WAVE-EQUATION SCATTERING IN EVEN-DIMENSIONAL SPACES

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1. Formulation of the Problem

Let G be a domain in m-dimensional Euclidean space R_m, the complement of which is contained in the sphere $\Omega_a = \{x: |x| < a\}$ and has a smooth boundary ∂G. Let $\sigma(x)$ be a smooth, nonnegative function on ∂G. By $H(g, \sigma)$ we denote the Hilbert space of ordered pairs of functions on $\overline{G}$ with norm defined by the expression

$$\|f\|^2_{H(G,\sigma)} = \frac{1}{2}\int_G \{|(\partial_x f_1)(x)|^2 + |f_2(x)|^2\}\,dx + \frac{1}{2}\int_{\partial G} \sigma(x)|f_1(x)|^2 d\Sigma_{\partial G}.$$

It is well-known that the resolvent group generated by the exterior Cauchy problem for the wave equation

$$\partial_t^2 v = \Delta v, \quad \frac{\partial v}{\partial n} + \sigma \cdot v\Big|_{\partial G} = 0, \tag{1}$$
$$(v, \partial_t v)_{t=0} = (f_1, f_2), \quad f_1, f_2 \in C^\infty(\overline{G}), \quad (f_1, f_2) \in H(G, \sigma)$$

can be extended by continuity to a group of unitary operators $\{U(t)\}$ in $H(G, \sigma)$.

In $H(G, \sigma)$, single out an outgoing subspace $\mathscr{D}_+^a$ and an incoming subspace $\mathscr{D}_-^a$ that consist of all initial values for which the (generalized) solutions of the Cauchy problem (1) vanish on the truncated cones $|x| < a+t, t > 0$, and $|x| < a - t$, $t < 0$, respectively. The subspaces $\mathscr{D}_\pm^a$ possess the following properties [1, 2]:

$$1)\ U(\pm t)\mathscr{D}_\pm^a \subset \mathscr{D}_\pm^a,\ t > 0, \quad \bigcap_t U(t)\mathscr{D}_\pm^a = \{0\}; \qquad 2)\ \overline{\bigcup_t U(t)\mathscr{D}_\pm^a} = H(G, \sigma). \tag{2}$$

By $L_2(N)$ we denote the space of measurable vector functions on $(-\infty, \infty)$ with values in some fixed Hilbert space N.† By virtue of properties (2), there exist isometric mappings $\mathscr{F}_\pm(G, \sigma)$ of $H(G, \sigma)$ onto $L_2(N)$ such that

$$1)\ (\mathscr{F}_\pm(G, \sigma)U(t)f)(\lambda) = e^{-i\lambda t}(\mathscr{F}_\pm(G, \sigma)f)(\lambda),$$
$$2)\ \mathscr{F}_\pm(G, \sigma)D_\pm^a = H_2^\pm(N), \tag{3}$$

where

$$H_2^\pm(N) = \left\{f: f(\lambda) = \text{l.i.m.}\int_0^\infty e^{\mp i\lambda t} g(t)\,dt,\ g \in L_2(N)\right\}.$$

The mapping $\mathscr{F}_-(G, \sigma)\mathscr{F}_+^*(G, \sigma)$ in $L_2(N)$ acts like "multiplication" by a measurable operator-function $S_a(G, \sigma/\lambda)$, the values of which are unitary operators in N. The function $S_a(G, \sigma|\lambda)$ is called the scattering matrix or the scattering suboperator. In scattering theory, singularities in wave propagation near scattering obstacles (within the sphere Ω_a in our case) as well as the structure of these obstacles can be successfully studied using the scattering matrix. Many important results concerning external problems for wave equations were obtained in this manner by Lax and Phillips and are presented in their monograph [1] and

†In the theory of wave-equation scattering, the natural realization of N is the space of functions $L_2(S_{m-1})$ on the unit sphere S_{m-1} in R_m.

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in their subsequent articles. In addition, in [1] Lax and Phillips describe in detail general systematic ways of investigating wave systems that are based on the analysis of special spectral representations of the group $\{U(t)\}$ generated by the outgoing and incoming subspaces.

In the case when the space R_m is of odd dimension, the subspaces $\mathscr{D}^a_\pm$ turn out to be orthogonal in $H(G, \sigma)$. This additional property of the outgoing and incoming subspaces leads to the situation where, for wave equations in odd-dimensional spaces, the scattering matrix is a (nontangential) boundary value of an interior operator-function [1]. Recall that a function $Q(z)$ that is holomorphic in the lower half-plane and the values of which are operators from N into N is called interior, if 1) $\|Q(z)\| < 1$ for $\operatorname{Im} z < 0$, and 2) $s\text{-}\lim_{\tau\downarrow 0} Q^* (\lambda - i\tau)Q(\lambda - i\tau) = s\text{-}\lim_{\tau\downarrow 0} Q(\lambda - i\tau)Q^*(\lambda - i\tau) = I|_N$ (almost everywhere).

If R_m is of even dimension, the subspaces $\mathscr{D}^a_\pm$ are not orthogonal in $H(G, \sigma)$. Nevertheless, for a fixed positive value of the parameter $a>0$ independent of the domain G $(R_m\setminus G \subset \Omega_a)$ and of the function σ on ∂G, in all spaces $H(G, \sigma)$, the subspaces $\mathscr{D}^a_\pm$ obviously consist of one and the same classes of functions, and the metrics of the spaces $H(G, \sigma)$ coincide on the lineal $\mathscr{D}^a_+ + \mathscr{D}^a_-$. In view of this fact, a natural question arises: Does there exist a universal formula univalently connecting the family of scattering matrices $S_a(G, \sigma|\cdot)$ for the wave equations (1) in even-dimensional spaces for a fixed radius of the sphere Ω_a that contains the scattering obstacles with some subset of the family of interior operator-functions? The following theorem gives a complete answer to this question.

THEOREM 1. Set

$$p(z) = \frac{\sqrt{\pi}}{2C_a} e^{i\left(za+\frac{\pi}{4}\right)}\sqrt{z}\left[\frac{H_1^{(2)}(za)}{H_1^{(2)}(-ia)} + \frac{H_0^{(2)}(za)}{H_0^{(2)}(-ia)}\right],$$

$$q(z) = i\frac{\sqrt{\pi}}{2C_a} e^{i\left(za+\frac{\pi}{4}\right)}\sqrt{z}\left[\frac{H_1^{(2)}(za)}{H_1^{(2)}(-ia)} - \frac{H_0^{(2)}(za)}{H_0^{(2)}(-ia)}\right], \tag{4}$$

$$-\pi \leqslant \arg z \leqslant 0,$$

where $H_1^{(2)}$ and $H_0^{(2)}$ are Hankel functions of the second kind, C_a is a nonnegative constant, and

$$C_a^{-2} = -iaH_1^{(2)}(-ia)H_0^{(2)}(-ia).$$

The scattering matrix $S_a(G, \sigma|\lambda)$ generated by the scattering problem for the wave equation (1) in an even-dimensional space can be represented to within multiplication on the left and right by arbitrary unitary operators that do not depend on λ in the form

$$S_a(G, \sigma|\lambda) = [\bar{q}(\lambda)I + \bar{p}(\lambda)\mathscr{E}(G, \sigma|\lambda)][p(\lambda)I + q(\lambda)\mathscr{E}(G, \sigma|\lambda)]^{-1}, \tag{5}$$

where $\mathscr{E}(G, \sigma|\lambda)$ is the boundary value of an interior operator-function.

In odd-dimensional spaces, the scattering matrix $S_a(G, \sigma|\lambda)$ for equation (1) in essence coincides with the boundary value of the characteristic operator-function $\Theta_B(z)$, $\operatorname{Im} z > 0$,† of the infinitesimal operator of the semigroup of contractions $\{Z(t)\}$, $t > 0$. This operator-function operates in the subspace $K = H(G, \sigma) \ominus [\mathscr{D}^a_+ \oplus \mathscr{D}^a_-]$ and is connected with the group $\{U(t)\}$ by the formula

$$Z(t) = P_K U(t)|_K, \quad t > 0, \tag{6}$$

where P_K is the orthoprojector onto the subspace K. In this case, the operators $Z(t)$ defined according to (6) form a semigroup of contractions due to the orthogonality of $\mathscr{D}^a_+$ and $\mathscr{D}^a_-$ (see [1]). At the same time, the function

$$\tilde{S}_a(G, \sigma|\zeta) = S_a\left(G, \sigma\,|\,i\frac{\zeta+1}{\zeta-1}\right), \quad |\zeta| = 1,$$

is actually the boundary value of the characteristic operator-function $\Theta_T(w)$, $|w| < 1$, of the cogenerator T of the semigroup $\{Z(t)\}$,

$$T = (B+iI)(B-iI)^{-1} = I - 2\int_0^\infty e^{-t}Z(t)\,dt = I|_K - 2\int_0^\infty e^{-t}P_K U(t)|_K dt.$$

†In fact, the functions $S_a(G, \sigma|\lambda)$ and $\Theta_B(\lambda) = s\text{-}\lim_{\tau\downarrow 0}\Theta_B(\lambda - i\tau)$ are connected by the equality

$$\Theta_B(\lambda) = O_1 S_a(G, \sigma|\lambda)O_0,$$

where O_1 and O_0 are arbitrary unitary mappings of N onto the space in which the operator-function $\Theta_B(z)$ operates.

For Eq. (1) in even-dimensional spaces, the boundary value of the characteristic contraction operator-function

$$T = I|_K - 2\int_0^\infty e^{-t} P_K U(t)|_K dt,$$

which operates in the subspace $K = H(G, \sigma) \ominus \overline{(\mathcal{D}_+^a + \mathcal{D}_-^a)}$, coincides with the function $\mathscr{E}(G, \sigma\,|\cdot)$ in expression (4) for the scattering matrix to within multiplication by constant, unitary operators and a change of argument $\lambda = i(\zeta + 1)(\zeta - 1)^{-1}$. But then the semigroup of contractions generated by the scattering operator $B = i(T + I)(T - I)^{-1}$ does not coincide with the operator-function $P_K U(t)|_K$, $t > 0$, that characterizes the dissipation of the energy from Ω_a. In fact, the values of this function no longer even form a semigroup. Nevertheless, the function $P_K U(t)|_K$ can be reconstructed from $\mathscr{E}(G, \sigma\,|\,\lambda)$ to within an isomorphism. That is, the following theorem holds.

THEOREM 2. There exists a unitary mapping W of the subspace K onto the subspace $H_2^+(N) \ominus \mathscr{E}(G, \sigma\,|\cdot) H_2^+(N)$ such that, for any $f \in K$ and any $t > 0$,

$$(WP_K U(t) f)(\lambda) = e^{-i\lambda t}(Wf)(\lambda) + $$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\mu t} - e^{-i\lambda t}}{\lambda - \mu} [\mathscr{E}(G, \sigma\,|\,\lambda)\,\overline{p(\mu)} - \overline{q(\mu)}\,I]\,[\mathscr{E}(G, \sigma\,|\,\mu)\,\overline{p(\mu)} - \overline{q(\mu)}I]^{-1}(Wf)(\mu)\,d\mu.$$

Also,

$$(W\exp(-iBt)f)(\lambda) = e^{-i\lambda t}(Wf)(\lambda) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\mu t} - e^{-i\lambda t}}{\lambda - \mu}\mathscr{E}(G, \sigma\,|\,\lambda)\,\mathscr{E}^*(G, \sigma\,|\,\mu)(Wf)(\mu)\,d\mu.$$

Theorem 2 is a corollary of general propositions of coupling theory **of semiunitary operators [3].** A description of the operator-theoretic constructions leading directly to the proof of this theorem will be presented elsewhere. In this article, only Theorem 1 is proved.

2. Preliminary Remarks

Let $R_m \setminus \overline{G}_1$ and $R_m \setminus \overline{G}_2$ be domains that have smooth boundaries ∂G_1 and ∂G_2 and are located in the sphere Ω_a. Let $\sigma_1(x)$ and $\sigma_2(x)$ be arbitrary smooth, nonnegative functions on ∂G_1 and ∂G_2. Let $\mathscr{F}_\pm(G_1, \sigma_1)$ and $\mathscr{F}_\pm(G_2, \sigma_2)$ be isometric mappings of the spaces $H(G_1, \sigma_1)$ and $H(G_2, \sigma_2)$ onto $L_2(N)$ that satisfy conditions (3). Since each of the spaces $H(G_1, \sigma_1)$ and $H(G_2, \sigma_2)$ is a completion of the lineal $\mathcal{D}_+^a + \mathcal{D}_-^a$, the operators $\mathscr{F}_\pm(G_2, \sigma_2)\mathscr{F}_\pm^*(G_1, \sigma_1)$ are uniquely defined on the subspaces $H_2^\pm(N)$, respectively. The operators $\mathscr{F}_\pm(G_2, \sigma_2)\mathscr{F}_\pm^*(G_1, \sigma_1)$ obviously map the subspaces $H_2^\pm(N)$ isometrically onto themselves and commute in $H_2^\pm(N)$ with the semigroups of operators of multiplication by the functions $e^{\pm \lambda t}$, $t > 0$. By means of arguments usually adduced in the proof of the Beurling–Lax theorem concerning invariant spaces of semigroups of translations, it is easy to verify that operators with the same properties as $\mathscr{F}_\pm(G_2, \sigma_2)\mathscr{F}_\pm^*(G_1, \sigma_1)$ operate on vector-functions in $H_2^\pm(N)$ like operators of "multiplication" by unitary operators in N that do not depend on the variable λ.

Since $\mathscr{F}_\pm^*(G_i, \sigma_i)\mathscr{F}_\pm(G_i, \sigma_i) = I|_{H(G_i, \sigma_i)}$, $i = 1, 2$, this last assertion means that there exist unitary operators $Q_\pm$ in N such that, for arbitrary vectors $f_\pm \in \mathcal{D}_\pm^a$, the equalities

$$(\mathscr{F}_\pm(G_2, \sigma_2)f_\pm)(\lambda) = Q_\pm(\mathscr{F}_\pm(G_1, \sigma_1)f_\pm)(\lambda) \tag{7}$$

are valid.

In particular, these equalities hold also in the case when one of the domains in (7) contains no points at all, i.e., in the case when one pair of the mappings $\overset{0}{\mathscr{F}}_\pm$ in (7) satisfying conditions (3) is constructed for the group $\{U_0(t)\}$ generated by the wave equation in free space.

Analogous considerations show that, in addition, any mappings $\mathscr{F}'_\pm(G, \sigma)$ and $\mathscr{F}''_\pm(G, \sigma)$ constructed for a general domain G and a general function σ that satisfy conditions (3) coincide for like "±" indices to within multiplication on the left by an arbitrary unitary operator in N.

Having fixed the mappings $\overset{0}{\mathscr{F}}_\pm$ constructed for the wave equation in a space with no obstructions in accordance with requirements (3), we will assume in what follows that the mappings $\mathscr{F}_\pm(G, \sigma)$ are subject to the conditions

$$\mathscr{F}_\pm(G, \sigma)\,|_{\mathcal{D}_\pm^a} = \overset{0}{\mathscr{F}}_\pm\,|_{\mathcal{D}_\pm^a} \tag{8}$$

and, therefore, that all of the mappings $\mathcal{F}_{\pm}(G,\sigma)$ coincide on the domains $\mathcal{D}^a_{\pm}$.

Denote by H_0 the Hilbert space of pairs of functions $f = (f_1, f_2)$ on R_m with the norm

$$\|f\|^2_{H_0} = \frac{1}{2} \int_{R_m} \{|(\partial_x f_1)(x)|^2 + |f_2(x)|^2\}\, dx.$$

From the definition of the subspaces $\mathcal{D}^a_{\pm}$, it follows that the supports of the functions in these subspaces are concentrated in $R_m \setminus \Omega_a$. Therefore, as we have already mentioned, the equalities

$$(f_+, f_-)_{H(G,\sigma)} = \frac{1}{2} \int_{|x|>a} \{((\partial_x f_{+,1})(x)\overline{(\partial_x f_{-,1})(x)}) + f_{+,2}(x)\overline{f_{-,2}(x)}\}\, dx = (f_+, f_-)_{H_0} \tag{9}$$

are valid for any vectors $f_{\pm} \in \mathcal{D}^a_{\pm}$ independently of the domain $R_m \setminus G$ in the sphere Ω_a and of the function σ on ∂G.

Let $S^0_a(\lambda)$ be the scattering matrix for the wave equation in free space. Recalling the definition of the scattering matrix $S_a(G, \sigma|\lambda)$, we have

$$(f_+, f_-)_{H(G,\sigma)} = \int_{-\infty}^{\infty} ([\mathcal{F}_-(G,\sigma) f_+](\lambda),\ [\mathcal{F}_-(G,\sigma) f_-](\lambda))\, d\lambda =$$

$$= \int_{-\infty}^{\infty} (S_a(G,\sigma|\lambda)[\mathcal{F}_+(G,\sigma) f_+](\lambda),\ [\mathcal{F}_-(G,\sigma) f_-](\lambda))\, d\lambda =$$

$$= \int_{-\infty}^{\infty} (S_a(G,\sigma|\lambda)[\mathcal{F}^0_+ f_+](\lambda),\ [\mathcal{F}^0_- f_-](\lambda))\, d\lambda$$

on the basis of the properties of the mappings $\mathcal{F}_{\pm}(G,\sigma)$ [including equalities (8)] for arbitrary vectors $f_{\pm} \in \mathcal{D}^a_{\pm}$. Since, on the other hand,

$$(f_+, f_-)_{H_0} = \int_{-\infty}^{\infty} (S^0_a(\lambda)[\mathcal{F}^0_+ f_+](\lambda),\ [\mathcal{F}^0_- f_-](\lambda))\, d\lambda$$

and since the vector-functions $\mathcal{F}^0_{\pm} f_{\pm}$, $f_{\pm} \in \mathcal{D}^a_{\pm}$, span the subspaces $H^{\pm}_2(N)$, we conclude on the basis of (9) that the difference $S_a(G, \sigma|\lambda) - S^0_a(\lambda)$ is the boundary value of a bounded operator-function that is holomorphic in the lower half-plane.

By $L_\infty([N, N])$, denote the space of essentially bounded operator-functions on the real axis that map N into N. In $L_\infty([N, N])$, single out the subspace $H^+_\infty([N, N])$ of boundary values of bounded operator-functions that are holomorphic in the lower half-plane. Define the functions C_1 and C_2 of $L_\infty([N, N])$ to be equivalent if $(C_1 - C_2) \in H^+_\infty([N, N])$. From the reasoning adduced above, it follows that all scattering matrices $S_a(G, \sigma|\lambda)$ belong to the same equivalence class as the function $S^0_a(\lambda)$.

Appropriately choosing the mappings $\mathcal{F}^0_{\pm}$, we have $S^0_a(\lambda) = e^{-2i\lambda a} I$, if R_m is of odd dimension, and $S^0_a(\lambda) = e^{-2i\lambda a}\,\mathrm{sign}\,\lambda I$, if R_m is of even dimension [2].

In the odd-dimensional case, the class of functions that are equivalent to the scattering matrix $S^0_a(\lambda)$ obviously coincides with the subspace $H^+_\infty([N, N])$, and, therefore, all scattering matrices $S_a(G, \sigma|\lambda)$, considered as unitary-valued functions in this class, turn out to be boundary values of interior operator-functions.

In the even-dimensional case, the scattering matrices $S_a(G, \sigma|\lambda)$ are unitary-valued functions equivalent to the function $e^{-2i\lambda a}\,\mathrm{sign}\,\lambda\, I$. To prove Theorem 1, it now remains only to verify that all such functions are described by formula (5) when the "parameter" $\mathcal{E}(G,\sigma|\lambda)$ runs through the set of boundary values of the interior operator-functions.

3. Completion of the Proof of Theorem 1

First of all, let us show that any scalar contraction function $s(\lambda)$ that is equivalent to the function $s_0(\lambda) = e^{-2i\lambda a}\,\mathrm{sign}\,\lambda$ in the sense indicated above can be represented as a linear-fractional transformation

$$s(\lambda) = [\bar{p}(\lambda) e(\lambda) + \bar{q}(\lambda)]\,[p(\lambda) + q(\lambda) e(\lambda)]^{-1} \tag{10}$$

of a contraction function $e(\lambda)$ in the subspace H^+_∞ $(= H^+_\infty([N, N])$, $\dim N = 1)$ of L_∞.

Let M_i be the set of scalar contraction functions equivalent to the function

$$s_i(\lambda) = \frac{i\bar{p}(\lambda) + \bar{q}(\lambda)}{p(\lambda) + iq(\lambda)} = -e^{-2i\lambda a}\,\mathrm{sign}\,\lambda\,\frac{H_0^{(1)}(\lambda a)}{H_0^{(2)}(\lambda a)}. \tag{11}$$

In addition to $s_i(\lambda)$, the function

$$s_{-i}(\lambda) = \frac{-i\bar{p}(\lambda) + \bar{q}(\lambda)}{p(\lambda) - iq(\lambda)} = -e^{-2i\lambda a}\,\mathrm{sign}\,\lambda\,\frac{H_1^{(1)}(\lambda a)}{H_1^{(2)}(\lambda a)} \tag{12}$$

also belongs to the family M_i. Indeed, by virtue of well-known properties of cyclindrical functions and, in particular, in view of the fact that, on the main branches of the functions $H_0^{(2)}(z)$ and $H_1^{(2)}(z)$, there are no zeros, the difference

$$s_i(\lambda) - s_{-i}(\lambda) = -\frac{4ie^{-2i\lambda a}}{\pi\lambda a}\cdot\frac{1}{H_0^{(2)}(\lambda a)\,H_1^{(2)}(\lambda a)}$$

turns out to be in H_∞^+.

Since the family of equivalent contraction functions M_i contains more than one element, by the basic result of [4] there exist functions $\tilde{p}(z)$ and $\tilde{q}(z)$ holomorphic in the lower half-plane satisfying the conditions:

1) $|\tilde{p}^{-1}(z)| < 1$, $|\tilde{q}(z)\tilde{p}^{-1}(z)| < 1$, $\mathrm{Im}\, z < 0$;

2) $\tilde{p}(-i) > 0$, $\tilde{q}(-i) = 0$;

3) $|\tilde{p}(\lambda)|^2 - |\tilde{q}(\lambda)|^2 = 1$, $\mathrm{Im}\,\lambda = 0$;

4) $\tilde{p}(z)$ is an exterior function, i.e.,

$$\ln|\tilde{p}(-i)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|\tilde{p}(\lambda)|}{1+\lambda^2}\,d\lambda.$$

Also, the functions $\tilde{p}(z)$ and $\tilde{q}(z)$ are such that a one-to-one correspondence between the contraction functions $e \in H_\infty^+$ and the functions of M_i is established by the formula

$$s_e(\lambda) = [\bar{\tilde{p}}(\lambda)e(\lambda) + \bar{\tilde{q}}(\lambda)]\,[\tilde{p}(\lambda) + \tilde{q}(\lambda)e(\lambda)]^{-1}. \tag{13}$$

In addition, in order that the function $s_e \in M_i$ be the image of a unitary constant ε under the mapping (11), i.e., that it be a so-called canonical function [4], it is necessary and sufficient that one can put in correspondence with the function $s_e \in M_i$ one and only one (to within multiplication by a positive, real number) function φ out of the Hardy subspace H_2^+ that is connected with s_e by the relation $s_e(\lambda) = (\lambda + i)\varphi_e(\lambda)/(\lambda - i)\bar{\varphi}_e(\lambda)$.

In such cases,

$$\varphi_e(\lambda) = \frac{1}{\lambda - i}\,\frac{1}{\bar{\theta}\tilde{p}(\lambda) + \theta\tilde{q}(\lambda)}, \qquad \theta^2 = \varepsilon, \tag{14}$$

to within multiplication by a real constant.

Analyzing the explicit expressions and the asymptotic expansions for the functions $H_0^{(2)}$ and $H_1^{(2)}$, it is easy to see that the functions $s_i(\lambda)$ and $s_{-i}(\lambda)$ are canonical. On the basis of Eqs. (11), (12), and (14), this assertion implies that

$$\begin{aligned} e^{-i\frac{\pi}{4}}p(\lambda) + e^{i\frac{\pi}{4}}q(\lambda) &= c_1[e^{-i\psi_1}\tilde{p}(\lambda) + e^{i\psi_1}\tilde{q}(\lambda)], \quad \mathrm{Im}\, c_1 = 0,\\ e^{i\frac{\pi}{4}}p(\lambda) + e^{-i\frac{\pi}{4}}q(\lambda) &= c_2[e^{-i\psi_2}\tilde{p}(\lambda) + e^{i\psi_2}\tilde{q}(\lambda)], \quad \mathrm{Im}\, c_2 = 0. \end{aligned} \tag{15}$$

The functions $p(\lambda)$ and $q(\lambda)$ have the following obvious properties:

$$1)\ p(-i) > 0,\quad q(-i) = 0; \qquad 2)\ |p(\lambda)|^2 - |q(\lambda)|^2 = 1. \tag{16}$$

Taking these properties as well as the properties of the functions $\tilde{p}(\lambda)$ and $\tilde{q}(\lambda)$ and those of equalities (15) into account, we arrive at the conclusion that $\tilde{p}(\lambda) = p(\lambda)$ and $\tilde{q}(\lambda) = q(\lambda)$.

It remains only to verify that $s_0(\lambda)$ belongs to Mi. But $s_0(\lambda)$ is a linear-fractional transformation (10) of the boundary value of the function

$$e_0(z) = i\frac{J_1(za)H_0^{(2)}(-ia) + J_0(za)H_1^{(2)}(-ia)}{J_1(za)H_0^{(2)}(-ia) - J_0(za)H_1^{(2)}(-ia)}, \quad \operatorname{Im} z < 0,$$

where $J_0(za)$ and $J_1(za)$ are Bessel functions. Note that $|e_0(z)| < 1$ if $\operatorname{Im} z < 0$ and that $|e_0(\lambda)| = 1$ if $\operatorname{Im} \lambda = 0$. Therefore, the set of contraction functions on L_∞ that are equivalent to $s_0(\lambda)$ coincides with the image of the set of contraction functions in H_∞^+ under the linear-fractional transformation (10).

As a by-product we have discovered that $p(\lambda)$ is the boundary value of an exterior function and that the function $\chi(\lambda) = q(\lambda)p^{-1}(\lambda)$ is a contraction function ($|\chi(\lambda)| < 1$ for $\lambda \neq 0$, $\chi(0) = -1$) and belongs to H_∞^+.

We can now complete the proof of Theorem 1. Let $S(\lambda)$ be a unitary-valued function in the equivalence class $S_a^0(\lambda)$ ($= s_0(\lambda)I$). On the basis of the equality 2) in (16), one can represent $S(\lambda)$ in the form of a linear-fractional transformation (5) of some unitary function $\xi \in L_\infty([N, N])$. For any vector $h \in N$, the scalar function $(S(\lambda)h, h)$ belongs to the same equivalence class as $s_0(\lambda)$ (h, h). Therefore, the scalar function

$$(S(\lambda)h, h) - ([\bar{p}(\lambda)\cdot 0 + \bar{q}(\lambda)][p(\lambda) + 0\cdot q(\lambda)]^{-1}h, h) =$$
$$= (S(\lambda)h, h) - (\bar{q}(\lambda)p^{-1}(\lambda)h, h) = \frac{1}{p^2(\lambda)}(\mathscr{E}(\lambda)[I + \chi(\lambda)\mathscr{E}(\lambda)]^{-1}h, h)$$

belongs to H_∞^+ and can be represented in the form

$$p^{-2}(\lambda)e_h(\lambda)[1 + \chi(\lambda)e_h(\lambda)]^{-1}(h, h),$$

where e_h is a function in H_∞^+ that depends on $h \in N$. This implies that, for $h \in N$, the function

$$([I - \chi(\lambda)\mathscr{E}(\lambda)][I + \chi(\lambda)\mathscr{E}(\lambda)]^{-1}h, h)$$

is the boundary value of the function

$$[1 - \chi(z)e_h(z)][1 + \chi(z)e_h(z)]^{-1}(h, h)$$

that is holomorphic in the lower half-plane and, obviously, has nonnegative real part there.

Thus, there exists an operator-function that is holomorphic in the lower half-plane and that has a nonnegative real part and weak boundary values that coincide almost everywhere with the function $[I - \chi(\lambda)\cdot\mathscr{E}(\lambda)][I + \chi(\lambda)\mathscr{E}(\lambda)]^{-1}$. This obviously implies that the function $\chi(\lambda)\mathscr{E}(\lambda)$ is a contraction function in $H_\infty^+([N, N])$.

Since $p(z)$ is an exterior function, for any h and g in N, the function

$$(\mathscr{E}(\lambda)h, g) = p^2(\lambda)([I + \chi(\lambda)\mathscr{E}(\lambda)][S(\lambda) - p^{-1}(\lambda)\bar{q}(\lambda)I]h, g),$$

which is a contraction function on the real axis, can be represented in the form of the product of a function in H_∞^+ and the boundary value of an exterior function. That is, for any h and g in N, the function $(\mathscr{E}(\cdot)h, g) \in H_\infty^+$. Therefore, the unitary-valued function $\mathscr{E}(\lambda)$ is the boundary value of an interior function in $H_\infty^+([N, N])$. Theorem 1 is proved.

Remark 1. Formula (5) was discovered by the author while studying the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x}$$

on the semiaxis using the theory of coupling of semiunitary operators [3].

Remark 2. Let H be an arbitrary Hilbert space that is a completion of the subspace $\overline{\mathscr{D}_+^a + \mathscr{D}_-^a}$. Let $\{U_0(t)\}$ be the group of unitary operators generated by the wave equation in free, even-dimensional space. Let $\{U(t)\}$ be any group of unitary operators in H that satisfy the conditions: 1) $U(t)|_{\mathscr{D}_+^a} = U_0(t)|_{\mathscr{D}_+^a}$, $t > 0$; 2) $U(t)|_{\mathscr{D}_-^a} = U_0(t)|_{\mathscr{D}_-^a}$, $t < 0$; and 3) $\overline{\bigcup_t U(t)D_\pm^a} = H$. Denote the scattering matrix of $\{U(t)\}$ by $S(\lambda)$. Just as is done above, define it by means of the special spectral representations of $\{U(t)\}$ connected with the subspaces $\mathscr{D}_\pm^a$. It is easy to see that the operator-function $S(\lambda)$ is unitary-valued and, to within multiplication by constant unitary operators, is equivalent to the scattering matrix $S_a^0(\lambda)$. Therefore, the assertion of Theorem 1 can be extended to $S(\lambda)$.

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