THEOREM 5. Homogeneous spaces of the form G/Ker $\bar{\eta}$, where $\bar{\eta}$ runs through the characters of all centralizers of tori, whose differentials have the form $\eta = \sum_{i=1}^{k} a_i\bar{\omega}_i$, where $a_i \neq 0$ for any i, and $\Psi - \Psi_K = \{\psi_1, \ldots, \psi_k\}$ is a basis of the center of this centralizer, exhaust homogeneous sasaki spaces with semisimple compact group G.

Theorem 5 completes the classification of spaces whose study was started in [5].

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