ON ERGODIC PROPERTIES OF CERTAIN BILLIARDS

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It is known (cf. [1] and [2]) that dispersing billiards (i.e., billiards in regions Q, located on a two-dimensional torus T^2 or on the Euclidean plane R^2, whose boundary consists of components convex inward from Q) are K systems in the sense of A. N. Kolmogorov [3, 4]. In [5], an analogous result was obtained for one class of regions $Q \subset R^2$ whose boundaries contain both dispersing and focusing components. On the other hand, billiards in a convex region with sufficiently smooth boundaries are not ergodic (cf., for example, [6]).

The aim of the present note is to show that there exist convex regions $Q \subset R^2$ such that a billiard in Q is a K system. To V. I. Arnol'd [7] is due the graphic explanation of the analogy between dispersing billiards and geodesic flows on surfaces of negative curvature. In this sense, the billiards considered by us are analogous to geodesic flows on surfaces whose curvature is not negative.

1. Description of Dynamic Systems. Let Q be a bounded closed region on the Euclidean plane whose boundary consists of a finite number of smooth (class C^3) components. We assume the boundary ∂Q to be fitted out with a field of internal (with respect to Q) normals n(q). Let the curvature on each regular component of the boundary be either sign-constant or identically zero.

Consider a billiard in region Q. The phase space of the billiard is the space of linear elements M. We denote by π the natural projection of M onto Q. In space M we introduce measure μ, setting $d\mu = dqd\omega$, where dq is the measure on Q induced by the Euclidean metric and $d\omega$ is the natural measure on strip $S^1(q)$. We assume measure μ to be normalized. Let $\{S_t\}$ be a single-parameter group of shifts along the trajectory of the billiard. It retains measure μ (cf. [8]) and, consequently, $\{S_t\}$ is a flow in the sense of ergodic theory.

We denote by ∂Q^+ the union of all the regular components of boundary ∂Q each of which has positive curvature at all points of the given component (with the field of normals we have selected). Analogously, ∂Q^- is the union of all regular components of boundary ∂Q with negative curvature, while ∂Q^0 is the boundary components with zero curvature. We shall henceforth call ∂Q^+, ∂Q^-, and ∂Q^0 the dispersing, focusing, and neutral parts of the boundary, respectively.

Consider the following special representation (cf. [9]) of flow $\{S_t\}$. We set $M_1 = \{x:(x,\ n(q)) < 0,\ \ q \in \partial Q,\ q = \pi(x)\}$, and let $\tau(x)$ be the closest negative moment of the reflection from ∂Q of the trajectory of point x. It is easy to see that $\tau(x) > -\infty$ and, for any $x \in M_1$, there is defined the transformation $Tx = S_{\tau(x)-0}x \in M_1$.

In this note we study billiards in regions whose boundaries do not contain dispersing components, i.e., $\partial Q^+ = \phi$. Let $\pi(x) \in \partial Q^-$, and let us denote by $\bar{\tau}(x)$ the closest negative moment of the reflection from ∂Q^- of the trajectory of point x.

2. Conditions on the Boundary of the Region. We consider regions Q satisfying the following conditions (we denote by Γ_i the regular components of ∂Q):

1) Each focusing component Γ_i has constant curvature, i.e., is an arc of some circle O_{Γ_i}.

2) For almost all points $x \in \Gamma_i \subset \partial Q^-$ such that $\pi(Tx) \notin \Gamma_i$, the length of the chord of circle O_{Γ_i} defined by line element x is strictly less than $|\tau(x)|$.

3) No two focusing components are arcs of one and the same circle.

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4) The ends of each focusing component belong to ∂Q^0.

3. The Basic Theorem. We denote by $\tilde{Q}$ a region satisfying conditions 1)-4), and whose boundary has exactly two focusing and two neutral components. theboundary,

THEOREM 1. A billiard in region $\tilde{Q}$ is a K system.

Consider a bundle of trajectories with zero curvature. Then (cf. [5]), if this bundle undergoes a series of successive reflections from some focusing component of the boundary then, after each reflection of this series, it becomes convergent (i.e., has negative curvature) after which, between two successive reflections it passes through the conjugate point and, having already become divergent, arrives at the boundary prior to the next reflection, with the time during which the bundle has positive curvature (diverges) being greater than half the total interval of time between the successive reflections from the focusing component. Upon reflection from the neutral part of the boundary, the curvature of the bundle is not changed.

Now, let the bundle in question be reflected from one focusing component, then, perhaps, undergo a series of successive reflections from the neutral part of the boundary and, finally, be again reflected from the focusing part of the boundary. Then, it follows from conditions 2)-4) that, during such a series of reflections, the bundle undergoes dilation (in the phase space of dynamic systems) with a coefficient strictly greater than unity.

This allows us (using the method of [1, 2, 5]) to construct a transversal strip for flow $\{S_t\}$. For this it is required to show that, for almost all trajectories, series of reflections similar to that considered above occur quite frequently, and each such series has a length which is not very great. Thus, those trajectories are "bad" which have long series of successive reflections from one and the same focusing component or from the neutral part of the boundary $\partial\tilde{Q}^0$ (this latter is possible if the neutral components are parallel). Using the geometric properties of region $\tilde{Q}$ with the help of the Borel–Cantelli Lemmas (cf. [10]), we can show that the set of "bad" trajectories has measure zero.

4. Absorbing Pockets. Let $\partial Q^+ = \phi$ and, for region Q, let conditions 1)-4) be met. For any natural n we consider the set

$$A_n^\alpha = \{x : \pi(x) \in \partial Q^-,\ \pi(T^i x) \in \partial Q^0 \ \text{ for } 0 < i \leqslant n^\alpha\}. \tag{a}$$

THEOREM 2. If there exists α $(0 < \alpha < 1)$ such that $\sum_{n=1}^{\infty} \mu(A_n^\alpha) < \infty$, then a billiard in region Q is a K system.

It is easy to see that, for region $\tilde{Q}$, it follows from Theorem 1 that the condition of Theorem 2 is met.

We call an absorbing pocket a boundary of region $\tilde{Q}$, from which a focusing component is excluded, lying further from the vertex of the angle formed by the neutral components (if the latter are parallel, we then eliminate any of the two focusing components). It can be shown that the condition of Theorem 2 holds for regions whose boundaries are unions of absorbing pockets.

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