VARIATIONAL PRINCIPLES FOR NONLINEAR EIGENVALUE PROBLEMS

Yu. Sh. Abramov

Let H be a real Hilbert space and S the set of all bounded symmetrical operators in H. In the present note the following nonlinear eigenvalue problem is considered:

$$L_\lambda x = 0, \quad \lambda \in (c, d), \qquad x \in H, \tag{1}$$

where L: (c, d) → S is a continuously differentiable (with respect to the operator norm) function of λ on the interval (c, d) Problem (1) has been discussed in many publications under various assumptions of the dependence of L_λ on λ, among other works in [1-7]. Variational principles are established below which are similar to the classical principles of Rayleigh, Fischer–Courant–Weyl, Poincaré–Ritz.

1. Definition [1]. Let p: $H\setminus\{0\}$ → (c, d) be a continuous functional. Let us suppose that the following conditions hold: 1) $p(\alpha x) = p(x), \alpha \in R, \alpha, x \neq 0$, 2) $(L_{p(x)}x, x) = 0$, 3) $(L'_{p(x)}x, x) > 0$, then p is called a Rayleigh functional for L_λ. The pair (L, p) is called a Rayleigh system (R.s.).

The set of all values of p is denoted by W_p. A number $\lambda \in W_p$ and a vector $x \neq 0$, which are solutions of the problem (1), are called an eigenvalue (e.v.) and the eigenvector of the R.s. corresponding to it. One denotes by $p\sigma$ the totality of the eigenvalues of a R.s. and by $\mathscr{P}_\lambda$ and P_λ the eigensubspace and the orthogonal projection onto it, $\lambda \in p\sigma$. The dimension of $\mathscr{P}_\lambda$ is the multiplicity of λ. One sets $\gamma_d = \sup p(x)$, $\gamma_c = \inf p(x)$. We shall consider the following point sets $\lambda \in \overline{W}_p$ (for each λ under consideration the existence is assumed of a sequence $\{x_n\}, \|x_n\| = 1$ with the following properties):

$$\sigma_1 = \{\lambda: L_\lambda x_n \to 0\}, \qquad \pi_1 = \{\lambda: L_\lambda x_n \to 0,\ x_n \rightharpoonup 0\},$$

$$\sigma_2 = \{\lambda: L_\lambda x_n \to 0,\ p(x_n) \to \lambda\}, \quad \pi_2 = \{\lambda: L_\lambda x_n \to 0,\ x_n \rightharpoonup 0,\ p(x_n) \to \lambda\},$$

(the arrow ⇀ indicates weak convergence in H). Moreover, if $\gamma_d = d(\gamma_c = c)$, it is assumed that d(c) belongs to all the sets enumerated above. It is not difficult to see that all these sets must be closed and also that $p\sigma \subseteq \sigma_2 \subseteq \sigma_1 \subseteq \overline{W_p}$, $\pi_2 \subseteq \pi_1$, $\pi_i \subseteq \sigma_i$, $i = 1, 2$. We further set

$$\Delta[\alpha, \beta] = (\alpha - \beta)^{-1}(L_\alpha - L_\beta), \quad \alpha \neq \beta, \quad \Delta[\alpha, \alpha] = L'_\alpha;$$

$$[x, y]_\lambda = (\Delta[\lambda, p(y)]x, y) \ \text{for} \ y \neq 0 \ \text{and} \ [x, 0]_\lambda = 0;\ [x, y] = [x, y]_{p(x)}.$$

It is observed that in the classical case of $L_\lambda = \lambda I - A$, the functional $p(x) = (Ax, x)/(x, x)$; W_p forms a numerical region, $\sigma_1 = \sigma_2$ is the spectrum, $\pi_1 = \pi_2$ is the limiting spectrum of A, $[x, y]_\lambda = (x, y)$, $\lambda \in (-\infty, \infty)$. It is assumed below that (L, p) is a R.s.

2. The following propositions are valid.

LEMMA 1 (analog of the Weyl criterion). 1) The point λ belongs to $\sigma_1\setminus\pi_1$ if and only if λ is an isolated point of σ_1, which is an e.v. of finite multiplicity and 0 is an isolated point of the spectrum of the operator L_λ. 2) If $\lambda \in \sigma_2\setminus\pi_2$, then λ is an isolated point of σ_2 and it is an eigenvalue of finite multiplicity.

LEMMA 2. Let $\lambda_0 \in \sigma_1\setminus\pi_1$; then in a neighborhood U_{λ_0} of a point λ_0 the following expansion for the resolvent $R_\lambda = L_\lambda^{-1}$ is valid:

$$R_\lambda = (\lambda - \lambda_0)^{-1} P_{\lambda_0} K_\lambda + Q_\lambda, \quad \lambda \in U_{\lambda_0} \setminus \{\lambda_0\} \subset (c, d),$$

in which the functions $K_\lambda^{(\lambda_0)} = K_\lambda$ and $Q_\lambda^{(\lambda_0)} = Q_\lambda$ are continuous in λ_0 and $K_{\lambda_0}x = L_{\lambda_0}Q_{\lambda_0}x$, $x \in \mathscr{P}_{\lambda_0}^{\perp}$.

Institute of Economics, Leningrad. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 7, No. 4, pp. 76-77, October-December, 1973. Original article submitted January 24, 1972.

The functions $Q_\lambda^{(\lambda_0)}$ and $K_\lambda^{(\lambda_0)}$ are defined for $\lambda_0 \in \sigma_1 \setminus \pi_1$ and $\lambda \in U\lambda_0$. At the points $\lambda, \lambda_0 \in [\gamma_c, \gamma_d] \setminus \pi_1$ they are defined by the formulas $Q_\lambda^{(\lambda_0)} = R_\lambda$, $K_\lambda^{(\lambda_0)} = I$. We now introduce the functions $\hat{R}_\lambda = Q_\lambda^{(\lambda)}$, $I_\lambda = K_\lambda^{(\lambda)}$, $F_\lambda = (I - P_\lambda I_\lambda L'_\lambda)\hat{R}_\lambda$, $\mathcal{R}(\lambda, x) = (\hat{R}_\lambda x, x)$, $\lambda \in [\gamma_c, \gamma_d] \setminus \pi_1$, $x \in H$. For $\lambda \in p\sigma$ we set $\mathcal{P}_\lambda = \{0\}$.

LEMMA 3. If $\lambda \in [\gamma_c, \gamma_d] \setminus \pi_1$ and $x \in \mathcal{P}_\lambda^\perp$, then $\mathcal{R}(\lambda, x) = (L_\lambda F_\lambda x, F_\lambda x)$ and $\mathcal{R}'(\lambda, x) = -(L'_\lambda F_\lambda x, F_\lambda x)$.

LEMMA 4. If $\alpha \in \overline{W}_p$, $(\alpha, \gamma_d] \cap \pi_1 = \phi$ and $y \in \mathcal{P}_\lambda^\perp \setminus \{0\}$, $\lambda \in (\alpha, \gamma_d]$, then $\mathcal{R}(\lambda, y) > 0$, $\lambda \in (\alpha, \gamma_d]$.

LEMMA 5. 1) Let γ_d and $\gamma_c \in \sigma_2$. 2) Let $E \subseteq H$, codim $E < \infty$ and $\gamma = \sup\{p(x), x \in E\} \notin \pi_2$; then there exists $y \in E \setminus \{0\}$, such that $p(y) = \gamma$ and $PL_\gamma y = 0$, where P is the projector onto E.

Let $\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant \ldots$, $\lambda_i \in p\sigma$ and let $x_1, \ldots, x_n, \ldots$ be linearly independent eigenelements which correspond to them. One sets

$$\underline{n} = \min\{i : \lambda_i = \lambda_n\}, \quad \bar{n} = \max\{i : \lambda_i = \lambda_n\}, \quad X_n = [x_1, \ldots, x_n],$$
$$X_n(\lambda) = [\Delta[\lambda, \lambda_1] x_1, \ldots, \Delta[\lambda, \lambda_n] x_n],$$
$$E^n(\lambda) = H \ominus X_n(\lambda), \quad \Gamma^n(\lambda) = \sup\{p(x), x \in E^n(\lambda)\},$$
$$\overline{\Gamma}^n(\lambda) = \overline{\lim_{\mu \to \lambda}} \Gamma^n(\mu), \quad \underline{\Gamma}^n(\lambda) = \underline{\lim_{\mu \to \lambda}} \Gamma^n(\mu).$$

The totality of subspaces of H of dimension (codimension) n is denoted by $\mathscr{E}_n$ ($\mathscr{E}^n$).

LEMMA 6. Let $\lambda \in W_p$ and $\lambda \leq \lambda_n$. Then: 1) $X_n(\lambda) \in \mathscr{E}_n$, $E^n(\lambda) \in \mathscr{E}^n$, 2) $H = X_n + E^n(\lambda)$, 3) if $[\underline{\Gamma}^n(\lambda), \overline{\Gamma}^n(\lambda)] \cap \pi_2 = \phi$, then the function Γ^n is continuous at the point λ.

LEMMA 7 (analog of the Weyl inequality). If $E \in \mathscr{E}^i$, $1 \leqslant i \leqslant \bar{n} - 1$, then $\sup\{p(x), x \in E\} \geq \lambda_n$.

LEMMA 8 (analog of the Poincaré inequality). If $E \in \mathscr{E}_i$, $i \geqslant \underline{n}$, then $\min\{p(x), x \in E\} \leq \lambda_n$.

3. THEOREM. Let $\beta < \gamma_d$ and $(\beta, \gamma_d] \cap \pi_2 = \phi$. Then: a) $(\beta, \gamma_d] \cap \sigma_2 \neq \phi$ and it consists of isolated eigenvalues of finite multiplicity of a R.s. $\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant \ldots (\lambda_1 = \gamma_d)$; b) the corresponding to them eigenelements $x_1, \ldots, x_n, \ldots$ can be selected as linearly independent; c) the following variational principles are valid:

$$\lambda_n = \max_{\substack{[x, x_i]_{\lambda_n} = 0 \\ i=1,\ldots,n-1}} p(x) = \min_{E \in \mathscr{E}^{n-1}} \max_E p(x) = \max_{E \in \mathscr{E}_n} \min_E p(x). \tag{2}$$

However, if $(\beta, \gamma_d] \cap \pi_1 = \phi$, then in addition one has

$$\lambda_n = \max_{\substack{[x, x_i] = 0 \\ i=1,\ldots,n-1}} p(x), \; [x_i, x_j] = \delta_{ij}, \; i, j = 1, 2, \ldots. \tag{3}$$

In the first relation of (2) and in (3) the maximum is attained on x_n.

4. From our theorem the results of [2, 4, and 5] as well as some results of [3] follow as particular cases. The results established in this note can be applied in the theory of elliptic differential operators which depend nonlinearly on a parameter.

The author is indebted to D. F. Kharazov for his assistance in this work.

LITERATURE CITED

1. E. H. Rogers, Arch. Rat. Mech. Anal., 16, No. 2, 89-96 (1964).
2. R. E. L. Turner, J. Math. Anal. Appl., 17, 151-160 (1967).
3. R. E. L. Turner, J. Func. Anal., 2, 297-322 (1968).
4. K. P. Hadeler, Arch. Rat. Mech. Anal., 30, 297-307 (1968).
5. R. Kühne, Acta Scient. Math. Szeged, 29, Nos. 1-2, 39-68 (1968).
6. D. F. Kharazov, Trudy Tbil. Matem. In-ta Akad. Nauk Gruz. SSSR, 26, 171-187 (1959).
7. M. V. Keldysh, Dokl. Akad. Nauk SSSR, 71, No. 4, 11-14 (1951).