LAGRANGIAN AND LEGENDRIAN SINGULARITIES

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INTRODUCTION

By a Lagrangian submanifold of a cotangent foliation, we mean a submanifold of the largest possible dimension on which the standard symplectic form of the cotangent foliation vanishes. Lagrangian mappings are projection mappings of Lagrangian submanifolds onto the base. Singularities of Lagrangian mappings are encountered in the study of the structure of caustics, in the study of the asymptotic behavior of integrals depending on parameters, and so on.

By a Legendrian submanifold of a projectivized cotangent foliation, we mean an integral manifold of the standard contact structure of the foliation having the largest possible dimension. Legendrian mappings are projection mappings of Legendrian manifolds onto the base. Singularities of Legendrian mappings are encountered in the study of the structure and bifurcations of wave fronts, in the study of singularities of solutions of partial differential equations, etc.

The purpose of this note is to construct local normal forms for Lagrangian and Legendrian singularities in general position when the dimension of the manifold being mapped does not exceed 10.

In §1 for a germ of a Lagrangian submanifold we construct a germ of a family of functions, depending on parameters and called generating, such that the action of the group of Lagrangian diffeomorphisms is equivalent to the action on the generating functions of the group consisting of right changes of coordinates and addition with functions of parameters.

In §2 generating families are constructed for Legendrian manifolds. Here, close germs of Legendrian manifolds are Legendre equivalent if and only if the germs of the generating families are contact equivalent.

Hence, one obtains theorems, stated by Arnol'd [2] and Guckenheimer [1], to the effect that Lagrangian (Legendrian) stabilityof Lagrangian (Legendrian) manifolds follows from infinitesimal Lagrangian (Legendrian) stability (§3).

In §4 we list the normal forms of generating families of Lagrangian and Legendrian mappings $R^n \to R^n$, $n < 11$ ($R^n \to R^{n+1}$ for the Legendrian case) in general position.

Starting with $n = 6$, we inherently encounter unstable germs. Here, since Lagrangian (Legendrian) diffeomorphisms preserve the affine (projective) structure of a fiber of a Lagrangian (Legendrian) foliation, the normal forms have moduli that are functions of parameters.

All objects are assumed to be C^∞ smooth.

The author expresses his sincere thanks to V. I. Arnol'd for his constant attention to this work.

§1. Lagrangian Generating Families

Recall that by a Lagrangian equivalence of a foliation T^*M^n, where M^n is a smooth manifold, we mean a diffeomorphism of T^*M^n that preserves the symplectic structure and structure of the foliation. Lagrangian mappings are said to be Lagrange equivalent if there exists a Lagrangian equivalence that carries the corresponding Lagrangian manifolds into each other. Henceforth, we shall talk about Lagrangian equivalence of Lagrangian manifolds.

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 10, No. 1, pp. 26-36, January-March, 1976. Original article submitted July 23, 1975.

According to Darboux's theorem, all Lagrangian foliations are locally Lagrange equivalent, and so we shall consider the standard foliation $\pi: T^*R^n \to R^n$ with coordinates $q \in R^n$, $p \in T^*_{q_0}R^n$ and form $\omega = dp \wedge dq$.

1. The following assertions follow from the definition of Lagrangian equivalence:

Assertion 1. Let $\mathcal{L}$ be a Lagrangian equivalence of T^*R^n, and let the form $\alpha = pdq$. Then:

1) There exists a function Φ: $R^n \to R$, unique up to addition of a constant, such that $\mathcal{L}^* \circ \alpha - \alpha = d\Phi$;

2) $\mathcal{L}$ is uniquely defined by the pair (Θ, Φ), where Θ: $R^n \to R^n$ and $\Theta \circ \pi = \pi \circ \mathcal{L}$ is the induced diffeomorphism of the base.

Proof. We shall find an explicit form for the Lagrangian equivalence in the coordinates p and q. If $\mathcal{L}: (p, q) \mapsto (P, Q)$, then $Q = \Theta(q), P = (\Theta^{*})^{-1}\left(p + \frac{\partial \Phi}{\partial q}\right)$. We shall write $\mathcal{L} = (\Theta, \Phi)$.

2. A germ of a Lagrangian manifold (L, m), $L \subset T^*R^n$, is well projected onto the base and defined in some neighborhood of a point m of the generating function $F_L(q)$ by $p = \partial F/\partial q$.

It can be verified immediately that a germ of the Lagrangian manifold $(\mathcal{L}(L), \mathcal{L}(m))$, where $\mathcal{L}$ is a Lagrangian equivalence, has generating function

$$F_{\mathcal{L}(L)} = (F_L + \Phi) \circ \Theta^{-1} \tag{1.1}$$

and, in particular, is well projected onto the base.

3. Hörmander's Construction [3]. Let ρ: $R^{n+k} \to R^n$ be the foliation $\rho: R^{n+k} \to R^n$. We denote by A^n the subfoliation in T^*R^{n+k}, induced by ρ. The fiber over $x = (q, u)$ is the set of forms $\beta \in T^*_x R^{n+k}$ that annihilate the tangent space to $\rho^{-1}(q)$. Let ρ_1: $A^n \to T^*R^n$ be the induced mapping of the foliations and $i_1: A^n \to T^*R^{n+k}$ the imbedding.

Assertion 2. Let $(\tilde{L}, w)$ be a germ of a Lagrangian manifold $\tilde{L} \subset T^*R^{n+k}$ which is well projected onto R^{n+k} and which intersects A^n transversely at w. Then: a) $(\rho_1(\tilde{L} \cap A^n), \rho_1(w))$ is a germ of a Lagrangian manifold $L \subset T^*R^n$; b) the generating function F(q, u) of $(\tilde{L}, w)$ at $\pi(w)$ satisfies the conditions

$$\left.\frac{\partial F}{\partial u}\right|_{\pi(w)} = 0, \quad \operatorname{rank}\left(\frac{\partial^2 F}{\partial u\, \partial u}, \frac{\partial^2 F}{\partial u\, \partial q}\right) = k.$$

Proof. Let p and v be the coordinates dual to q and u in T^*R^{n+k}. Then A^n is defined in T^*R^{n+k} by $v = 0$. By virtue of the transversality, $\tilde{L} \cap A^n$ is a submanifold in A^n. Let us prove that ρ_1 is regular in $\tilde{L} \cap A^n$. For otherwise there would exist a vector ξ tangent to $\tilde{L} \cap A^n$ with coordinates $p_\xi = q_\xi = 0$, $v_\xi = 0$. The hyperplane Ann ξ of vectors skew-orthogonal to ξ is not transversal to A^n, and so $T_w(\tilde{L}) \subset \text{Ann}\ \xi$ would not be transversal to A^n. The obvious relation $p_1^* \circ \omega = 0$ completes the proof of a. Condition b is the coordinate form of the hypotheses of the assertion.

Definition. A germ (F(q, u), x) of the family of functions of $u \in R^k$ with parameters $q \in R^n$ satisfying Assertion 2b at x is called a generating family of the germ of the Lagrangian manifold $(L, m) = (\rho_1(\tilde{L} \cap A^n), \rho_1(w))$, where $(\tilde{L}, w)$ has the generating function F(q, u).

4. Consider the subgroup Λ of Lagrangian equivalences of T^*R^{n+k} that leave A^n invariant. Now $\Lambda = \{(\Theta, \Phi)\}$, where Θ and Φ satisfy the following condition:

(B) Θ: $R^{n+k} \to R^{n+k}$ is a diffeomorphism, and Φ: $R^{n+k} \to R$ preserves ρ; i.e., there exist a diffeomorphism $\tilde{\Theta}$: $R^n \to R^n$ and a function $\tilde{\Phi}$: $R^n \to R$ such that $\Phi = \tilde{\Phi} \circ \rho$, $\rho \circ \Theta = \tilde{\Theta} \circ \rho$.

Definition. Germs of the families $(F_i(q, u), x_i)$, $i = 1, 2$, are said to be R^+ equivalent if there exist mappings Θ and Φ such that (B) holds and $F_2 \circ \Theta = F_1 + \Phi$.

Λ acts on the generating function of the manifold according to (1.1), and so the germs of (L_i, w_i), $i = 1, 2$, $\tilde{L}_i \subset T^*R^{n+k}$, with generating functions $(F_i(q, u), w_i)$ are Λ equivalent if and only if the $F_i(q, u)$ are R^+ equivalent.

5. A Lagrangian equivalence $\tilde{\mathcal{L}} \in \Lambda$ that preserves A^n induces a Lagrangian equivalence of T^*R^n. In the notation of Paragraph 4 we have $\mathcal{L} = (\tilde{\Theta}, \tilde{\Phi})$. In the notation of Para. 3 we obtain the following assertion:

Assertion 3. The manifold $\tilde{\mathscr{L}}(\tilde{L})$ intersects A^n transversely at $\tilde{\mathscr{L}}(w)$ and $\rho_1(\tilde{\mathscr{L}}(\tilde{L}) \cap A^n) = \mathscr{L}(L)$.

6. Any germ (L, m) of a Lagrangian manifold $L \subset T^*R^n$ is well projected onto at least one of the 2^n n-dimensional coordinate subspaces p_I, q_J ($I \cup J = \{1, \ldots, n\}$, $I \cap J = \phi$). In this case there exists a unique, up to addition of a constant, function $F(p_I, q_J)$ such that (L, m) is defined by the equations

$$-p_J = \frac{\partial F}{\partial q_J}, \quad q_I = \frac{\partial F}{\partial p_I}.$$

It is easy to verify that the germ at $x = (q_0, p_{I_0})$, where $m = (q_0, p_{I_0}, p_{J_0})$, of the family $G_L = p_I q_I - F(p_I, q_J)$ is generating for (L, m). If the number of elements of I is minimal for (L, m) and $k(I) = \dim\ker \pi_*|_{TL}$, then $(\partial^2 F/\partial p_I \partial p_I) = 0$ (see [2]).

7. Definition. The families $F_1(q, u)$, $q \in R^n$, $u \in R^k$, and $F_2(q, v)$, $v \in R^l$, are said to be R^+-stably equivalent if there exists a family $F_3(q, w)$, $w \in R^s$, $s \le l$, k, such that the F_i, i = 1, 2, are R^+ equivalent to the families $F_3 + Q_i$, where Q_i is a nondegenerate quadratic form in the appropriate number of variables u or v.

Assertion 4. All generating families of (L, m) are mutually R^+-stably equivalent.

Proof. Let $(F(q, v), (q_0, v_0))$ be a generating family of (L, m), $m = (q_0, p_0)$. Then, by the generalized Morse lemma for functions depending on parameters, there exists a diffeomorphism Θ_1: $(q, v) \to (q, V(q, v))$, that induces the identity change of parameters q such that $F \circ \Theta_1 = F_1(q, u) + Q$, where $v = (u, w)$, $u \in R^k$, and Q is a nondegenerate quadratic form in w and $(\partial^2 F/\partial u \partial u)_{(q_0, u_0)} = 0$.

According to Assertion 3, $F_1 + Q$, and therefore, F_1 are generating germs for (L, m).

If L is well projected onto (p_I, q_J), where $k(I) = k_{\min}$, then $\det\left(\frac{\partial^2 F}{\partial u\, \partial q_I}\right)_{(q_0, u_0)} \neq 0$, and the mapping Θ_2: $(q, u) \to (q, \partial F/\partial q_I)$ defines a diffeomorphism of a neighborhood of (q_0, u_0) into a neighborhood of (q_0, p_{I_0}). The germ of $G = F_1 \circ \Theta_2^{-1}$ at (q_0, p_{I_0}) generates (L, m), where ρ_1 (see Para. 3) has the form

$$\rho_1\colon \left(q, p_I, \frac{\partial G}{\partial q}\right) \mapsto \left(q, p_I, \frac{\partial G}{\partial q_J}\right).$$

Thus, $d(G_L - G)|\pi_1 L = 0$, where π_1 is the projection π_1: $(q, p) \to (q, p_I)$. The manifold $\pi_1 L$ is defined in a neighborhood of $\pi_1(m)$ by $\partial G_L/\partial p_I = 0$, and so the germ of $G_L - G + c_1$ at $\pi_1(m)$, where c_1 is a constant, belongs to $\mathfrak{U}^2$, where $\mathfrak{U}$ is an ideal in $C_{\pi_1(m)}(n + k, 1)$ – the ring of germs at $\pi_1(m)$ of functions $R^{n+k} \to R$,

$$\mathfrak{U} = C_{\pi_1(m)}(n + k, 1)\left\{\frac{\partial G_L}{\partial p_I}\right\}.$$

Consider the homotopy G_t, $G_t = G_L + t(G - G_L)$, $t \in [0, 1]$. It follows from the relation $(\partial^2 G/\partial p_I \partial p_I)_{\pi_1(m)} = 0$ that there exist smooth functions $h_{\alpha,\beta}(q, p_I, t)$, $\alpha, \beta = 1, \ldots, k$, defined in $U \times [0, 1]$, where U is a neighborhood of $\pi_1(m)$, such that

$$\frac{\partial G_L}{\partial p_\alpha} = \sum_\beta \frac{\partial G_t}{\partial p_\beta} h_{\alpha,\beta}. \tag{1.2}$$

It follows from (1.2) that there exist smooth functions $H_\alpha(q, p_I, t)$, defined in $U \times [0, 1]$, such that

$$H_\alpha|_{\pi_1 L} = 0, \quad \frac{\partial G_t}{\partial t} = -\sum_\alpha \frac{\partial G_t}{\partial p_\alpha} H_\alpha.$$

The field $(\dot{q}, \dot{p}_I) = (0, H_I)$ defines a one-parameter family of diffeomorphisms $\tilde{\Theta}_t$ of some neighborhood of $\pi_1(m)$, that are identical on $\pi_1 L$ and carry G_L into G_t.

The composition $\Theta_1^{-1} \circ \Theta_2^{-1} \circ \tilde{\Theta}_1$ sets up an R^+-stable equivalence of the generating family F with the fixed family G_L. This proves the assertion.

THEOREM 1. Germs of Lagrangian manifolds (L_i, m_i), i = 1, 2, are Lagrange equivalent if and only if the germs of the corresponding generating families $F_i(q, u_i)$, $u_i \in R^{k_i}$, are R^+-stably equivalent (and R^+ equivalent if $k_i = \dim\ker \pi_*|_{TL_i}$).

The proof of the theorem follows from Assertions 3 and 4.

§2. Legendrian Generating Families

Definitions. By a Legendrian foliation, we mean a foliation $\pi: M^{2n+1} \to B^{n+1}$ whose space is a contact manifold and whose fibers are Legendrian submanifolds. The definitions of Legendrian equivalence, Legendrian mapping, and equivalent Legendrian mappings are similar to the Lagrangian definitions (see [3]).

Locally, all Legendrian foliations are Legendre equivalent. We shall consider two local models of Legendrian foliations connected with the contactization and symplectization functors of the standard Lagrangian foliation, respectively:

1) $(J^1(R^n, R), \tilde{\pi}, \alpha)$, where $J^1(R^n, R)$ is the space of 1 jets of functions $R^n \to R$ with coordinates $q \in R^n$, $p \in T^*_{q_0}R^n$, $z \in R$†; the projection $\tilde{\pi}$: $(p_1 q, z) \to (q, z)$ and hyperplane of zeros of $\alpha = dz - pdq$ define a Legendrian foliation structure;

2) $(PT^*R^{n+1}, \bar{\pi}, \beta)$, where PT^*R^{n+1} is the projectivization of T^*R^{n+1} with coordinates $x \in R^{n+1}$ and $y \in T^*_{x_0}R^{n+1}$ – the homogeneous coordinates in the fiber; the projection $\bar{\pi}$: $(x, y) \to (x)$ and form $\beta = ydx$ on T^*R^{n+1} define a Legendrian foliation structure in PT^*R^{n+1}.

We denote by $T^*R^{n+1} \setminus R^{n+1} \to PT^*R^{n+1}$ the projectivization and by $*\lambda$ ($\lambda \in R \setminus \{0\}$) the mapping $*\lambda$: $T^*R^{n+1} \to T^*R^{n+1}$, $*\lambda$: $(x, y) \mapsto (x, \lambda y)$. Then the mapping

$$l\colon J^1_q(f(q)) \mapsto \mathrm{pr}\,(d(z - f(q)),\ l\colon J^1(R^n, R) \mapsto PT^*R^n,$$

realizes a Legendrian equivalence of 1) and 2).

Assertion 1. A Legendrian equivalence is uniquely defined by the induced diffeomorphism of the base.

Proof. A Legendrian equivalence $\mathscr{L}$ of PT^*R^{n+1} has the form $\mathscr{L} = \mathrm{pr} \circ \tilde{L} \circ \mathrm{pr}^{-1}$, where $\tilde{L}$ is a fiber-homogeneous Lagrangian equivalence of T^*R^{n+1}, i.e., $*\lambda \circ \tilde{L} = \tilde{L} \circ (*\lambda)$. Now $\tilde{L}$ is uniquely defined by the induced diffeomorphism of the base (see §1).

1. If a germ of a Legendrian manifold $(L, m) \subset PT^*R^{n+1}$ is well projected onto the base, then there exists a germ of the generating function $(\Phi_L(x), \bar{\pi}(m))$ such that L is defined by

$$\Phi_L(x) = 0, \quad y = \frac{\partial \Phi_L}{\partial x} \quad \text{and} \quad \left.\frac{\partial \Phi}{\partial x}\right|_{\bar{\pi}(m)} \neq 0.$$

The function $\Phi(x)$ is defined up to multiplication by $\Psi(x)$, $\Psi(\bar{\pi}(m)) \neq 0$.

The Legendrian equivalence $\mathscr{L}$ defined by a diffeomorphism of the base Θ acts on Φ_L by the formula

$$\Phi_{\mathscr{L}(L)} = \Psi(x)(\Phi_L \circ \Theta). \tag{2.1}$$

2. Hörmander's Construction. Consider the foliation $\tilde{\rho}$: $R^{n+k+1} \to R^{n+1}$ and the subfoliation A^{n+1} (see §1). In the diagram

$$\begin{array}{ccccc} T^*R^{n+k+1} & \xrightarrow{i_1} & A^{n+1} & \xrightarrow{\rho_1} & T^*R^{n+1} \\ & \searrow & \downarrow \pi & & \downarrow \pi \\ & & R^{n+k+1} & \xrightarrow{\tilde{\rho}} & R^{n+1} \end{array} \tag{2.2}$$

the mappings i_1 and ρ_1 defined in §1 commute with the mappings $*\lambda$ in T^*R^{n+k+1} and T^*R^{n+1}, and so, projectivizing T^*R^{n+1} and T^*R^{n+k+1}, we obtain the commutative diagram

$$\begin{array}{ccccc} PT^*R^{n+k+1} & \xleftarrow{i_1} & PA^{n+1} & \xrightarrow{\xi} & PT^*R^{n+1} \\ & \searrow_{\bar{\pi}} & \downarrow \bar{\pi} & & \downarrow \bar{\pi} \\ & & R^{n+k+1} & \xrightarrow{\tilde{\rho}} & R^{n+1} \end{array}, \tag{2.3}$$

where $\xi \circ \mathrm{pr} \circ \rho_1$.

Assertion 2. Let $(\tilde{L}, m)$ be a Legendrian manifold in PT^*R^{n+k+1} which is well projected onto the base and which intersects PA^{n+1} transversely at m. Then $\xi(\tilde{L} \cap PA^{n+1})$ is a Legendrian manifold in PT^*R^{n+1}.

The proof follows from the fact that $\mathrm{pr}^{-1}(\tilde{L} \cap PA^{n+1})$ is a conic Lagrangian manifold in T^*R^{n+k+1} that satisfies the hypotheses of Assertion 2 in §1 at $\bar{m} = \mathrm{pr}^{-1}(m)$, and also from the fact that ρ_1 in (2.2) commutes with c $*\lambda$.

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†$J^1(R^n, R)$ is naturally isomorphic to $T^*R^n \times R$.

Definition. A germ of the family $(F(x, u), (x_0, u_0))$ of functions of $u \in R^k$ with parameters $x \in R^{n+1}$, satisfying the conditions: a) $\Phi(u_0, x_0) = 0$, b) $\left.\frac{\partial \Phi}{\partial u}\right|_{(x_0, u_0)} = 0$, c) rank $\eta|_{(x_0, u_0)} = k + 1$, where $\eta: (x, u) \mapsto \left(\Phi, \frac{\partial \Phi}{\partial u}\right)$, i.e., which is a generating function of some Legendrian manifold $(\tilde{L}, m)$, satisfying the hypotheses of Assertion 2, is called a germ of the generating family of a germ of the Legendrian manifold $(L, m_1) = (\xi(\tilde{L} \cap PA^{n+1}), \xi(m))$.

3. Let Λ be the subgroup of Legendrian equivalences of PT^*R^{n+k+1} that preserve $\tilde{\rho}$. Let $\mathcal{L} \in \Lambda$. Then there corresponds to $\mathcal{L}$ a diffeomorphism of the base $\tilde{\Theta}: R^{n+k+1} \to R^{n+k+1}$ that preserves $\tilde{\rho}$ (see §1.4).

Definition. Germs of the families $F_i(x, u)$, $i = 1, 2$, are said to be K equivalent if there exist a diffeomorphism $\tilde{\Theta}$, that preserves $\tilde{\rho}$, and a function $\Phi(x, u)$, $\Phi(x_0, u_0) \neq 0$, such that $F_2 = \Phi(F_1 \circ \tilde{\Theta})$.

The definition of K-stably equivalent families is introduced in the corresponding way.

Λ acts on the generating function of $\tilde{L}$ according to (2.1), and so the germs of Legendrian manifolds well projected onto R^{n+k+1} are Legendre equivalent if and only if their generating functions, regarded as families, are Λ equivalent.

4. Any germ of a Legendrian manifold has a generating family. For, a germ (L, m) of a Legendrian manifold in $J^1(R^n, R)$ is defined by the generating function $F(p_I, q_J)$ by

$$-p_J = \frac{\partial F}{\partial q_J}, \quad q_I = \frac{\partial F}{\partial p_I}, \quad -z = p_I q_I - F(p_I, q_J).$$

In this case $F_L = z + p_I q_I - F(p_I, q_J)$ is a generating family for (L, m).

5. Assertion 3. The germs of the generating families of (L, m) are mutually K-stably equivalent.

Proof. Let $x = (q, z)$ be coordinates in R^{n+1} and U_z^{n+1} the affine chart of PT^*R^{n+1}, $U_z^{n+1} = \{(x, y), y_{n+1} \neq 0\}$. Then $\psi_n: U_z^{n+1} \to J^1(R^n, R)$, $\psi_n: (x, y) \mapsto \left(x, \frac{y}{y_{n+1}}\right)$ is a Legendrian equivalence.

In some neighborhood of $\pi(m)$ in R^{n+1} the coordinates q and z can be chosen so that $\psi_n L$ is well projected onto T^*R^n (see preceding footnote).

Let the generating family F(x, u) of (L, m) be a generating function of a germ of the Legendrian manifold $(\tilde{L}, w) \subset PT^*R^{n+k+1}$ that satisfies the hypothesis of Assertion 2, and let $w \in U_z^{n+k+1}$. Then there exist an imbedding i_3 and a projection ξ such that the following diagram commutes:

$$\begin{array}{ccccc} J^1(R^{n+k}, R) & \xleftarrow{i_3} & \tilde{A} & \xrightarrow{\xi} & J^1(R^n, R) \\ \uparrow \psi_{n+k} & & \uparrow \tilde{\psi}_{n+k} & & \uparrow \psi_n \\ U^{n+k+1} & \xleftarrow{i_2} & PA^{n+1} & \xrightarrow{\xi} & U_z^{n+1}. \end{array} \quad (2.4)$$

Here $\tilde{\psi}_{n+k} = \psi_{n+k}|_{U_z^{n+k+1} \cap PA^{n+1}}$ and $\tilde{A} = \operatorname{Im}(\tilde{\psi}_{n+k})$. It is easy to see that by means of the natural projection $\pi_2^n: J^1(R^n, R) \to T^*R^n$, $\pi_2^n: (p, q, z) \mapsto (p, q)$, the upper row of (2.4) can be completed to the commutative diagram

$$\begin{array}{ccccc} T^*R^{n+k} & \xleftarrow{i_1} & A^n & \xrightarrow{\rho_1} & T^*R^n \\ \uparrow \pi_2^{n+k} & & \uparrow \pi_2^{n+k} & & \uparrow \pi_2^n \\ J^1(R^{n+k}, R) & \xleftarrow{i_3} & \tilde{A} & \xrightarrow{\xi} & J^1(R^n, R), \end{array} \quad (2.5)$$

where i_1 and ρ_1 are defined in §1.

$\pi_2^{n+k}(L')$ – the projection of $\tilde{L}' = \psi_{n+k}\tilde{L}$ under π_2^{n+k} – is a Lagrangian manifold that satisfies the hypothesis of Assertion 2 in §1, and, since (2.5) is commutative, $\rho_1(\pi_2^{n+k}L' \cap A^n) = \pi_2^n \psi_n(L)$.

A generating function of a germ of L' has the form $\Phi(u, q, z)\,(z + \tilde{F}(q, u))$, where $\Phi(\psi_{n+k}(w)) \neq 0$ and $\tilde{F}(q, u)$ is a generating function of $\pi_2^{n+k}L'$. Assertion 3 now follows from the fact that $\tilde{F}(q, u)$ is a generating family of $\pi_2^n L$, i.e., belongs to some fixed orbit of the group of R^+-stable equivalences.

The next theorem follows from Assertions 2 and 3.

THEOREM 2. The germs of Legendrian manifolds are Legendre equivalent if and only if the corresponding generating families are K-stably equivalent.

§3. The Stability of Lagrangian (Legendrian) Mappings

We provide the space of functions F: $R^{n+k} \to R$ and the space of mappings i: $R^n \to T^*R^n$, where i is a Lagrangian-manifold imbedding, with the Whitney C^∞ topology. We say that Lagrangian manifolds are close if there exist close imbeddings of them.

The following assertion follows from the definition of a generating family:

Assertion 1. If generating families are close, then so are the corresponding Lagrangian manifolds, and if a Lagrangian manifold (L_1, m_1) is close to a Lagrangian manifold (L_2, m_2) with generating function F_2, then there exists a generating function F_{L_1}, close to F_2. If, in addition, the germs (L_1, m_1) and (L_2, m_2) at the close points m_1 and m_2 are Lagrange equivalent, then (F_{L_1}, m_1) and (F_2, m_2) are R^+ equivalent.

A topology is also introduced in the space of Legendrian mappings.

Definition. A germ of a Lagrangian (Legendrian) manifold (L, m) is said to be Lagrange (Legendre) stable if for any Lagrangian (Legendrian) manifold close to L there exists a point m_1 close to m such that (M, m_1) is Lagrange (Legendre) equivalent to (L, m).

THEOREM 3 (Arnol'd [2], Guckenheimer [1]). A germ of a Lagrangian manifold (L, m) is Lagrange stable if and only if the generating family F(q, u) + z with the additional parameter $z \in R$ is a versal deformation of the germ at $(0, u_0)$ of $f(q) = F(q + q_0, u_0)$, $m = \rho_1(q_0, u_0)$.

LEMMA 1. Let G(x, ε) be the family of functions of $x \in R^n$ with parameters $\varepsilon \in R^r$. Then the following conditions are equivalent:

1) G(x, ε) is a versal deformation of the germ of $f(x) = G(x, \varepsilon_0)$;

2) the mapping ${}^rG\colon R^{n+r} \to J_0^r(n, 1)$,* ${}^rG\colon (x_1, \varepsilon) \mapsto J_0^r(G(x + x_1, \varepsilon))$, is transversal at $(0, \varepsilon_0)$ to the orbit ${}^rOf$ of r jets of f under the action of the group of right substitutions $(R^n, 0) \to (R^n, 0)$.

3) there exists a neighborhood of (x_0, ε_0), in which for any family F close to G there exists a point (x_1, ε_1) such that the germs of $(G, (x_0, \varepsilon_0))$ and $(F, (x_1, \varepsilon_1))$ are R equivalent.

The proof of the lemma follows from the versality theorem [5] and from Assertion 1.6 of [6] (see also [7]).

The theorem follows from Assertion 1, Lemma 1, and the following remark:

Assertion 2. If the versal deformations $G_i(u, q) + z$, $z \in R$, i = 1, 2, of the functions f_i are R equivalent, then the families $G_i(u, q)$ are R^+ equivalent.

THEOREM 4. A germ of a Legendrian manifold (L, m) is Legendre stable if and only if the generating family F(x, u) is a versal deformation for the levels of $f(u) = F(x_0, u + u_0)$; i.e., for any germ $\alpha \in C(u)$ there exists a decomposition

$$\alpha = \varphi \cdot f + \frac{\partial f}{\partial u} \cdot \psi + \left.\frac{\partial F}{\partial x}\right|_{x=x_0} \cdot \chi,$$

where $\varphi, \psi_i \in C(u)$, $\chi_j \in R$, $i = 1, \ldots, k$, $j = 1, \ldots, n+1$.

The theorem follows from the versality theorem for levels and §2.

§4. Normal Forms of Lagrangian (Legendrian) Mappings

A generating family F(q, u) of a germ (L, m) of a Lagrangian manifold is induced by a versal deformation of $f(u) = F(q_0, u)$, $m = (q_0, p_0)$ (see §1); i.e., there exists a mapping $\Xi\colon (q, u) \mapsto (y(q), \tilde{u}(u, q))$, $y \in R^\mu$, such that

$$F(q, u) = f(\tilde{u}) + \sum_i \varphi_i(\tilde{u})\, y_i(q), \tag{4.1}$$

where φ_i, $1 \le i \le \mu$, are generators of the R-module $C(u)/\{\partial f/\partial u\}$.

If (L, m) is a stable germ, then the inducing mapping Ξ is a diffeomorphism, and (4.1) is a normal form of a stable generating family.

*$J_0^r(n, 1)$ is the space of r jets at 0 of functions f: $R^n \to R$.

THEOREM 5 (on Semiuniversality).* Let $G(x, y)$, $x \in R^k$, $y \in R^n$, be a miniversal deformation of $f(x)$. Let Θ be a family diffeomorphism, $G \circ \Theta = G$ and $\tilde{\Theta}$ the corresponding parameter diffeomorphism, $\tilde{\Theta} \circ \pi = \pi \circ \Theta$. Let K be the stationary group of diffeomorphisms, $f \circ h = f$, h: $(R^k, 0) \to (R^k, 0)$ and $\tilde{K}$ the discrete group of connection components of K. Then $\{\tilde{\Theta}\}$ is isomorphic to $\tilde{K}$.

LEMMA 1. Let $G(x, y) = f(x) + y_i\varphi_i(x)$, and let Θ_t be a one-parameter family of diffeomorphisms, $t \in [0, 1]$, $G \circ \Theta_t = G$. Then $\tilde{\Theta}_t = \mathrm{id}_{R^n}$.

Proof. The field $(\overset{0}{x}, \overset{0}{y})$ generated by Θ_t satisfies the relation

$$0 = \sum_j \left(\frac{\partial f}{\partial x_j} + \sum_i y_i \frac{\partial \varphi_i}{\partial x_j}\right) \overset{0}{x}_j + \sum_i \varphi_i \overset{0}{y}_i.$$

In the space of functions $\overset{0}{x}$ and $\overset{0}{y}$ consider the following grading with respect to powers of y:

$$\overset{0}{x}_j = \overset{0}{x}_{j,0} + \overset{0}{x}_{j,1} + \ldots, \qquad \overset{0}{y}_i = \overset{0}{y}_{i,0} + \overset{0}{y}_{i,1} + \ldots.$$

Let us prove that all the $\overset{0}{y}_{i,s} = 0$.

Since φ_i is a minimal system of generators of the R-module $C(x)/\{\partial f/\partial x\}$, we have $\overset{0}{y}_{i,0} = 0$. By induction, from the assumption $\overset{0}{y}_{i,m} = 0$ we obtain

$$\sum \frac{\partial f}{\partial x_j} \overset{0}{x}_{j,m} = 0, \quad \sum_{i,j} y_i \frac{\partial \varphi_i}{\partial x_j} \overset{0}{x}_{j,m} + \sum_i \varphi_i \overset{0}{y}_{i,m+1} = 0. \tag{4.2}$$

Since the Koszul complex of the gradient of f is acyclic, it follows that $\overset{0}{x}_{\alpha,m} = \sum_{\alpha,\beta} a_{\alpha,\beta} \frac{\partial f}{\partial x_\beta}$, where $a_{\alpha,\beta} \in C(x, y)$ and $a_{\alpha,\beta} + a_{\beta,\alpha} = 0$. It follows from (4.2) that $\overset{0}{y}_{i,m+1} = 0$. This proves the lemma.

LEMMA 2. Let Θ be a diffeomorphism, $G \circ \Theta = G$, and let $h_1 = \Theta|_{y=0}$ be a mapping such that there exists a homotopy h_t, $t \in [0, 1]$, $f \circ h_t = f$, $h_0 = \mathrm{id}_{R^k}$. Then $\tilde{\Theta} = \mathrm{id}_{R^n}$.

Proof. Θ can be joined by a homotopy Θ_t, $t \in [0, 1]$, with $\mathrm{id}_{R^{n+k}}$ so that $\Theta_t|_{y=0} = h_t$. Set $G_t = G \circ \Theta_t$. Then $G_t|_{y=0} = f(x)$. According to Lemma 1 of [5], there exists a family of diffeomorphisms $\Theta^1_{t,\tau}$, $\tau \in [0, 1]$, smoothly depending on t and τ, such that $\Theta^1_{t,0} = \mathrm{id}_{R^{n+k}}$, and

$$h^1_{t,\tau} = \Theta^1_{t,\tau}|_{y=0} = \mathrm{id}_{R^k}, \quad \tilde{G}_t = G_t \circ \Theta^1_{t,1} = f + \sum \frac{\partial G_t}{\partial y_i}\bigg|_{y=0} \cdot y_i, \tag{4.3}$$

and if $G_{t_0} = f + \sum \frac{\partial G}{\partial y}\Big|_{y=0} y_i$, then

$$\Theta_{t_0,\tau} = \mathrm{id}_{R^{n+k}}, \quad \tau \in [0, 1]. \tag{4.4}$$

By Lemma 2 in [5], there exists a family $\Theta^2_{t,\tau}$ satisfying (4.3) and (4.4) and such that $\tilde{G}_t \circ \Theta^2_{t,1} = G$. Thus, $\tilde{\Theta} \circ \tilde{\Theta}^1_{1,1} \circ \Theta^2_{1,1} = \mathrm{id}_{R^n}$, and it follows from (4.2) that $\tilde{\Theta} = \mathrm{id}_{R^n}$. This proves the lemma.

Proof of Theorem 5. Suppose that Θ_i, $i = 1, 2$, preserves G and that $G, h_i = \Theta_i|_{y=0}$. Suppose that h_1 and h_2 lie in the same connection component of K. Since $h_3 = \Theta_1 \circ \Theta_2^{-1}|_{y=0}$ lies in id_{R^k} in K, by Lemma 2 we obtain $\tilde{\Theta}_1 \circ \tilde{\Theta}_2^{-1} = \mathrm{id}_{R^n}$. This proves the theorem.

The proof of the following theorem is similar:

THEOREM 6. In the hypotheses of the theorem on semiuniversality, suppose that ξ: $R^s \to R^n$, $y = \xi(q)$ is a regular mapping and that $dy_I/dq \neq 0$, $I \subset \{1, \ldots, n\}$, $I = \{i_1, \ldots, i_k\}$. Then the family $F(\xi(q), x)$ is R^+ equivalent to the family

$$f(x) + \sum_{i \in I} q_i \varphi_i + \sum_{j \notin I} \eta_j(q) \varphi_j,$$

where the $\eta_j(q)$ are smooth functions defined by ξ up to the action of $\{\tilde{\Theta}\}$.

THEOREM 7. The mappings which, in a neighborhood of each of its points of Lagrangian equivalence, reduce to Lagrangian mappings having the following generating families form an everywhere dense open

*All objects under consideration in this theorem are assumed to be real (complex) analytic.

set in the fine C^∞ topology in the space of Lagrangian mappings $\pi \circ i$: $R^n \to R^n$, $n < 11$:

for $n \leqslant 5$ see [2];
for $n = 6$ also ${}^0A_7$, ${}^0D_7$, ${}^0E_7$, ${}^1P_8$;
for $n = 7$ also ${}^0A_8$, ${}^0D_8$, ${}^0E_8$, ${}^0P_8$, ${}^1P_9$, ${}^1X_9$;
for $n = 8$ also ${}^0A_9$, ${}^0D_9$, ${}^0P_9$, ${}^0X_9$, ${}^1P_{10}$, ${}^1Q_{10}$, ${}^1R_{4,4}$, ${}^1X_{10}$, ${}^1J_{10}$;
for $n = 9$ also ${}^0A_{10}$, ${}^0D_{10}$, ${}^0P_{10}$, ${}^0Q_{10}$, ${}^0R_{4,4}$, ${}^0X_{10}$, ${}^0J_{10}$, ${}^1P_{11}$, ${}^1Q_{11}$, ${}^1R_{4,5}$, ${}^1S_{11}$, ${}^1T_{4,4,4}$, ${}^1X_{11}$, ${}^1Y_{5,5}$, ${}^1Z_{11}$, ${}^1J_{11}$, ${}^0K_{10}$;
for $n = 10$ also ${}^0A_{11}$, ${}^0D_{11}$, ${}^0P_{11}$, ${}^0Q_{11}$, ${}^0R_{4,5}$, ${}^0S_{11}$, ${}^0T_{4,4,4}$, ${}^0X_{11}$, ${}^0Y_{5,5}$, ${}^0Z_{11}$, ${}^0J_{11}$, ${}^1P_{12}$, ${}^1Q_{12}$, ${}^1R_{4,6}$, ${}^1R_{5,5}$, ${}^1S_{12}$, ${}^1T_{4,4,5}$, ${}^1U_{12}$, ${}^1X_{12}$, ${}^1Y_{5,6}$, ${}^1Y_{4,7}$, ${}^1Z_{12}$, ${}^1W_{12}$, ${}^1J_{12}$, ${}^1K_{12}$, ${}^5O_{16}$.

Here ${}^l\Phi_\mu$ denotes the generating families

$${}^l\Phi_\mu = f(u) + \sum_{i=1}^{l} y_i(q)\varphi_i(u) + \sum_{j=l+1}^{\mu-1} q_{j-l}\varphi_j(u),$$

where: a) the $y_i(q)$ are smooth functions;

b) f(u) belongs to the class Φ_μ of singularities of functions (see [8, 9]);

c) the stratum $\mu = \mu(\Phi)$ in the space $\mathfrak{M}^2 \subset C(u)$ containing f has the following form in a neighborhood of f:

$$f(u) + \sum_{1}^{r} a_i\varphi_i(u) \qquad (a_i \text{ are moduli});$$

d) the functions 1, $\varphi_i(u)$ $(i = 1, \ldots, r)$, $\varphi_j(u)$ $(j = r + 1, \ldots, \mu - 1)$ are generators of $C(u)/\{\partial f/\partial u\}$;

e) $l \leq r$.

For example, ${}^1P_8$ has the form

$${}^1P_8 = \pm u_1^3 \pm u_2^3 \pm u_3^3 + y_1(q)\, u_1u_2u_3 + q_1u_1^2 + q_2u_2^2 + q_3u_3^2 + q_4u_1 + q_5u_2 + q_6u_3.$$

Proof. The smooth strata μ = const, codim < 11 and union of the strata codim ≥ 11 form a stratification satisfying Whitney's first condition. The mappings transversal to a stratification form an everywhere dense open set in the space of mappings (u, q) → C(u), transversal to $\mathfrak{M}^2$ (i.e., Lagrangian mappings; see §1).

The form of the normal forms follows from Theorem 6.

A similar list of normal forms of Legendrian mappings corresponds to the contact stratification of C(u).

THEOREM 8. The mappings which, in a neighborhood of each of its points of Legendrian equivalence, reduce to Legendrian mappings having the following generating families form an everywhere dense open set in the fine C^∞ topology in the space of Legendrian mappings $R^n \to R^{n+1}$, $n \leq 11$:

for $n = 1$ ${}^0A_2^+$; for $n = 2$ also ${}^0A_3^+$;
for $n = 3$ also ${}^0A_4^+$, ${}^0D_4^+$; for $n = 4$ also ${}^0A_5^+$, ${}^0D_5^+$; for $n = 5$ also ${}^0A_6^+$, ${}^0D_6^+$, ${}^0E_6^+$;
for $n = 6$ also ${}^0A_7^+$, ${}^0D_7^+$, ${}^0E_7^+$, ${}^1P_8^+$;
for $n = 7$ also ${}^0A_8^+$, ${}^0D_8^+$, ${}^0E_8^+$, ${}^0P_8^+$, ${}^1X_9^+$, ${}^0\bar{P}_9^+$;
for $n = 8$ also ${}^0A_9^+$, ${}^0D_9^+$, ${}^0X_9^+$, ${}^1J_{10}^+$, ${}^0\bar{P}_{10}^+$, ${}^0\bar{X}_{10}^+$, ${}^0\bar{Q}_{10}^+$, ${}^0\bar{R}_{4,4}^+$;
for $n = 9$ also ${}^0A_{10}^+$, ${}^0D_{10}^+$, ${}^0J_{10}^+$, ${}^0K_{10}^+$, ${}^0\bar{P}_{11}^+$, ${}^0\bar{Q}_{11}^+$, ${}^0\bar{R}_{4,5}^+$, ${}^0\bar{S}_{11}^+$, ${}^0\bar{T}_{4,4,4}^+$, ${}^0\bar{X}_{11}^+$, ${}^0\bar{Y}_{5,5}^+$, ${}^0\bar{Z}_{11}^+$, ${}^0\bar{J}_{11}^+$;
for $n = 10$ also ${}^0A_{11}^+$, ${}^0D_{11}^+$, ${}^0\bar{P}_{12}^+$, ${}^0\bar{Q}_{12}^+$, ${}^0\bar{R}_{4,6}^+$, ${}^0\bar{R}_{5,5}^+$, ${}^0\bar{S}_{12}^+$, ${}^0\bar{T}_{4,4,5}^+$, ${}^0\bar{U}_{12}^+$, ${}^0\bar{X}_{12}^+$, ${}^0\bar{Y}_{5,6}^+$, ${}^0\bar{Y}_{4,7}^+$, ${}^0\bar{Z}_{12}^+$, ${}^0\bar{W}_{12}^+$, ${}^0\bar{J}_{12}^+$, ${}^0\bar{K}_{12}^+$, ${}^4\bar{\bar{O}}_{16}^+$.

Here ${}^l\Phi_\mu^+ = {}^l\Phi_\mu(u, q_1, \ldots, q_n) + q_{n+1}$ (see Theorem 3); for unimodal singularities (r = 1)

$${}^0\bar{\Phi}_\mu^+ = f(u) + a_0\varphi_1(u) + \sum_{j=2}^{\mu-1} q_{j-1}\varphi_j(u) + q_{n+1},$$

where a_0 is a fixed value of the modulus, and ${}^4\bar{\bar{O}}_{16}^+$ is a four-modal family of generating families:

$${}^4\bar{\bar{O}}_{16}^+ = u_1^3 + u_2^3 + u_3^3 + u_4^3 + (a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4)^3 + u_1u_2u_3u_4 + q_1u_1^2 + q_2u_2^2 + q_3u_3^2 + q_4u_4^2 + q_5u_1u_2 + q_6u_3u_4 + q_7u_1 + q_8u_2 + q_7u_3 + q_{10}u_4 + q_{11}.$$

The proof of the theorem is similar to that of Theorem 7.

In conclusion, note that it would be interesting to classify Lagrangian or Legendrian mappings with respect to wider groups of equivalences, especially when the Lagrangian mapping depends on parameters. The definitions can be found in [10, 11-14].

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