A SCHEME FOR INTEGRATING THE NONLINEAR EQUATIONS OF MATHEMATICAL PHYSICS BY THE METHOD OF THE INVERSE SCATTERING PROBLEM. I

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1. General Outline of the Method

In 1967 a group of Princeton theoretical physicists (Gardner, Green, Kruskal, and Miura [1]) discovered a new method of mathematical physics, the method of the inverse scattering problem. The new method enabled its inventors to integrate an equation long known in the theory of nonlinear waves, namely, the Korteweg–deVries (KV) equation

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

In 1970 the authors of the present paper, using ideas advanced in a paper by P. Lax [2], integrated (see [3, 4]), with the aid of the inverse problem method, the equation

$$iu_t + u_{xx} \pm |u|^2 u = 0, \tag{2}$$

also widely used in the physics of waves in nonlinear media. In the succeeding years new examples were found of nonlinear equations integrable by the inverse problem method, over twenty of them having physical meaning (for a survey of integrable equations, see [5]).

In the present paper we give a general scheme for applying the inverse scattering problem method to integrate nonlinear differential equations, and we also present an algorithm for finding equations which can be so integrated. The original version of this scheme was presented by one of us in [6].

We consider a linear integral operator $\hat{F}$ acting on vector-valued functions $\psi = \{\psi_1, \ldots, \psi_N\}$ of the variable $x(-\infty < x < +\infty)$

$$\hat{F}\psi = \int_{-\infty}^{\infty} F(x, z)\psi(z)\,dz. \tag{3}$$

The vector ψ and the $N \times N$ matrix-function F depend additionally on the two parameters t and y. In the sequel we assume that

$$\sup_{x>x_0} \int_{x_0}^{\infty} |F(x, z)|\,dz < \infty \quad \text{for all} \quad x_0 > -\infty. \tag{4}$$

We consider the problem of representing the operator $\hat{F}$ in the following "factorized" form:

$$1 + \hat{F} = (1 + \hat{K}_+)^{-1}(1 + \hat{K}_-); \tag{5}$$

Here $\hat{K}_+$ and $\hat{K}_-$ are Volterra operators, where $K_+(x, z) = 0$ for $z < x$, and $K_-(x, z) = 0$ for $z > x$. The operator $1 + \hat{K}_+$ is invertible. Multiplying Eq. (5) by $1 + \hat{K}_+$ and assuming $z > x$, we can verify that the kernel K_+ satisfies the Gel'fand–Levitan equation

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$$F(x,z)+K_+(x,z)+\int_x^\infty K_+(x,s)F(s,z)\,ds=0. \tag{6}$$

Further, assuming $z < x$, we obtain $K_-(x, z)$:

$$K_-(x,z)=F(x,z)+\int_x^\infty K_+(x,s)F(s,z)\,ds.$$

If Eq. (6) is solvable (a sufficient condition for which, for example, is the possibility of representing the kernel F in the form $F = F_1 + F_2$, where F_1 is a positive operator and $\|F_2\| < 1$), then the kernels K_+ and K_- also satisfy the condition (4). We define on the functions $\psi(x, t, y)$ the operator:

$$\hat{M}=\alpha\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial y}+\hat{L}_{0n},\qquad \hat{L}_{0n}=l\frac{\partial^n}{\partial x^n}.$$

Here α and β are constants and l is a constant matrix. We consider the class of differential operators $\tilde{M}$, connected with $\hat{M}$ by the transformation

$$\hat{\tilde{M}}=(1+\hat{K}_+)\,\hat{M}\,(1+\hat{K}_+)^{-1}, \tag{7}$$

with a Volterra [see Eq. (5)] operator of the transformation $1 + K_+$. Multiplying Eq. (7) on the left by $(1 + \hat{K}_+)$, we obtain

$$\hat{\tilde{M}}(1+\hat{K}_+)-(1+\hat{K}_+)\hat{M}=0. \tag{8}$$

The condition of equating to zero the differential parts in the operator relation (8) enables us to calculate $\tilde{M}$:

$$\hat{\tilde{M}}=\alpha\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial y}+\hat{L}_n,\qquad \hat{L}_n=l\frac{\partial^n}{\partial x^n}+\sum_{k=0}^{n-1}u_k(x)\frac{\partial^{n-k-1}}{\partial x^{n-k-1}}. \tag{9}$$

The coefficients $u_k(x)$ of the operator $\hat{L}$ can be found in the form of a set of recursion relations

$$\begin{aligned}
u_0(x)&=[l,\ \xi_0],\\
u_1(x)&=(n-1)!\,l\frac{d\xi_0}{dx}+\frac{1}{2}\left\{\frac{d\xi_0}{dx},l\right\}+\frac{1}{2}[l,\xi_1]+u_0\xi_0,\\
u_2(x)&=\frac{n(n-2)}{2}\,l\,\frac{d^2\xi_0}{dx^2}+\frac{1}{4}\left[l,\frac{d^2\xi_0}{dx^2}\right]+\frac{n-2}{2}\,l\,\frac{d\xi_1}{dx}+\\
&+\frac{1}{2}\frac{d}{dx}[l,\xi_1]+\frac{1}{4}[l,\xi_2]+u_0\left[\left(n-\frac{3}{2}\right)\frac{d\xi_0}{dx}+\frac{1}{2}\xi_1\right]+u_1\xi_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{10}$$

Here $\xi_i(x)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial z}\right)^i K(x,z)\big|_{z=x},\quad \xi_0(x)=K(x,x).$

Since the relation (7) is linear in M and $\tilde{M}$, we can take $\hat{M}$ to be the operator

$$\hat{M}=\alpha\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial y}+\hat{L}_0,\qquad \hat{L}_0=\sum_n l_n\frac{\partial^n}{\partial x^n}. \tag{11}$$

Then $\hat{\tilde{M}}=\alpha\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial y}+\hat{L},\quad \hat{L}=\sum_n\hat{L}_n$. Equation (8) then has the form

$$\alpha\frac{\partial K_+}{\partial t}+\beta\frac{\partial K_+}{\partial y}+LK_++\sum(-1)^{n-1}\frac{\partial^n}{\partial z^n}K_+l_n=0. \tag{12}$$

For a given operator $\hat{M}$ of the type (11), Eq. (12) and the condition (10) define the class of Volterra operators $1 + \hat{K}_+$, transforming $\hat{M}$ into $\hat{\tilde{M}}$.

The following two theorems play a fundamental role in the sequel.

THEOREM 1. If the operator $\hat{F}$, written in the form (5), satisfies the condition

$$[\hat{M}, \hat{F}] = \hat{M}\hat{F} - \hat{F}\hat{M} = 0, \tag{13}$$

then the operators $(1+\hat{K}_{\pm})$, which accomplish the factorization of $\hat{F}$, satisfy the relations

$$\hat{\hat{M}}(1+\hat{K}_{\pm}) - (1+\hat{K}_{\pm})\hat{M} = 0 \tag{14}$$

and are operators transforming $\hat{M}$ into $\hat{\hat{M}}$.

To prove this we write the identity

$$\hat{\hat{M}}(1+\hat{K}_-) - (1+\hat{K}_-)\hat{M} = \hat{\hat{M}}(1+\hat{K}_+)(1+\hat{F}) - (1+\hat{K}_+)(1+\hat{F})\hat{M} = \\ = \{\hat{\hat{M}}(1+\hat{K}_+) - (1+\hat{K}_+)\hat{M}\}(1+\hat{F}) + (1+\hat{K}_+)\{\hat{M}\hat{F} - \hat{F}\hat{M}\}. \tag{15}$$

By virtue of the relations (10), the differential operator on the right side and, hence, also on the left side of the identity (15) is absent. For $z > x$ the left side in the relation (15) vanishes, and from the invertibility of the operator $1 + F$ it follows that $\hat{\hat{M}}(1+\hat{K}_+) - (1+\hat{K}_+)\hat{M} = 0$. Further, it is obvious that $\hat{\hat{M}}(1+\hat{K}_-) - (1+\hat{K}_-)\hat{M} = 0$.

THEOREM 2. Suppose that the operator $\hat{F}$ satisfies simultaneously the two relations

$$[\hat{M}_1, \hat{F}] = 0, \quad M_1 = \alpha\frac{\partial}{\partial t} + \hat{L}_0^{(1)}, \tag{16}$$

$$[\hat{M}_2, \hat{F}] = 0, \quad M_2 = \beta\frac{\partial}{\partial y} + \hat{L}_0^{(2)}. \tag{17}$$

Here $\hat{L}_0^{(1)}$ and $\hat{L}_0^{(2)}$ are operators of the form

$$\hat{L}_0^{(1)} = \sum_{n=1}^{N_1} l_n^{(1)}\frac{\partial^n}{\partial x^n}, \quad \hat{L}_0^{(2)} = \sum_{n=1}^{N_2} l_n^{(2)}\frac{\partial^n}{\partial x^n}, \tag{18}$$

satisfying the condition

$$[\hat{L}_0^{(1)}, \hat{L}_0^{(2)}] = 0. \tag{19}$$

Then the operators $L^{(1)}$ and $L^{(2)}$ satisfy the relation

$$\beta\frac{\partial\hat{L}^{(1)}}{\partial y} - \alpha\frac{\partial\hat{L}^{(2)}}{\partial t} = [\hat{L}^{(1)}, \hat{L}^{(2)}]. \tag{20}$$

Proof. By virtue of Theorem 1 we have the following relations from Eqs. (17):

$$\hat{\hat{M}}_{1,2}(1+\hat{K}_+) = (1+\hat{K}_+)\hat{M}_{1,2}, \quad \hat{\hat{M}}_1 = \alpha\frac{\partial}{\partial t} + \hat{L}^{(1)}, \quad \hat{\hat{M}}_2 = \beta\frac{\partial}{\partial y} + \hat{L}^{(2)}. \tag{21}$$

Writing out the relations (21) in explicit form, applying to the first of them the operator $\beta(\partial/\partial y)$ and to the second the operator $\alpha(\partial/\partial t)$, and then subtracting the second from the first, we obtain

$$\left(\beta\frac{\partial\hat{L}^{(1)}}{\partial y} - \alpha\frac{\partial\hat{L}^{(2)}}{\partial t} + [\hat{L}^{(2)}, \hat{L}^{(1)}]\right)(1+\hat{K}_+) = 0.$$

From this, using the invertibility of the operator $1 + \hat{K}_+$, we obtain the relation (20).

The relation (20) constitutes a system of nonlinear differential equations in N_3 matrix variables (N_3 is the larger of the numbers N_1 and N_2). Thus, we have shown that an arbitrary solution $F(x, z, t, y)$ of the system (16), (17) of two linear equations with constant coefficients generates, once the kernel $K_+(x, z, t, y)$ of the Gel'fand–Levitan equation (6) has been found, the exact solution of the system (20). Equations (16) and (17) can be solved by the Fourier method; Eq. (19) guarantees their compatibility. Thus, to each pair of operators $\hat{L}_0^{(1)}$, $\hat{L}_0^{(2)}$ there corresponds an integrable system of the type (20). Putting $\alpha = 0$ or $\beta = 0$, we obtain the equations

$$\beta \frac{\partial \hat{L}^{(1)}}{\partial y} = [\hat{L}^{(1)}, \hat{L}^{(2)}], \tag{22}$$

$$\alpha \frac{\partial \hat{L}^{(2)}}{\partial t} = [\hat{L}^{(2)}, \hat{L}^{(1)}], \tag{23}$$

which are also naturally related to the given pair of operators $\hat{L}_0^{(1)}$, $\hat{L}_0^{(2)}$.

2. Scalar Operators

As a first example of an application of the proposed scheme we consider the case of scalar operators $\hat{L}$. Using such operators, we can integrate the equations which arise in physics in connection with problems concerning the propagation of nonlinear waves in media with a weak dispersion. In the scalar case Eqs. (10) have the form

$$u_0(x) = 0, \quad u_1(x) = n\frac{d\xi_0}{dx}, \quad u_2(x) = \frac{n}{2}(n-2)\frac{d^2\xi_0}{dx^2} + \frac{n}{2}\frac{d}{dx}\xi_1 + n\xi_0\frac{d\xi_0}{dx}. \tag{24}$$

From Eqs. (24) it follows that $\hat{L}_1 = \hat{L}_{01}$. As the first nontrivial operators we have

$$\hat{L}_{02} = \frac{\partial^2}{\partial x^2}, \quad \hat{L}_2 = \frac{\partial^2}{\partial x^2} + u(x), \quad u = 2\frac{d}{dx}\xi_0(x) = 2\frac{d}{dx}K(x, x),$$

$$\hat{L}_{03} = \frac{\partial^3}{\partial x^3} + \lambda\frac{\partial}{\partial x}, \quad \hat{L}_3 = \frac{\partial^3}{\partial x^3} + \frac{3}{4}\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right) + w + \lambda\frac{\partial}{\partial x},$$

$$w = \frac{3}{2}\frac{d}{dx}(\xi_1 + \xi_0^2). \tag{25}$$

The commutator of the operators $\hat{L}_2$, $\hat{L}_3$ has the form

$$[\hat{L}_2, \hat{L}_3] = \frac{\partial}{\partial x}w + w\frac{\partial}{\partial x} - \left[\frac{1}{4}(u_{xxx} + 6uu_x) + \lambda u_x\right]. \tag{26}$$

As $\hat{M}_1$, $\hat{M}_2$ we consider the following operators:

$$\hat{M}_1 = \alpha\frac{\partial}{\partial t} + \hat{L}_{03}, \quad \hat{M}_2 = \hat{L}_{02}. \tag{27}$$

Here the kernel F satisfies the equations

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0, \quad \alpha\frac{\partial F}{\partial t} + \frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3} + \lambda\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\right) = 0. \tag{28}$$

The kernel K satisfies the equation

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial z^2} + u(x)K = 0, \tag{29}$$

from which there follows a condition on the characteristic $z = x$

$$\frac{d}{dx}(\xi_1 + \xi_0^2) = 0, \tag{30}$$

indicating that $w = 0$. Equation (23) now has the form ($\hat{L}^{(1)} = \hat{L}_3$, $\hat{L}^{(2)} = \hat{L}_2$)

$$\alpha\frac{\partial u}{\partial t} + \frac{1}{4}(u_{xxx} + 6uu_x + \lambda u_x) = 0 \tag{31}$$

and constitutes an unknown Korteweg–de Vries equation. It should be remarked that in this example both the function u(x, t) as well as the constants α and λ can be complex. This whole discussion can also be carried through assuming u(x, t) to be a matrix belonging to an arbitrary associative matrix algebra. Moreover, Eq. (31) can be written in the form

$$\alpha \frac{\partial u}{\partial t} + \frac{1}{4}(u_{xxx} + 3uu_x + 3u_x u + \lambda u_x) = 0.$$

We now consider the operators $\hat{M}_1$ and $\hat{M}_2$ in the form

$$\hat{M}_1 = \hat{L}_{03}, \qquad \hat{M}_2 = \beta \frac{\partial}{\partial y} + \hat{L}_{02}. \tag{32}$$

The kernel F now obeys the equations

$$\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3} + \lambda\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\right) = 0, \quad \beta \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0, \tag{33}$$

and Eq. (22) has the form

$$\frac{3}{4}\beta \frac{\partial u}{\partial y} = -\frac{\partial w}{\partial x}, \quad \beta \frac{\partial w}{\partial y} = \frac{1}{4}(u_{xxx} + 6uu_x) + \lambda u_x \tag{34}$$

or

$$\frac{3}{4}\beta^2 \frac{\partial^2 u}{\partial y^2} + \lambda u_{xx} + \frac{1}{4}u_{xxxx} + \frac{3}{2}(uu_x)_x = 0.$$

Putting $3/4\ \beta^2 = \pm 1$, $\lambda = \pm 1$, we obtain four real equations

$$\pm \frac{\partial^2 u}{\partial y^2} \pm \frac{\partial^2 u}{\partial x^2} + \frac{1}{4}u_{xxxx} + \frac{3}{2}(uu_x)_x = 0. \tag{35}$$

Two of them,

$$\pm\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{1}{4}u_{xxxx} + \frac{3}{2}(uu_x)_x = 0,$$

represent versions of the equation for a nonlinear jet (see [7]). Finally, considering the general case and putting

$$M_1 = \alpha \frac{\partial}{\partial t} + L_{03}, \quad M_2 = \beta \frac{\partial}{\partial y} + L_{02},$$

we obtain from the relation (20) the system of equations

$$\frac{3}{4}\beta \frac{\partial u}{\partial y} = -\frac{\partial w}{\partial x}, \quad \beta \frac{\partial w}{\partial y} = \alpha \frac{\partial u}{\partial t} + \lambda u_x + \frac{1}{4}(u_{xxx} + 6uu_x), \tag{36}$$

equivalent to the equation

$$\frac{3}{4}\beta^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left\{\alpha \frac{\partial u}{\partial t} + \lambda u_x + \frac{1}{4}(u_{xxx} + 6uu_x)\right\} = 0. \tag{37}$$

In addition the kernel F satisfies the two equations

$$\alpha \frac{\partial F}{\partial t} + \frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3} + \lambda\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\right) = 0, \quad \beta \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0. \tag{38}$$

Equation (37) was obtained for the first time in [8]; we shall call it the Kadomtsev–Petviashvili equation. This equation describes a two-dimensional nonstationary problem concerning waves in a dispersive medium, providing that the scale crosswise to the propagation of the wave (along the y axis) is much larger than the longitudinal scale (along the x axis).

From what has been said it is clear that each pair of scalar operators $L_0^{(1)}$, $L_0^{(2)}$ generates in each matrix algebra an equation of type (20).

3. N-Soliton Solutions

In this section we describe an important class of particular solutions of the Kadomtsev–Petviashvili equation (37), the N-soliton solutions. We restrict our consideration to the special case $\alpha = \beta = 1$ $(\lambda < 0)$. Equation (37) then describes waves on the surface of a shallow liquid; it is written in a frame of reference moving with a speed exceeding the speed of infinitely long waves. We put

$$F = e^{-\varkappa x - \eta z} M(t, y). \tag{39}$$

From Eqs. (38) it follows that

$$M = M(t, y) = M_0 \exp \{(\eta^2 - \varkappa^2) y + (\varkappa^3 + \eta^3 + \lambda(\varkappa + \eta))t\}. \tag{40}$$

Solving the Gel'fand–Levitan equation (6), we find

$$\begin{gathered} K(x, z) = -\frac{M e^{-\varkappa x - \eta z}}{1 + \frac{M}{\varkappa + \eta} e^{-(\varkappa+\eta)x}}, \\ u(x) = 2\frac{d}{dx} K(x, x) = \frac{1}{2} \frac{(\varkappa + \eta)^2}{\operatorname{ch}^2 (\varkappa + \eta)(x - x_0)}, \\ x_0 = (\eta - \varkappa) y + (\eta^2 - \varkappa\eta + \varkappa^2 + \lambda) t + \frac{1}{\varkappa + \eta} \ln \frac{M_0}{\varkappa + \eta}. \end{gathered} \tag{41}$$

The expressions (41) describe a soliton, i.e., a solitary wave propagating at an angle to the x axis. The soliton is characterized by three parameters: $\varkappa$ and η characterizing its amplitude, its angle of inclination to the x axis, and its velocity; the parameter M_0 characterizes the position of the center of the soliton when $t = 0$ and $y = 0$. When $\varkappa = \eta$, the soliton propagates strictly along the x axis; when $\eta^2 - \varkappa\eta + \varkappa^2 + \lambda = 0$ $(\lambda < 0)$, the soliton is stationary in the given frame of reference.

Assume now that

$$F = \sum_n M_n e^{-\varkappa_n x - \eta_n z}. \tag{42}$$

As before, we have

$$M_n(t, y) = M_n(0) \exp \{(\eta_n^2 - \varkappa_n^2) y + (\varkappa_n^3 + \eta_n^3 + \lambda(\eta_n + \varkappa_n)) t\}.$$

Putting, as we did earlier, $K(x, z) = \sum_n K_n(x) e^{-\eta_n z}$, we obtain a system of equations for the $K_n(x)$:

$$K_n(x) + M_n e^{-\varkappa_n x} + M_n \sum_m \frac{e^{-(\varkappa_n + \eta_m)x}}{\varkappa_n + \eta_m} K_m(x). \tag{43}$$

The general solution of this system has the form

$$u(x) = 2\frac{d}{dx} \sum K_n(x) e^{-\eta_n x} = 2\frac{d^2}{dx^2} \ln \Delta,$$

$$\Delta = \det \left\| M_n \delta_{nm} + M_n \frac{e^{-(\varkappa_n - \eta_m)x}}{\varkappa_n + \eta_m} \right\|. \tag{44}$$

Equations (44) give an explicit form of the solution, which for $t \to \pm\infty$, $y \to \pm\infty$ breaks up asymptotically into an aggregate of noninteracting solitons propagating at arbitrary angles to the x axis. Analogous expressions for the KV equation were found in [9]. A natural generalization of the solution (41) is a solution for which

$$F = \varphi(x, y, t) e^{-\eta z}. \tag{45}$$

Here the function φ satisfies the equations

$$\frac{\partial\varphi}{\partial t}+\frac{\partial^3\varphi}{\partial x^3}+\lambda\frac{\partial\varphi}{\partial x}-(\eta^3+\lambda\eta)\varphi=0,\quad \frac{\partial\varphi}{\partial y}+\frac{\partial^2\varphi}{\partial x^2}-\eta^2\varphi=0 \tag{46}$$

and has the form

$$\varphi=\int_{-\infty}^{+\infty} c(k)\exp\{iky-\varkappa x+(\varkappa^3+\eta^3+\lambda(\varkappa+\eta))t\}\,dk,$$

$$c(-k)=c^*(k),\ \varkappa^2=\eta^2-ik. \tag{47}$$

Here

$$K(x,z)=\frac{\varphi(x)e^{-\eta z}}{1+\int_x^\infty \varphi(s)e^{-\eta s}\,ds},\qquad u(x,y)=2\frac{d}{dx}\frac{\varphi(x)e^{-\eta x}}{1+\int_x^\infty \varphi(s)e^{-\eta s}\,ds}. \tag{48}$$

The expressions (48) describe an "oscillating" soliton; we can in an analogous way construct generalizations of N-soliton solutions.

4. Matrix Equations

Assume now that $\hat{L}_0^{(1)}$ and $\hat{L}_0^{(2)}$ are $N\times N$ matrix operators of the first order

$$\hat{L}_0^{(1)}=l_1\frac{\partial}{\partial x},\quad \hat{L}_0^{(2)}=l_2\frac{\partial}{\partial x},\quad [l_1,l_2]=0.$$

Then

$$\hat{L}^{(1)}=l_1\frac{\partial}{\partial x}+[l_1,\xi_0],\qquad \hat{L}^{(2)}=l_2\frac{\partial}{\partial x}+[l_2,\xi_0]. \tag{49}$$

Choosing the matrices (49) to be diagonal matrices: $l_1=b_n\delta_{nm}$, $l_2=a_n\delta_{nm}$ $(a_i\neq 0)$, we obtain for Eq. (21) (see [10])

$$(a_i-a_j)\frac{\partial\xi_{ij}}{\partial t}=(b_i-b_j)\frac{\partial\xi_{ij}}{\partial y}+s_{ij}\frac{\partial\xi_{ij}}{\partial x}+\sum_{k\neq i,j}(s_{ik}+s_{kj}+s_{ji})\xi_{ik}\xi_{kj}\quad \xi_{ij}=\xi_{0ij}. \tag{50}$$

The system (50), beginning with N = 3, is nontrivial. When N = 3, of physical interest are the "reductions" of the system (50) to matrices containing half the number of independent functions, these being derivable with the help of the relations $\xi^+=I\xi I$, $I^2=1$, where I is a diagonal matrix. If I = 1, then the system (50), by means of the substitution $\xi_{ij}=\lambda_{ij}u_{ij}$ (λ_{ij} are constants), is reduced to the form

$$\frac{\partial u_{12}}{\partial t}+\mathbf{V}_{12}\nabla u_{12}=iu_{13}u_{23}^*,\quad \frac{\partial u_{23}}{\partial t}+\mathbf{V}_{23}\nabla u_{23}=iu_{13}u_{12}^*,\quad \frac{\partial u_{13}}{\partial t}+\mathbf{V}_{13}\nabla u_{13}=iu_{12}u_{23}. \tag{51}$$

Here the $\mathbf{V}_{ij}$ are constant two-dimensional vectors. The system (51) describes the resonance interaction of three waves in a nonlinear medium and plays an important role in nonlinear optics. Choosing the matrix $I=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$, we come to the system

$$\frac{\partial u_{12}}{\partial t}+\mathbf{V}_{12}\nabla u_{12}=iu_{13}^*u_{23}^*,\quad \frac{\partial u_{23}}{\partial t}+\mathbf{V}_{23}\nabla u_{23}=iu_{13}^*u_{12}^*,\quad \frac{\partial u_{13}}{\partial t}+\mathbf{V}_{13}\nabla u_{13}=iu_{12}^*u_{23}^*, \tag{52}$$

describing "explosive instability" of a nonlinear medium (see [10]). The remaining ways of choosing the matrix I lead to systems equivalent to the systems (51) or (52).

In the case N > 3, more involved reductions of the system (50) are possible; their classification would be of great interest for the applications.

The case in which some of the $a_i = b_i = 0$ is a degenerate case and is also of great interest. In this case the operators $\hat{L}^{(1)}$, $\hat{L}^{(2)}$ can be represented in the form

$$\hat{L}^{(2)} = \begin{bmatrix} \hat{L}_0 & \hat{L}_1 \\ \hat{L}_2 & 0 \end{bmatrix}, \quad \begin{array}{l} \hat{L}_0 = l_2 \frac{\partial}{\partial x} + [l_2, \xi_{11}], \\ \hat{L}_1 = l_2\xi_{12}, \quad L_2 = \xi_{12} l_2, \end{array}$$

$$\hat{L}^{(1)} = \begin{bmatrix} \hat{A}_0 & \hat{A}_1 \\ \hat{A}_2 & 0 \end{bmatrix}, \quad \begin{array}{l} \hat{A}_0 = l_1 \frac{\partial}{\partial x} + [l_1, \xi_{11}], \\ \hat{A}_1 = l_1\xi_{12}; \quad A_2 = \xi_{21} l_1. \end{array}$$

Here l_1 and l_2 are diagonal $M \times M$ matrices not containing zeros, and the ξ_{ij} are cells of the matrix ξ. Considering only the case in which there is no dependence on y, we readily see that now the system (50) reduces to the form

$$\alpha \frac{\partial \hat{L}_0}{\partial t} = [\hat{L}_0, \hat{A}_0] + [\hat{G}, q], \qquad \hat{G} = \hat{L}_1 \hat{L}_2,$$
$$\alpha \frac{\partial \hat{G}}{\partial t} = [\hat{G}, \hat{A}_0] + \hat{L}_0 q \hat{G} - \hat{G} q \hat{L}_0; \qquad q = l_1 l_2^{-1}. \tag{53}$$

In the case $N = 4,\ M = 2,\ l_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ l_1 = 1$ the system (53) leads to the known "sin–gordon" equation (see [12])

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0.$$

It should be noted that in the degenerate case (53) the procedure we have presented is only suitable for finding integrable equations, since the formal application of the Gel'fand–Levitan equation (6) in the degenerate case leads to divergent expressions. For the "sin–gordon" equation this difficulty was surmounted in [12].

Matrix operators of much higher orders have so far not been studied. An exception here is the case in which $\hat{L}_0^{(2)} = l(\partial/\partial x)$, where l is a matrix of the form $l = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix}$, l_1 and l_2 are constants, and $\hat{L}_0^{(1)}$ is an arbitrary scalar operator. This case coincides in its essential features with the purely scalar case. Calculating $\hat{L}^{(1)}$ and $\hat{L}^{(2)}$, we have

$$\hat{L}^{(1)} = \frac{\partial^2}{\partial x^2} + 2\xi_x, \qquad L^{(2)} = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} + (l_1 - l_2) \begin{bmatrix} 0 & \xi_{12} \\ \xi_{21} & 0 \end{bmatrix},$$

and we obtain from the relations (21) for $\beta = 0$ a system of equations for the antidiagonal cells of the matrix ξ

$$\left(-\alpha \frac{\partial}{\partial t} + \frac{l_1 + l_2}{l_1 - l_2} \frac{\partial^2}{\partial x^2}\right) \xi_{12} - 2 \frac{l_1 + l_2}{l_1 l_2} \xi_{12}\xi_{21}\xi_{12} = 0,$$
$$\left(\alpha \frac{\partial}{\partial t} + \frac{l_1 + l_2}{l_1 - l_2} \frac{\partial^2}{\partial x^2}\right) \xi_{21} - 2 \frac{l_1 + l_2}{l_1 l_2} \xi_{21}\xi_{12}\xi_{21} = 0; \tag{54}$$

for $\alpha = i$ and real l_1, l_2 the system (54) admits the natural reduction $\xi_{21} = \pm \xi_{12}^{+}$ and reduces to the single equation

$$-i\xi_{12t} + \frac{l_1 + l_2}{l_1 - l_2} \xi_{12xx} \mp 2 \frac{l_1 + l_2}{l_1 l_2} \xi_{12}\xi_{12}^{+}\xi_{12} = 0. \tag{55}$$

In the scalar case Eq. (55) coincides with Eq. (2). The matrix case of Eq. (55) was considered by S. V. Manakov in [13]. In the case in which $\hat{L}_0^{(1)} = \partial^3/\partial x^3$ the Eq. (23) leads to the "modified" Korteweg–de Vries equation (see [14])

$$u_t \pm |u|^2 u_x + u_{xxx} = 0. \tag{56}$$

Cases in which the operator $\hat{L}_0^{(1)}$ is of an order higher than the third are not of interest from the point of view of the applications.

5. Solution Scheme for the Cauchy Problem

We now pose the question as to the possibility of solving the Cauchy problem for the Eqs. (20) and (23) with initial conditions in t. Let $\hat{L}_0^{(2)}$ be the operator defined in Eqs. (18), having order N_2. The corresponding operator $\hat{L}^{(2)}$ contains N_2-1 variable coefficients $u_i(x, y, t)$, which we assign at $t = 0$. We shall assume here that

$$u_i(\pm\infty, y, 0) = 0. \tag{57}$$

From Eq. (12) it follows that the kernel K^+ obeys the equation

$$\beta\frac{\partial K^+}{\partial y} + \hat{L}^{(2)}K^+ + \sum_{n=1}^{N_2-2}(-1)^{n-1}\frac{\partial^n}{\partial z^n}K^+ l_n = 0. \tag{58}$$

On its solution we impose the restriction

$$K^+(x,\ x+s) \to 0 \quad \text{for} \quad x \to +\infty. \tag{59}$$

With respect to the functions $u_i(x, y, 0)$ we can, using the Eqs. (10), find the derivatives of the kernel $K^\pm$ along the normal to the line $x = z-\xi^i(x, y, 0)$ $(i = 0, \ldots, N_2-2)$, whose assignment, along with the condition (59), defines a Cauchy–Goursat problem for Eq. (58). The solution of this problem determines the kernel $K^+(x, z)$ for $z > x$. In proceeding further we follow the scheme

$$u_i(x,y,0) \xrightarrow{\text{I}} K^\pm(x,z,y,0) \xrightarrow{\text{II}} F(x,z,y,0) \xrightarrow{\text{III}} F(x,z,y,t) \xrightarrow{\text{IV}} K^+(x,z,y,t) \xrightarrow{\text{V}} u_i(x,y,t). \tag{60}$$

In the first stage of this scheme we solve the Cauchy–Goursat problem for Eq. (58); in the second stage, with the help of the Gel'fand–Levitan equation (6), we determine the kernel F at t = 0. The evolution of the kernel F in time is described by the linear equation (16) with constant coefficients, constituting the third stage of the scheme. In the fourth stage we use Eq. (6) to obtain the kernel $K^+(x, z, y, t)$, and finally, at the last stage we apply Eqs. (10), enabling us to find $u(x, y, t)$.

A fundamental difficulty in applying the scheme (60) is that the kernel $K_+(x, z, y, t)$ obtained from the solution of the Cauchy–Goursat problem (58), (59) cannot satisfy the condition (4) of decrease with respect to z.

We consider this situation by an example in which $L_0^{(2)} = l\frac{\partial}{\partial x}$, $l = \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{bmatrix}$, $l_1 > l_2 > l_3 > 0$, and $\beta = 0$. In this case Eq. (58) has the form

$$l_i\frac{\partial K_{ij}^+}{\partial x} + l_j\frac{\partial K_{ij}^+}{\partial z} + \sum_k (l_i - l_k)\,\xi_{ik}(x)\,K_{kj} = 0. \tag{61}$$

Using $K_{ij} = \varphi_{ij}(x)e^{i\lambda z}$, we obtain a set of three spectral problems for φ_{ij}, which we number by means of the subscript j,

$$l_i\frac{\partial\varphi_{ij}}{\partial x} + i\lambda l_j\varphi_{ij} + \sum_k (l_i - l_k)\,\xi_{ik}(x)\,\varphi_{kj} = 0, \tag{62}$$

with the conditions

$$\begin{aligned} &\varphi_{ij}e^{\lambda x} \to 0 \quad \text{for} \quad x \to \pm\infty, \\ &\varphi_{ji}e^{\lambda x} \to 0 \quad \text{for} \quad x \to +\infty. \end{aligned} \tag{63}$$

The asymptotic behavior of the kernel K_{ij} for $z \to +\infty$ is determined by the discrete characteristic numbers of the spectral problems (62), (63), lying in the complex plane. For j = 1, 3 it follows from the conditions (63) that $\text{Im}\,\lambda > 0$ for discrete values of λ, and the asymptotic behavior is decreasing. However, when j = 2 the conditions (63) do not impose single-valued restrictions on the position of the discrete spectrum of the corresponding spectral problem and, for $z \to +\infty$, the asymptotic behavior of the kernel K_+ may turn out to be exponentially increasing.

This difficulty does not appear for scalar operators and, also, if $L_0^{(2)} = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} \frac{\partial}{\partial x}$ (l_1, l_2 are constants). Obviously, there is also no difficulty if the characteristic values in the spectral problem for j = 2 are absent [this can happen if the norm of the initial conditions $\xi_{ij}(x, y, 0)$ is sufficiently small]. The methods for overcoming these difficulties associated with the exponential growth of the kernel K_+ and correct results relative to the Cauchy problem for certain classes of integrable systems will be the subjects of the second part of our paper.

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