# THE THEORY OF LINEAR RELATIONS AND SPACES WITH AN INDEFINITE METRIC

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Let H be a Hilbert space with scalar product (x, y); let $H = H_+ \oplus H_-$ be an orthogonal decomposition of the space; let $P_\pm$ be the orthoprojectors onto $H_\pm$ ($P_+ + P_- = I$). We set $J = P_+ - P_-$ and introduce in H a new scalar product $[x, y] = (Jx, y)$, which is in general indefinite. A Hilbert space H in which along with the scalar product (x, y) we consider such an indefinite scalar product [x, y] will be called a J-space.

There are many works which deal with the geometry of and operator theory in J-spaces (see [1-7]). We will preserve the notation and terminology in these works.

Many assertions in the theory of operators take on a more complete character if we introduce the concept of a linear relation, which generalizes the concept of the graph of an operator. We mention here, for example, the theory of extensions of operators (see [8-11]) and the theory of extensions of differential operators in a space of vector functions (see [12]). The theory of linear relations in linear spaces was developed by MacLane in [13] and Arens in [14], and in Hilbert spaces and J-spaces it was developed by Arens in [14], Coddington in [9-11], Bennevitz in [15], and Glukhov in [16] and [17].

The present work is devoted to an investigation of some important classes of linear relations in J-spaces.

## §1. Linear Relations in Linear Spaces

1. Let E and E' be linear spaces, and let $\mathbf{E} = E \dotplus E'$ be their direct sum, defined as the collection of pairs $\langle x, x' \rangle$ ($x \in E$, $x' \in E'$) with the natural linear operations. By a linear relation (hereafter denoted by l.r.) $E \to E'$ we mean an arbitrary lineal in $\mathbf{E}$. For an arbitrary l.r. $A$: $E \to E'$ we call $\mathfrak{D}(A) = \{x \in E: \exists x' \in E', \langle x, x' \rangle \in A\}$ the domain of A; $\ker A = \{x \in E: \langle x, 0 \rangle \in A\}$ is called the kernel of A; $\mathfrak{R}(A) = \{x' \in E': \exists x \in E, \langle x, x' \rangle \in A\}$ is called the range of A; $\operatorname{ind} A = \langle x' \in E': \{0, x'\rangle \in A\}$ is called the indeterminacy of A.

An l.r. A: $E \to E'$ can be regarded as a many-valued mapping from E into E' if to each $x \in \mathfrak{D}(A)$ ($\subset E$) we associate all $x' \in E'$ such that $\langle x, x' \rangle \in A$. In particular, $A0 = \operatorname{ind} A$. If L is a lineal in E, then by definition AL is the union of all $Ax$ ($\forall x \in \mathfrak{D}(A) \cap L$). The set AL is a lineal in E' which contains ind A.

Let $\mathbf{E}^{\#} = E' \dotplus E$. If A is an l.r. $E \to E'$, then the inverse l.r. $A^{-1}$: $E' \to E$ is defined as the set of all pairs $\langle x', x \rangle \in \mathbf{E}^{\#}$ such that $\langle x, x' \rangle \in A$. We note that $\mathfrak{D}(A) = \mathfrak{R}(A^{-1})$, $\ker A = \operatorname{ind} A^{-1}$.

2. Hereafter we assume that all linear spaces which we encounter are Hilbert spaces. A direct sum $\mathbf{H} = H \dotplus H'$ of two such spaces H and H' is assumed to be orthogonal and will be written as $\mathbf{H} = H \oplus H'$. An l.r. A: $H \to H'$ is said to be closed if the corresponding lineal is closed. For such an l.r. the sets ker A and ind A are also closed.

## §2. Linear Relations in J-Spaces

1. An orthogonal decomposition $H = H_+ \oplus H_-$ of a J-space H as described in the introduction is said to be canonical.

Let H and H' be J-spaces and let $H = H_+ \oplus H_-$ and $H' = H'_+ \oplus H'_-$ be canonical decompositions of them. In the Hilbert space $\mathbf{H} = H \oplus H'$, which consists of all pairs $\langle x, x' \rangle$ ($x \in H$, $x' \in H'$), we introduce an

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Odessa Institute of Marine Engineers. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 10, No. 1, pp. 67-72, January-March, 1976. Original article submitted August 20, 1974.

indefinite scalar product as follows: if $\mathrm{x} = \langle x, x' \rangle$, $\mathrm{y} = \langle y, y' \rangle \in \mathrm{H}$, then

$$[\mathrm{x}, \mathrm{y}] = [x, y] - [x', y']. \tag{1}$$

It is easy to show that H is a J-space and the components of its canonical decomposition $\mathrm{H} = \mathrm{H}_+ \oplus \mathrm{H}_-$ are of the form $\mathrm{H}_+ = H_+ \oplus H'_-$, $\mathrm{H}_- = H'_+ \oplus H_-$.

The J-space H in which the indefinite metric is given by formula (1) will be denoted by $H \tilde{\oplus} H$ (and then the canonical decomposition of an arbitrary J-space H should be written in the form $H = H_+ \tilde{\oplus} H_-$).

The J-space $H' \tilde{\oplus} H$ will be denoted by $H^{\#}$. The operation #, which maps H onto $H^{\#}$ by the formula $\langle x, x' \rangle^{\#} = \langle x', x \rangle$, is anti-isometric:

$$[\langle x, x' \rangle, \langle y, y' \rangle]_{H \tilde{\oplus} H'} = -[\langle x', x \rangle, \langle y', y \rangle]_{H' \tilde{\oplus} H}.$$

2. In geometry and the theory of operators we will use the terminology and notation in [5-7]. In particular, J-orthogonality of vectors and lineals will be denoted by the symbol $\perp$; the J-orthogonal complement of a lineal L will be denoted by $L^{\perp}$.

Let H and H' be J-spaces, with $\mathrm{H} = H \tilde{\oplus} H'$. The graphs of operators from H into H' in some class or other can be interpreted conveniently in terms of the geometry of J-spaces. Thus, an operator T is J-expanding (J-contracting, J-isometric) if and only if its graph is a nonpositive (nonnegative, neutral) lineal in H. In this regard we introduce the following definition.

Definition. An l.r. T: $H \to H'$ is said to be J-expanding (J-contracting, J-isometric) if the lineal T ($\subset$ H) is nonpositive (nonnegative, neutral).

Such an l.r. is characterized by the fact that for an arbitrary $\langle x, x' \rangle \in T$ we have $[x', x'] \geq (\leq, =) [x, x]$. Therefore, the lineal ker T is nonpositive (nonnegative, neutral), and the lineal ind T is nonnegative (nonpositive, neutral).

If T is a J-contracting (J-expanding, J-isometric) l.r. $H \to H'$, then the l.r. $T^{-1}$: $H' \to H$ is J-expanding (J-contracting, J-isometric).

Let T be an l.r. $H \to H'$. The subspace $(T^{\perp})^{-1}$ in the J-space $H^{\#}$ is called the linear relation which is conjugate to T and is denoted by $T^c$.

Definition. A closed l.r. T: $H \to H'$ is said to be J-biexpanding (J-bicontracting) if T and $T^c$ are J-expanding (J-contracting) l.r.'s.

THEOREM 1. An l.r. T is J-biexpanding (J-bicontracting) if and only if the subspace T is maximal nonpositive (maximal nonnegative) in H.

We will carry out the proof for the J-biexpanding case. A subspace T is maximal nonpositive if and only if T is nonpositive and $T^{\perp}$ is nonnegative; nonnegativity of $T^{\perp}$ is equivalent to nonpositivity of $T^c = (T^{\perp})^{-1}$.

Definition. An l.r. T: $H \to H'$ is said to be J-semi-unitary (J-unitary) if T is a maximal neutral (hypermaximal neutral) subspace of $H \tilde{\oplus} H'$.

A J-unitary l.r. T is characterized by each of the following conditions: 1) $T^{\perp} = T$; 2) $T^c = T^{-1}$ (see [14, 17]). If T is J-unitary, then so is $T^{-1}$.

THEOREM 2. If T is a J-unitary l.r., then a) $\ker T = \mathfrak{D}(T)^{\perp}$; b) $\operatorname{ind} T = \mathfrak{R}(T)^{\perp}$.

Proof. It was shown in [14] and [16] that $\mathfrak{D}(T)^{\perp} = \operatorname{ind} T^c$. Since $T^c = T^{-1}$, assertion a) is proved. Assertion b) is obtained by considering the J-unitary l.r. $T^{-1}$.

## §3. Fractional-Linear Transformations of Linear Relations in J-Spaces

In [18] Potapov considered a fractional-linear transformation (f.l.t.) which takes J-contracting matrices to contracting matrices. This transformation was generalized by Ginzburg in [1] and [19] to the case of bounded J-bicontracting operators in an infinite-dimensional J-space. But even in the finite-dimensional case not every contraction is the image of some J-bicontracting operator. In the present section we will show that the aforementioned f.l.t. can be extended to all J-bicontracting l.r.'s. Here the images of these l.r.'s exhaust the set of all contractions. And the transformation has a natural interpretation in terms of the geometry of J-spaces.

Let T be a J-bicontracting l.r. $H \to H'$, that is, a maximal nonnegative subspace in $\mathbf{H} = H \tilde{\oplus} H'$. Let $K = K_T$ be the angular operator of T with respect to $\mathbf{H}_+$. This operator is a contraction from $\mathbf{H}_+ = H_+ \oplus H'_-$ into $\mathbf{H}_- = H'_+ \oplus H_-$. If $\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is the matrix representation of the operator K with respect to the above decompositions of $\mathbf{H}_+$ and $\mathbf{H}_-$, then the l.r. T consists of precisely those pairs $\langle x, x' \rangle$ which admit a representation

$$x = z_1 + K_{21}z_1 + K_{22}z_2, \quad x' = z_2 + K_{11}z_1 + K_{12}z_2, \tag{2}$$

where $z_1$ and $z_2$ run independently through $H_+$ and $H'_-$, respectively. If we set $z_1 + z_2 = z \ (\in \mathbf{H}_+)$, we obtain the equivalent formulas

$$x = \begin{pmatrix} I & 0 \\ K_{21} & K_{22} \end{pmatrix} z, \quad x' = \begin{pmatrix} K_{11} & K_{12} \\ 0 & I \end{pmatrix} z. \tag{3}$$

Conversely, each contraction $K$, $\mathfrak{D}(K) = \mathbf{H}_+$, $\mathfrak{R}(K) \subset \mathbf{H}_-$, generates by means of (3) a set of pairs $\langle x, x' \rangle$ which constitutes a J-bicontracting l.r.

From (3) it is easy to deduce equations for the lineals associated with the l.r. T:

$$\mathfrak{D}(T) = \mathfrak{R}\left(\begin{pmatrix} I & 0 \\ K_{21} & K_{22} \end{pmatrix}\right), \quad \mathfrak{R}(T) = \mathfrak{R}\left(\begin{pmatrix} K_{11} & K_{12} \\ 0 & I \end{pmatrix}\right);$$

$$\ker T = \{z_1 + K_{21}z_1 \colon z_1 \in \ker K_{11}\}, \quad \operatorname{ind} T = \{z_2 + K_{12}z_2 \colon z_2 \in \ker K_{22}\};$$
$$\dim \ker T = \dim \ker K_{11}, \quad \dim \operatorname{ind} T = \dim \ker K_{22}.$$

From these equations we can determine the connection between the properties of a J-bicontracting l.r. T and the corresponding contraction K.

THEOREM 3. a) T is an operator (i.e., ind $T = 0$) $\Leftrightarrow$ $K_{22}$ is a monomorphism;

b) $\mathfrak{D}(T) = H \Leftrightarrow \mathfrak{R}(K_{22}) = H_-$;

c) $\overline{\mathfrak{D}(T)} = H \Leftrightarrow \overline{\mathfrak{R}(K_{22})} = H_- \Leftrightarrow K^*_{22}$ is a monomorphism;

d) $\mathfrak{D}(T)$ is closed $\Leftrightarrow \mathfrak{R}(K_{22})$ is closed;

e) $\ker T = 0 \Leftrightarrow K_{11}$ is a monomorphism;

f) $\mathfrak{R}(T) = H' \Leftrightarrow \mathfrak{R}(K_{11}) = H'_+$;

g) $\overline{\mathfrak{R}(T)} = H' \Leftrightarrow \overline{\mathfrak{R}(K_{11})} = H'_+ \Leftrightarrow K^*_{11}$ is a monomorphism;

h) $\mathfrak{R}(T)$ is closed $\Leftrightarrow \mathfrak{R}(K_{11})$ is closed.

COROLLARY. An l.r. T is a bounded operator which is defined everywhere if and only if $K_{22}$ is an isomorphism of $H'_-$ onto $H_-$.

In this case it follows from (3) that

$$\begin{pmatrix} K_{11} & K_{12} \\ 0 & I \end{pmatrix} z = x' = Tx = T \begin{pmatrix} I & 0 \\ K_{21} & K_{22} \end{pmatrix} z,$$

i.e.,

$$T = \begin{pmatrix} K_{11} & K_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ K_{21} & K_{22} \end{pmatrix}^{-1} = \begin{pmatrix} K_{11} - K_{12}K_{22}^{-1}K_{21} & K_{12}K_{22}^{-1} \\ -K_{22}^{-1}K_{21} & K_{22}^{-1} \end{pmatrix}.$$

This last formula is a fractional-linear transformation of the contraction K with a continuously invertible $K_{22}$ into a J-bicontracting operator T. The inverse transformation is of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \to \begin{pmatrix} T_{11} - T_{12}T_{22}^{-1}T_{21} & T_{12}T_{22}^{-1} \\ -T_{22}^{-1}T_{21} & T_{22}^{-1} \end{pmatrix}.$$

The correspondence $T \leftrightarrow K_T$ is the aforementioned fractional-linear transformation of Potapov and Ginzburg which connects bounded J-bicontracting operators with contractions.

Remark. It is precisely J-unitary l.r.'s T which correspond to unitary operators K from $\mathbf{H}_+$ onto $\mathbf{H}_-$. From an assertion of Spitkovskii in [20]* and parts d) and h) of Theorem 3 it follows that for a J-

*In [20] the author considered matrices of unitary operators acting in a single Hilbert space. But the result which we mention here carries over easily to the case of unitary operators acting from one Hilbert space into another.

unitary l.r. T the lineals $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ can only be closed simultaneously. By virtue of parts b) and f) of Theorem 3 we know that a J-unitary l.r. T is the graph of a J-unitary operator from H onto H' if and only if $\mathfrak{R}(K_{11}) = H'_+$, $\mathfrak{R}(K_{22}) = H_-$.

## §4. A Generalized Fractional-Linear Transformation Generated by a Linear Relation

Let $H = H_+ \tilde{\oplus} H_-$ be a canonical decomposition of the J-space H, and let $\mathfrak{K}$ be the set of contractions from $H_-$ into $H_+$. In [21] it was shown that each bounded, everywhere-defined J-bicontracting operator T in H generates an f.l.t. $Z \to Z' = \varphi_T(Z)$ of the set $\mathfrak{K}$ into itself. Here, if $L_Z$ is a maximal nonpositive subspace with angular operator Z with respect to $H_-$, then $\varphi_T(Z)$ is the angular operator of the subspace $TL_Z$ with respect to $H_-$. In the present section we will generalize this assertion to J-bicontracting l.r.'s.

Let $H = H_+ \tilde{\oplus} H_-$ and $H' = H'_+ \tilde{\oplus} H'_-$ be canonical decompositions of two J-spaces H and H'; let $\mathfrak{K}$ and $\mathfrak{K}'$ respectively be the sets of contractions from $H_-$ into $H_+$ and from $H'_-$ into $H'_+$; let T be a J-bicontracting l.r. $H \to H'$, and let K be its angular operator, as in §3. We let $\mathfrak{K}_T$ denote the set of all $Z \in \mathfrak{K}$ for which the operator $I - ZK_{21}$ is continuously invertible. In particular, this set contains all $Z \in \mathfrak{K}$ for which $\|Z\| < 1$. (If $\|K_{21}\| < 1$, then $\mathfrak{K}_T = \mathfrak{K}$.)

Let $Z \in \mathfrak{K}_T$. The lineal $TL_Z = T(L_Z \cap \mathfrak{D}(T))$ is clearly nonpositive. We will show that this lineal is a maximal nonpositive subspace in H', and we will find its angular operator with respect to $H'_-$. By virtue of (2) the set $L_Z \cap \mathfrak{D}(T)$ consists of those $x = u + Zu \in L_Z$, for which $u = K_{21}z_1 + K_{22}z_2$, $Zu = z_1$. Therefore, $z_1 = ZK_{21}z_1 + ZK_{22}z_2$, $(I - ZK_{21})z_1 = ZK_{22}z_2$, and thus $z_1 = (I - ZK_{21})^{-1} \times ZK_{22}z_2$, where $z_2$ runs through $H'_-$. According to (2) the lineal $T(L_Z \cap \mathfrak{D}(T))$ consists of all x' of the form

$$x' = z_2 + K_{11}(I - ZK_{21})^{-1}ZK_{22}z_2 + K_{12}z_2 = z_2 + Z'z_2,$$

where

$$Z' = K_{12} + K_{11}(I - ZK_{21})^{-1}ZK_{22} \ (\in \mathfrak{K}'). \tag{4}$$

Since $z_2$ runs through all of $H'_-$, we know that $TL_Z$ is a maximal nonpositive subspace in H' with angular operator Z' with respect to $H'_-$.

Remark 1. Formula (4) can be written in the form

$$Z' = K_{12} + K_{11}Z(I - K_{21}Z)^{-1}K_{22},$$

where the operator $I - K_{21}Z$ is continuously invertible together with $I - ZK_{21}$.

Remark 2. Formula (4) is a generalization of the f.l.t. formula

$$Z' = \varphi_T(Z) = (T_{11}Z + T_{12})(T_{21}Z + T_{22})^{-1}$$

in [21] and can be transformed into it if $K_{22}$ is a continuously invertible operator (that is, if T is an everywhere-defined, bounded J-bicontracting operator). Here we use the formulas in §3 which connect the matrix entries of the operators T and $K_T$.

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