THE PROBLEM OF UNIQUENESS OF A GIBBSIAN RANDOM FIELD AND THE PROBLEM OF PHASE TRANSITIONS

R. L. Dobrushin

1. In a previous paper by the author [1] the following concepts were introduced. It was assumed that the potential U(t), $t \in T^\nu$, was specified, where T^ν is a ν-dimensional integer lattice such that a) for a certain $d < \infty$ we have

$$\sum_{t:|t|>d} |U(t)| < \infty \tag{1.1}$$

and b) $U(t) = U(-t)$, $t \in T^\nu$, the chemical potential μ, satisfies the conditions $-\infty < \mu < \infty$, and the constant β, satisfies the condition $0 < \beta < \infty$, (this constant is inversely proportional to the temperature). The random field $\xi(t)$, $t \in T^\nu$, which has the values in the space X consisting of two points 0 and 1, is specified by a set of finitely-dimensioned distributions $P = \{P_V(x_1, \ldots, x_{|V|}),\ V \subset T^\nu\}$, where V is a finite set of $|V|$ elements and $x_i \in X$. The random field and its distribution are called "Gibbsian" if the conditional probability satisfies the condition

$$P\{\xi(t_1) = x_1, \ldots, \xi(t_{|V|}) = x_{|V|}/\xi(t) = x(t),\ t \in T^\nu \setminus V\} = q_V(x_1, \ldots, x_{|V|}/x(t)) \tag{1.2}$$

with probability 1 for all finite $V = \{t_1, \ldots, t_{|V|}\} \subset T^\nu$, $x_1 \in X, \ldots, x_{|V|} \in X$ and functions $x(t)$, $t \in T^\nu \setminus V$, with values in X; here

$$q_V(x_1, \ldots, x_{|V|}/x(t)) = \frac{\exp\{-\beta U_V(x_1, \ldots, x_{|V|}/x(t))\}}{\sum\limits_{x_1 \in X, \ldots, x_{|V|} \in X} \exp\{-\beta U_V(x_1, \ldots, x_{|V|}/x(t))\}} \tag{1.3}$$

and

$$U_V(x_1, \ldots, x_{|V|}/x(t)) = -\mu \sum_{i=1}^{|V|} x_i + \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1,\, j \neq i} x_i x_j U(t_i - t_j) + \sum_{i=1}^{|V|} \sum_{t \in T^\nu \setminus V} x_i x(t) U(t_i - t). \tag{1.4}$$

A Gibbsian distribution in a vessel V with the boundary conditions $x(t)$, $t \in T^\nu \setminus V$, is defined as a random field $\xi(t)$ such that

$$P\{\xi(t_1) = x_1, \ldots, \xi(t_{|V|}) = x_{|V|}\} = q_V(x_1, \ldots, x_{|V|}/x(t)),$$
$$P\{\xi(t) = x(t)\} = 1,\ t \in T^\nu \setminus V. \tag{1.5}$$

In [1] concepts were developed demonstrating the fact that Gibbsian distributions specify methods of describing physical systems in an "infinite" vessel. It was also shown that the case when the Gibbsian distribution with the specified parameters (μ, β, $U(\cdot)$) is unique corresponds to the case when there is no separation of phases.

The purpose of this paper is to indicate the explicit conditions for which it is possible to establish the uniqueness or, conversely, the nonuniqueness of a Gibbsian distribution with specified parameters; we also use examples of nonuniqueness to discuss the consequences that derive from it with respect to phase transitions in the system.

2. First we indicate the conditions governing uniqueness of the Gibbsian distribution.

THEOREM 1. Assume

Institute of Information-Transmission Problems, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 2, No. 4, pp. 44-57, October-December, 1968. Original article submitted February 23, 1968.

$$\frac{1}{2}\sum_{s\in T^\nu,\, s\neq 0}\ \sup_{x(t),\ \tilde{x}(t):\ x(t)=\tilde{x}(t),\ t\neq s}\ \sum_{x\in X}|q_{\{0\}}(x/x(t))-q_{\{0\}}(x/\tilde{x}(t))|<1, \tag{2.1}$$

where $\{0\}$ is a set of one point 0 and the upper boundary is taken over all pairs of the functions $x(t)$, $\tilde{x}(t)$, $t \in T^\nu \setminus \{0\}$, such that $x(t) = \tilde{x}(t)$, $t \neq s$. Then there exists only one Gibbsian distribution. The properties of dimensional regularity from within and from without are satisfied for it (see [1]). If the potential U(t) is finite (i.e., $U(t) = 0$, $|t| > C$), then the functions $\varphi_V(\cdot)$ and $\tilde{\psi}_{\tilde{V}}(\cdot)$ in (5.3) and (5.4) taken from [1] must be assumed equal:

$$\varphi_V(d) = e^{-\alpha_V d}, \quad \psi_{\tilde{V}}(d) = e^{-\tilde{\alpha}_{\tilde{V}} d}, \quad \alpha_V > 0, \quad \tilde{\alpha}_{\tilde{V}} > 0. \tag{2.2}$$

If the potential $U(t) < \infty$ for all $t \in T^\nu$, then condition (2.1) is certainly satisfied for the values of (μ, β) such that $|\mu| \geq \mu_0$ or $\beta \leq \beta_0$, where $\mu_0 < \infty$, $\beta_0 > 0$ are certain constants. For an arbitrary potential U(t) Condition (2.1) is satisfied for values (μ, β) such that $\mu \leq \mu_0(\beta)$, where the continuous function $\mu_0(\beta)$ has a finite limit for $\beta \to \infty$ and where $\mu_0(\beta) \sim c/\beta$ for $\beta \to 0$, where the constant $c > 0$.

Proof. The basic postulate of the theorem formulated above is a particular case of a more general theorem (in which it is not assumed that the conditional distributions are Gibbsian), which was proved in detail in [2]. Here we briefly indicate only the basic formulation used in its proof; we assume for simplicity that the potential is finite. Each point $t \in T^\nu$ is juxtaposed with an operator Q_t that operates in the distribution space of the random fields and converts the distribution $P = \{P_V(\cdot),\ V \subset T^\nu\}$ into the distribution $\tilde{P} = \{\tilde{P}_V(\cdot),\ V \subset T^\nu$. We will define that operator on the basis of the proposition that

$$\tilde{P}_V(x_1, \ldots, x_{|V|}) = \begin{cases} P_V(x_1, \ldots, x_{|V|}), & t \notin V, \\ P_{V\setminus\{t\}}(x_1, \ldots, x_{|V|-1})\, q_{\{t\}}(x_{|V|}/x_1, \ldots, x_{|V|-1}), & \end{cases}$$
$$V = \{t_1, \ldots, t_{|V|-1}, t\},\ d(R^\nu \setminus V, \{t\}) > C, \tag{2.3}$$

where $q_{\{t\}}(x_{|V|}/x_1, \ldots, x_{|V|-1})$ is the value of $q_{\{t\}}(x_{|V|}/x(t))$, that is general according to the assumption of finiteness for all functions x(t) with $x(t_i) = x_i$, $i = 1, \ldots, |V|-1$. For the remaining V the value of $\tilde{P}_V(\cdot)$ is determined from the matching condition for the finitely dimensioned distributions (see (2.3) in [1]). It is obvious that the Gibbsian distributions are invariant relative to the operators Q_t. Assume

$$Q = \prod_{t\in T^\nu} Q_t. \tag{2.4}$$

(The sequence in which the noncommutative operators Q_t are multiplied is not essential.) We are able to prove that for condition (2.1) the operator Q is compressive (in a certain generalized sense), whence the statement of the theorem follows.

Here we need merely establish the range of values of (μ, β) for which condition (2.1) is satisfied. For this purpose we note that in accordance with (1.3) we have the following result for $x(t) = \tilde{x}(t)$, $t \neq s$, and $U(t) < \infty$, $t \in T^\nu$,

$$\frac{1}{2}\sum_{x\in X}|q_{\{0\}}(x/x(t))-q_{\{0\}}(x/\tilde{x}(t))| = \left|\frac{1}{1+\exp\left\{-\beta\left[-\mu+\sum_{t\neq 0}U(t)\,x(t)\right]\right\}} - \frac{1}{1+\exp\left\{-\beta\left[-\mu+\sum_{t\neq 0}U(t)\,\tilde{x}(t)\right]\right\}}\right|$$
$$\leqslant \frac{\exp\left\{\beta\left(\mu+\sum_{t\neq 0}|U(t)|\right)\right\}|1-\exp\{-\beta U(s)\}|}{\left(1+\exp\left\{-\beta\left[-\mu+\sum_{t\neq 0}|U(t)|\right]\right\}\right)^2}. \tag{2.5}$$

Since from (1.1) it follows that

$$\sum_{s\neq 0}|1-\exp\{-\beta U(s)\}|<\infty, \tag{2.6}$$

and the sum of that series tends to zero for $\beta \to 0$, it follows from (2.5) that condition (2.1) is satisfied for $\beta \leq \beta_0$. Since $|x + y| \leq |x|\,|y|$, if $|x| \geq 2$, $y \geq 2$ and $|1-e^x| \leq g|x|$, where $g < \infty$, if $e^x \leq 2$, it follows that

$$\sum_{s\neq 0} |1-\exp\{-\beta U(s)\}| \leqslant g\beta \sum_{\exp\{-\beta|U(s)|\}\leqslant 2} |U(s)|$$

$$+\sum_{\exp\{-\beta|U(s)|\}\geqslant 2} \exp\{-\beta U(s)\} \leqslant g\beta \sum_{s\neq 0} |U(s)| + \exp\left\{\beta \sum_{s\neq 0} |U(s)|\right\}. \quad (2.7)$$

From (2.7) and (2.5) it follows that for $\beta \to \infty$ and sufficiently large $|\mu|$ the sum with respect to s of the right sides in (2.5) tends to zero (for large positive μ due to the fact that the numerator of the fraction tends to ∞ more slowly than its denominator, and for large negative μ due to the fact that the numerator tends to 0 and the denominator tends to 1). Therefore, Condition (2.1) is satisfied for $|\mu| \geq \mu_0$.

When U(t) can acquire the values ∞, Inequality (2.5) and the subsequent estimates are applicable if $x(t) = \tilde{x}(t) = 0$ wherever $U(t) = \infty$. If $x(t) = \tilde{x}(t) = 1$ for $U(t) = \infty$, then the left side in (2.5) is simply equal to zero. There still remains the case when $U(s) = \infty$, $x(s) = 1$, $\tilde{x}(s) = 0$ and $x(t) = \tilde{x}(t) = 0$ for $U(t) = \infty$, $t \neq s$. For that case

$$\frac{1}{2}\sum_{x\in X} |q_{\{0\}}(x/x(t)) - q_{\{0\}}(x/\tilde{x}(t))| = \frac{\exp\left\{-\beta\left[-\mu + \sum_{t\neq 0} U(t)\tilde{x}(t)\right]\right\}}{1+\exp\left\{-\beta\left[-\mu+\sum_{t\neq 0} U(t)\tilde{x}(t)\right]\right\}}, \quad (2.8)$$

and this expression tends to 0 for $\mu \to -\infty$ uniformly with respect to x(t), $\tilde{x}(t)$ and all sufficiently large β. Finally, for $\beta \to 0$, $\mu\beta = \text{const}$ the expression in (2.8) tends to $e^{\beta\mu}/(1+e^{\beta\mu})$, whence it follows that $\mu_0(\beta) \sim c/\beta$.

It can be stated conveniently that Condition (2.1) means that the conditional Gibbsian probabilities depend weakly on the conditions. The satisfaction of Condition (2.1) for small β is related to the fact that for $U(t) \neq \infty$, $t \in T^\nu$, the conditional probabilities of both values of $x \in X$ prove to be close to their probabilities in the case of an ideal gas. For μ close to $-\infty$ the conditional probability that x = 0 is close to 1, and for μ close to $+\infty$ it is close to 0 uniformly with respect to all conditions. If U(t) can acquire infinite values and $U(t)x(t) = \infty$ for a certain t, then for all μ and β the conditional probability that x = 0 is equal to 1, and the reasoning is performed only in the case of sufficiently large negative μ. Typical ranges of the values (β, μ) for which conditions (2.1) are satisfied have been shaded in Figs. 1 and 2. In Fig. 1 we took into account the fact that when x(t) is replaced with $x'(t) = 1 - x(t)$ and μ is replaced with $\mu' = -\mu + \sum_{t\in T^\nu} U(t)$ the conditional probability $q_{\{0\}}(0/x(t))$ becomes $q_{\{0\}}(1/x'(t))$; therefore, the region in which Condition (2.1) is satisfied can be assumed symmetrical relative to the line

$$\mu = \bar{\mu}, \text{ где } \bar{\mu} = \frac{1}{2}\sum_{t\in T^\nu} U(t). \quad (2.9)$$

Note that a result analogous to Theorem 1 can be proved by the Ruelle-Minlos method (see [3]). Note also that from the results of [2] it is possible to extract explicit estimates for $\varphi_V(d)$ and $\psi_{\tilde{V}}(d)$ even in the case of nonfinite potentials.

3. Whereas in the multidimensional case uniqueness is proved only for a certain subrange of the values of the parameters (β, μ), in the one-dimensional case $\nu = 1$, uniqueness usually applies for all (β, μ) if the potential decreases sufficiently rapidly; this corresponds to the physical concept that phase transitions are absent in one-dimensional systems.

THEOREM 2. Assume $\nu = 1$. Assume there exists a sequence of numbers $\rho_k < 1$, k = 0, 1, . . ., such that $\rho_k \to 0$ for $k \to \infty$ and

$$\sup_{n=0,1,\ldots}\ \sup_{x(t),\,\tilde{x}(t):x(t)=\tilde{x}(t),\,-k\leqslant t<0,\,n<t} \frac{1}{2}\sum_{x_1\in X,\ldots,x_{n+1}\in X} |q_{[0,n]}(x_1,\ldots,x_{n+1}/x(t)) - q_{[0,n]}(x_1,\ldots,x_{n+1}/\tilde{x}(t))| \leqslant \rho_k,\ k=0,1,\ldots, \quad (3.1)$$

where the upper bound is chosen over all n = 0, 1, . . . and over all pairs of functions x(t), $\tilde{x}(t)$, $t \in T^1 \setminus [0, n]$, such that $x(t) = \tilde{x}(t)$, if $k \leq t < 0$ and $n < t$. Then there exists only one Gibbsian distribution. For this distribution the condition of uniform strong mixing is satisfied; this condition indicates the existence of a function $\chi(d) \to 0$ for $d \to \infty$, such that for all finite $V \subset T^1$, $\tilde{V} \subset T^1$

$$\sup_{A\in\mathfrak{B}_V, B\in\mathfrak{B}_{\tilde V}} |\Pr\{A/B\} - \Pr\{B\}| \leqslant \chi(d(V, \tilde V)) \tag{3.2}$$

(see [1], section 5); furthermore, if the potential is finite, it is possible to place $\chi(d) = e^{-\alpha d}$, $\alpha > 0$. If the sum

$$\sum_{t:|t|\geqslant d} |U(t)||t| < \infty, \tag{3.3}$$

then Condition (3.1) is satisfied for all (β, μ).

Note that the property of uniform strong mixing (3.2) brings with it the properties of uniform regularity from within and from without.

Proof. The basic postulate of the formulated theorem is a particular case of a more general theorem that was proved in detail in [2]. In the case of a finite potential it is derived simply from the ergodic theorem for Markov chains. Here we will merely verify the fact that from Condition (3.3) we obtain satisfaction of Condition (3.1). For this purpose we note that from (1.3) it follows that if $x(t) = \tilde x(t)$, $-k \leq t < 0$, $n < t$, then the ratio

$$\frac{q_{[0,n]}(x_1, \ldots, x_{n+1}/x(t))}{q_{[0,n]}(x_1, \ldots, x_{n+1}/\tilde x(t))} \geqslant \exp\Big\{-2\beta \sum_{\substack{0\leqslant \tilde s < n,\\ s<-k}} |U(\tilde s - s)|\Big\}$$

$$\geqslant \exp\Big\{-2\beta \sum_{|t|>k} |t||U(t)|\Big\}. \tag{3.4}$$

Therefore,

$$\begin{aligned} &\frac{1}{2}\sum_{x_1\in X,\ldots,x_{n+1}\in X} |q_{[0,n]}(x_1, \ldots, x_{n+1}/x(t)) - q_{[0,n]}(x_1, \ldots, x_{n+1}/\tilde x(t))| \\ &= 1 - \sum_{x_1\in X,\ldots,x_{n+1}\in X} \min(q_{[0,n]}(x_1, \ldots, x_{n+1}/x(t)), q_{[0,n]}(x_1, \ldots, x_{n+1}/\tilde x(t))) \\ &\leqslant 1 - \exp\Big\{-2\beta \sum_{|t|>k} |t||U(t)|\Big\} \sum_{x_1\in X,\ldots,x_{n+1}\in X} q_{[0,n]}(x_1, \ldots, x_{n+1}/\tilde x(t)) \\ &\qquad = 1 - \exp\Big\{-2\beta \sum_{|t|>k} |t||U(t)|\Big\}; \end{aligned} \tag{3.5}$$

the right side in this inequality tends to zero for $k \to \infty$. It is less than 1 for all k if $|U(t)| < \infty$ for all $t \in T^\nu$; this proves satisfaction of Condition (3.1) in this case. In the general case, when only (3.3) is valid, it is still necessary to check whether the right side in (3.5) is less than 1. Proceeding in a manner analogous to that used in obtaining the estimate (3.5), it is demonstrated that for $x(t) = \tilde x(t)$, $t > n$, we have

$$\begin{aligned} &\frac{1}{2}\sum_{x_1\in X,\ldots,x_{n+1}\in X} |q_{[0,n]}(x_1, \ldots, x_{n+1}/x(t)) - q_{[0,n]}(x_1, \ldots, x_{n+1}/\tilde x(t))| \\ &\geqslant \exp\Big\{-2\beta \sum_{|t|>d} |U(t)||t|\Big\} \sum_{x_{d+1}\in X,\ldots,x_{n+1}\in X} q_{[0,n]}(0, \ldots, 0, x_{d+1}, \ldots, x_{n+1}/\tilde x(t)). \end{aligned} \tag{3.6}$$

Furthermore (see (1.3), (1.4)):

$$\begin{aligned} &\sum_{x_{d+1}\in X,\ldots,x_{n+1}\in X} q_{[0,n]}(0, \ldots, 0, x_{d+1}, \ldots, x_{n+1}/\tilde x(t)) \\ &= \frac{1}{2^d} \sum_{x_1\in X,\ldots,x_{n+1}\in X} q_{[0,n]}(0, \ldots, 0, x_{d+1}, \ldots, x_{n+1}/\tilde x(t)) \\ &\geqslant \frac{1}{2^d} \min_{x_1\in X,\ldots,x_{n+1}\in X} \frac{U_{[0,n]}(0, \ldots 0, x_{d+1}, \ldots, x_{n+1}/\tilde x(t))}{U_{[0,n]}(x_1, \ldots, x_{n+1}/\tilde x(t))} \sum_{x_1\in X,\ldots,x_{n+1}\in X} q_{[0,n]}(x_1, \ldots, x_{n+1}/\tilde x(t)) \\ &\geqslant \frac{1}{2^d} \exp\Big\{ -\beta\Big(|\mu| d + |\inf_{t\in T^\nu} U(t)| \frac{d(d-1)}{2} - d \sum_{t\in T^\nu} \min(U(t), 0)\Big)\Big\}. \end{aligned} \tag{3.7}$$

Since the right side of Inequality (3.7) is positive and independent of n and $\tilde x(t)$, it follows from (3.7) and (3.6) that the coefficients ρ_k in the formulation of the theorem can be made less than 1; this proves satisfaction of Condition (3.1) when (3.3) is valid.

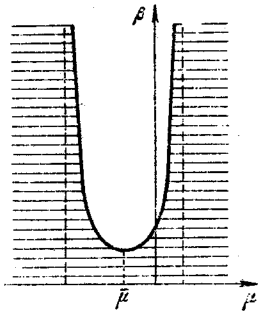

Fig. 1. Region in which Condition (2.1) is satisfied. The potential $U(t) \neq \infty$, $t \in T^{\nu}$.

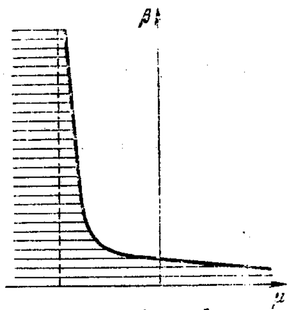

Fig. 2. Region where Condition (2.1) is satisfied. The potential U(t) can acquire the value $+\infty$.

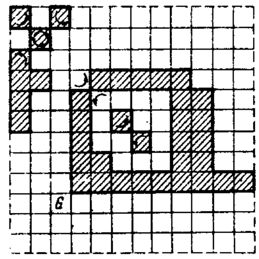

Fig. 3. Contours in the case $\bar{x}(t) = x_0(t)$ for the Ising model with attraction. The arrows isolate the direction of contour rotation.

In the case of finite U(t) the results given in Theorem 2 can be obtained from known explicit formulas for the correlation functions; it is new for nonfinite U(t).* The question as to whether Condition (3.3) can be replaced with the weaker condition (1.1) remains open. (On this subject see the discussion in [2], section 2.)

4. We will now proceed to examples of nonuniqueness of the Gibbsian distribution $\nu > 1$. First we will study the Ising model with attraction in which

$$U(t) = \begin{cases} a < 0, & \text{if } \quad |t| = 1, \\ 0, & \text{if } \quad |t| \neq 1. \end{cases} \tag{4.1}$$

We find that for such a potential there exist at least two different Gibbsian distributions for (see (2.9))

$$\mu = \bar{\mu} = \nu a, \quad \beta > -a^{-1} s_{\nu}, \tag{4.2}$$

where the constant s_{ν}, which depends only on the dimensionality ν, will be defined below during the calculations. This fact was first derived by R. A. Minlos and Ya. G. Sinai (oral communication) on the basis of the method developed in [4] based on equations for contours and probability estimates of a contour (see Lemma 1—Peierls [10], Griffiths [5], and R. L. Dobrushin [6]). The concepts developed in this paper make it possible to obtain this result in a considerably simpler manner as a direct corollary of Lemma 1.

All formulations in this and subsequent sections are analogous for any dimensionality $\nu \geq 2$. For purposes of geometric convenience we will assume, however, that $\nu = 2$ (the multidimensional generalization has been studied in detail in [6]). Below we shall study just two cases of the boundary conditions:

$$x(t) = \begin{cases} x_0(t) \equiv 0, & t \in T^2 \setminus V, \\ x_1(t) \equiv 1, & t \in T^2 \setminus V, \end{cases} \tag{4.3}$$

and shall assume that $V = \{t_1, \ldots, t_{|V|}\}$ is a certain square. We shall associate each point $t \in T^2$ of the two-dimensional lattice with a square having sides of length 1 parallel to the coordinate axes with its center at that point. We will also designate it with the symbol t. Each function $\bar{x}(t)$, $t \in T^2$, $\bar{x}(t) = 0, 1$, is juxtaposed with a geometric pattern (see Fig. 3) that is obtained if we shade the squares for which $\bar{x}(t) = 1$. We will define a boundary side as the side of a square that separates shaded and unshaded squares. Geometrically it is obvious that an even (0, 2, or 4) number of boundary sides emanate from each vertex of the squares, and that each ensemble of sides of squares within V having that property corresponds to a certain function $\bar{x}(t)$; furthermore, for a specified boundary function x(t) this amounts to a one-to-one relationship if $\bar{x}(t) = x(t)$, $t \in T^2 \setminus V$. We will define the length of the situation boundary $(x_1, \ldots, x_{|V|})$, $x_i \in X$, $i = 1, \ldots, |V|$, for a specified function x(t) as the number $\Gamma(x_1, \ldots, x_{|V|}/x(t))$, equal to the sum of the number of pairs of points t_i, t_j, $i \neq j$, such that $|t_i - t_j| = 1$, $x_i \neq x_j$, and the number of pairs of points $t_i, t \in T^2 \setminus V$, such that $|t_i - t| = 1$, $x_i \neq x(t)$. It can be seen that $\Gamma(x_1, \ldots, x_{|V|}/x(t))$ is the over-all number

*After this paper had been submitted for publication, the author received a preprint of a paper by Ruelle [9] in which a result close to that contained in Theorem 2 was proved.

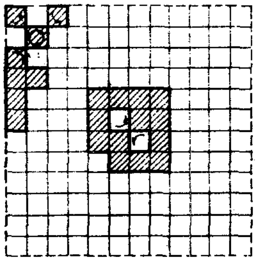

Fig. 4. Result of applying the transformation T_G to the situation in Fig. 3, where G is the contour of maximum length.

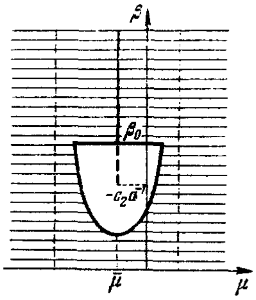

Fig. 5. Regions of uniqueness and nonuniqueness in the Ising model with attraction. The hypothetical continuation of the nonuniqueness line is shown dashed.

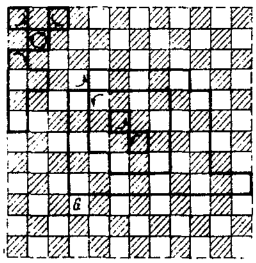

Fig. 6. Contours for the Ising model with repulsion.

of boundary sides for the function $\bar{x}(t) = x(t)$, $t \in T^2 \setminus V$, $\bar{x}(t_i) = x^i$, $t_i \in V$. The situation boundary $(x_1, \ldots, x_{|V|})$ is defined as the ensemble of boundary sides for $\bar{x}(t)$. Note that from (1.3) we obtain the following result for $\mu = \bar{\mu} = 2a$:

$$q_V(x_1, \ldots, x_{|V|}/x(t)) = \frac{\exp\left\{\frac{\beta a}{2}\Gamma(x_1, \ldots, x_{|V|}/x(t))\right\}}{\sum\limits_{x_1\in X, \ldots, x_{|V|}\in X} \exp\left\{\frac{\beta a}{2}\Gamma(x_1, \ldots, x_{|V|}/x(t))\right\}}. \tag{4.4}$$

Geometrically it is evident that the ensemble of all boundary sides can be subdivided into non-self-intersecting closed broken lines that we will call contours (Fig. 3 shows eight contours); furthermore, it is convenient to assume that all contours perform turns at vertices where four boundary sides converge. Under these conditions two contours either do not intersect, or one of them is inserted within the other. (The ambiguity of the subdivision of a boundary into contours is insignificant for our purposes.)

LEMMA 1. For a Gibbsian distribution in the square V with boundary conditions of the type (4.3) the probability $\Pr\{G\}$ that the situation $(x_1, \ldots, x_{|V|})$ contains a certain contour G (i.e., more precisely, that each of the sides of the contour G is included in the situation boundary $(x_1, \ldots, x_{|V|})$) is such that

$$\Pr\{G\} \leqslant \exp\left\{\frac{\beta a}{2}|G|\right\}, \tag{4.5}$$

where $|G|$ is the length of the contour G.

This lemma was proved in [5] and [6]. We will give a simple proof of it, since it will be generalized in the subsequent sections. Assume $\mathfrak{H}(G)$ is an ensemble of situations $(x_1, \ldots, x_{|V|})$ containing the contour G. Then from (4.4) it follows that

$$\Pr\{G\} = \frac{\sum\limits_{(x_1,\ldots,x_{|V|})\in\mathfrak{H}(G)} \exp\left\{\frac{\beta a}{2}\Gamma(x_1, \ldots, x_{|V|}/x(t))\right\}}{\sum\limits_{x_1\in X,\ldots,x_{|V|}\in X} \exp\left\{\frac{\beta a}{2}\Gamma(x_1, \ldots, x_{|V|}/x(t))\right\}}. \tag{4.6}$$

We introduce the transformation T_G that juxtaposes each situation $(x_1, \ldots, x_{|V|}) \in \mathfrak{H}(g)$ according to the following rule:

$$\begin{aligned} \tilde{x}_i &= x_i, && \text{if } t_i \text{ does not lie outside the contour G,} \\ \tilde{x}_i &= 1 - x_i, && \text{if } t_i \text{ lies inside the contour G} \end{aligned} \tag{4.7}$$

(Fig. 4). In other words, the transformation T_G annihilates the contour G and does not change other contours. It is evident that

$$\Gamma(\tilde{x}_1, \ldots, \tilde{x}_{|V|}/x(t)) = \Gamma(x_1, \ldots, x_{|V|}/x(t)) - |G| \tag{4.8}$$

and that different situations $(x_1, \ldots, x_{|V|})$ yield different situations $(\tilde{x}_1, \ldots, \tilde{x}_{|V|})$. Therefore,

$$\Pr\{G\} \leqslant \frac{\sum\limits_{(x_1,\ldots,x_{|V|})\in\mathfrak{H}(G)} \exp\left\{\frac{\beta a}{2}\Gamma(x_1,\ldots,x_{|V|}/x(t))\right\}}{\sum\limits_{(\tilde{x}_1,\ldots,\tilde{x}_{|V|})\in T_G\mathfrak{H}(G)} \exp\left\{\frac{\beta a}{2}\Gamma(\tilde{x}_1,\ldots,\tilde{x}_{|V|}/x(t))\right\}} = \exp\left\{\frac{\beta a}{2}|G|\right\}, \tag{4.9}$$

which proves the lemma.

Note that if the point $t_0 = (t_0^1, t_0^2) \in T^2$ lies inside the contour G and this contour contains a side of a square with its center at $t = (t^1, t^2)$, inside G, then the length of the contour G must be no less than $2(|t_0^1 - t^1| + |t_0^2 - t^2| + 2)$. Furthermore, no more than 3^{d-1} broken lines d pass through the given side. Assume $\pi(t_0)$ is the probability that the point t_0 lies inside at least one of the contours specified by the situation $(x_1, \ldots, x_{|V|})$. Summing over all contours G that contain t_0 inside them and pass along a side of the square t, and taking into account the fact that under these conditions a contour of length d will be counted d times and that the length d of a contour is always a multiple of 2 and each square has four sides, we obtain the following result from (4.5):

$$\pi(t_0) \leqslant 4 \sum_{t\in T^2} \sum_{k \geqslant \frac{|t_0^1 - t^1| + |t_0^2 - t^2| + 2}{2}} \frac{1}{3}\frac{1}{2k}\exp\{(\beta a + 2\ln 3)k\}$$

$$= \frac{2}{3}\sum_{k=1}^{\infty} k^{-1}[k^2 + (k-1)^2]\exp\{(\beta a + 2\ln 3)k\} \leqslant \frac{4}{3}\frac{\exp\{\beta a + 2\ln 3\}}{(1 - \exp\{\beta a + 2\ln 3\})^2},$$

if $\exp\{\beta a + 2\ln 3\} < 1$. Assume $s_2 > 0$ is the solution of the equation

$$\frac{4}{3}\frac{\exp\{-s_2 + 2\ln 3\}}{(1 - \exp\{-s_2 + 2\ln 3\})^2} = \frac{1}{2}. \tag{4.10}$$

Then, if Condition (4.2) is valid, it follows that $\pi(t_0) \leq \gamma < 1/2$ for all $t_0 \in T^\nu$. But if $x(t) = x_0(t)$ and the point t_0 lies outside any contour, then the field value is $\xi(t_0) = 0$. From this it follows that the following relationship is valid for a Gibbsian distribution with the boundary conditions $x_0(t)$:

$$\Pr\{\xi(t_i) = 1\} \leqslant \gamma < \frac{1}{2}, \quad t_i \in V. \tag{4.11}$$

It is obvious that Condition (4.11) isolates a closed set of distributions in the metric introduced in [1] (see [1], section 2). Therefore, in view of compactness (see [1]) and Theorem 1 from [1] there exists a Gibbsian field $\bar{\xi}(t)$ such that

$$\Pr\{\bar{\xi}(t) = 1\} \leqslant \gamma < \frac{1}{2}, \quad t \in T^\nu. \tag{4.12}$$

Analogous reasoning based on the boundary conditions $x(t) = x_1(t)$ leads to the conclusion that a Gibbsian field $\tilde{\xi}(t)$ exists such that

$$\Pr\{\tilde{\xi}(t) = 1\} \geqslant 1 - \gamma > \frac{1}{2}, \quad t \in T^\nu, \tag{4.13}$$

and that the Gibbsian fields $\bar{\xi}(t)$ and $\tilde{\xi}(t)$ cannot coincide. Taking into account the fact that Condition (4.11) is invariant relative to shifts and applying reasoning that is analogous to that used in deriving Theorem 1 from [1], we also find that noncoinciding translation-invariant distributions exist for which (4.12) and (4.13) are valid, respectively. These two distributions are obtained from one another by replacing 0 with 1 and 1 with 0.

The method of correlation functions for Minlos-Sinai contours [4] can be used to prove that the formulated distributions are the limits of the Gibbsian distributions $q_{V_i}(\cdot/x_0(t))$ and $q_{V_i}(\cdot/x_1(t))$, respectively, for any expanding sequence of cubes V_i. Their method also makes it possible to prove the uniqueness of the Gibbsian distribution for $\beta > \beta_0$, where $\beta_0 < \infty$ is a certain constant, and for all $\mu \neq \mu$. In Fig. 5 the region for which uniqueness of the Gibbsian distribution is proved has been shaded, and the line on which, as demonstrated, there is no uniqueness is designated by a thick line.

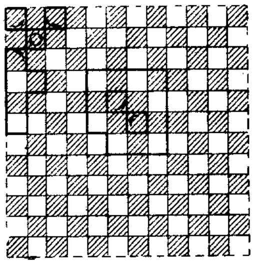

Fig. 7. Result of applying the transformation $\overline{T}_G$ to the situation in Fig. 6, where G is the contour of maximum length.

The natural hypothesis consists of the fact that the Gibbsian distribution is unique for all $(\bar{\mu}, \beta)$, with the exception of pairs $(\bar{\mu}, \beta$, where $\beta > c_2(-a^{-1})$ and c_2 is the value found by Onsager (see [7]). The important question of whether the class of Gibbsian distributions is exhausted by the two distributions found above and their linear combinations also remains open. The following hypotheses can be advanced. There are certain conclusions substantiating the proposition that this is so for $\nu = 2$. However, for $\nu > 2$ this is evidently no longer so. It is possible to assume that for $\nu = 3$ there is an entire additional family of extreme points in the ensemble of Gibbsian distributions that are limiting for distributions with boundary conditions $x(t)$, $t \in T^3$, such that $x(t^1, t^2, t^3) = 1$ for $t_1 \geq c$ and $x(t^1, t^2, t^3) = 0$ for $t_1 < c$, as well as analogous families that are obtained when t^1 is replaced by t^2 and t^3; it is assumed that these exhaust all the extreme points. Such distributions could describe surface phenomena on the boundary of the phases.

All that was said above in connection with the Ising model with attraction is not difficult to extend to the more general case of arbitrary potentials $U(t)$ for which the negative part is in a certain sense (precisely defined in a previous paper by the author [8]) greater than the positive part. In [8] it was proved that for such potentials and for sufficiently large β an estimate analogous to the estimate in Lemma 1 is valid.

5. We will study the Ising model with repulsion, where

$$U(t) = \begin{cases} a > 0, & \text{if } |t| = 1, \\ 0, & \text{if } |t| \neq 1. \end{cases} \tag{5.1}$$

We find that for such a potential at least two different Gibbsian distributions exist for $a > |\tilde{\mu}|/\nu$,

$$a > |\tilde{\mu}|/\nu, \quad \beta > \left(a - \frac{|\tilde{\mu}|}{\nu}\right)^{-1} s_\nu, \quad \text{where } \tilde{\mu} = \mu - \bar{\mu} = \mu - \nu a, \tag{5.2}$$

This result is evidently new from the mathematical standpoint.

We shall study the geometric interpretation introduced in the previous section, but we shall now define a boundary side as any side that separates two shaded or two unshaded squares. Correspondingly, a long situation boundary $(x_1, \ldots, x_{|V|})$ with the boundary function $x(t)$ is defined as the number $\tilde{\Gamma}(x_1, \ldots, x_{|V|}/x(t))$, equal to the sum of the number of pairs of points t_i, t_j, $i \neq j$, such that $|t_i - t_j| = 1$, $x_i = x_j$, and the number of pairs of points $t_i \in V$, $t \in T^2 \setminus V$, such that $|t_i - t| = 1$, $x_i = x(t)$. The obvious meaning of $\tilde{\Gamma}(x_1, \ldots, x_{|V|}/x(t))$ is the same as it is in the previous example. From (1.2) it follows that

$$q_V(x_1, \ldots, x_{|V|}/x(t)) = \frac{\exp\left\{-\beta\left(-\tilde{\mu}\sum_{i=1}^{|V|} x_i + \frac{a}{2}\tilde{\Gamma}(x_1, \ldots, x_{|V|}/x(t))\right)\right\}}{\sum_{x_1 \in X, \ldots, x_{|V|} \in X} \exp\left\{-\beta\left(-\tilde{\mu}\sum_{i=1}^{|V|} x_i + \frac{a}{2}\tilde{\Gamma}(x_1, \ldots, x_{|V|}/x(t))\right)\right\}}. \tag{5.3}$$

This formula can easily be obtained from (4.4) and from the fact that the sum $\Gamma(x_1, \ldots, x_{|V|}/x(t)) + \tilde{\Gamma}(x_1, \ldots, x_{|V|}/x(t))$ is independent of $(x_1, \ldots, x_{|V|})$.

We will call the point $t = (t^1, t^2)$ even if $|t^1 + t^2|$ is an even number, and odd if $|t^1 + t^2|$ is an odd number. It is obvious that separation into even and odd squares corresponds to the separation of a chessboard into white and black squares. In our subsequent analysis we will assume that $x(t)$ acquires the values $x_0(t)$ or $x_1(t)$, where

$$x_0(t) = \begin{cases} 0, & t \in T^2 \setminus V, \text{ if } t \text{ even}, \\ 1, & t \in T^2 \setminus V, \text{ if } t \text{ odd}, \end{cases} \qquad x_1(t) = \begin{cases} 0, & t \in T^2 \setminus V, \text{ if } t \text{ odd}, \\ 1, & t \in T^2 \setminus V, \text{ if } t \text{ even}. \end{cases} \tag{5.4}$$

Geometrically it is obvious that in these cases the ensemble of all sides has the same properties as it does in Section 4 of this paper; in particular, it can be split up into contours (Fig. 6).

LEMMA 2. For a Gibbsian distribution in the square V with boundary conditions of the form (5.4) the probability $\Pr\{G\}$ that the arrangement $(x_1, \ldots, x_{|V|})$ contains a certain contour G is such that

$$\Pr\{G\} \leqslant \exp\left\{-\beta\left(\frac{a}{2}-\frac{|\tilde{\mu}|}{4}\right)|G|\right\}, \tag{5.5}$$

where $|G|$ is the length of the contour.

The proof is performed in a manner analogous to the proof of Lemma 1. In particular, proceeding analogously to (4.6), we use (5.3) to obtain

$$\Pr\{G\} = \frac{\sum\limits_{(x_1,\ldots,x_{|V|})\in\mathfrak{H}(G)} \exp\left\{-\beta\left(-\tilde{\mu}\sum\limits_{i=1}^{|V|} x_i + \frac{a}{2}\tilde{\Gamma}(x_1,\ldots,x_{|V|}/x(t))\right)\right\}}{\sum\limits_{x_1\in X,\ldots,x_{|V|}\in X} \exp\left\{-\beta\left(-\tilde{\mu}\sum\limits_{i=1}^{|V|} x_i + \frac{a}{2}\tilde{\Gamma}(x_1,\ldots,x_{|V|}/x(t))\right)\right\}}. \tag{5.6}$$

The principal difference lies in the fact that instead of a transformation T_G we introduce a different transformation $\tilde{T}_G(x_1, \ldots, x_{|V|}) = (\tilde{x}_1, \ldots, \tilde{x}_{|V|})$, where $(x_1, \ldots, x_{|V|}) \in \mathfrak{H}(G)$; we define this transformation (Fig. 7) according to the following rule. Assume t_{j_i} is a square that is directly below the square t_i. Then

$$\begin{aligned} &\tilde{x}_i = x_i, \text{ if } t_i \text{ lies outside the contour } G,\\ &\tilde{x}_i = x_{j_i}, \text{ if } t_i \text{ and } t_{j_i} \text{ lie inside the contour } G,\\ &\tilde{x}_i = 1 - x_{j_i}, \text{ if } t_i \text{ lies inside } G, \text{ and } t_{j_i} \text{ lies outside } G. \end{aligned} \tag{5.7}$$

In other words, it can be stated that the transformation $\tilde{T}_G$ annihilates the contour G, does not change contours lying outside G, and raises all contours lying inside G upward by one (see Fig. 7). Under these conditions

$$\tilde{\Gamma}(\tilde{x}_1,\ldots,\tilde{x}_{|V|}/x(t)) = \tilde{\Gamma}(x_1,\ldots,x_{|V|}/x(t)) - |G|, \quad \left|\sum_{i=1}^{|V|}\tilde{x}_i - \sum_{i=1}^{|V|}x_i\right| \leqslant \frac{|G|_{hor}}{2}, \tag{5.8}$$

where $|G|_{hor}$ is the number of horizontal sides of the contour G. Concepts analogous to (4.9) now make it possible to derive the following result from (5.6):

$$\Pr\{G\} \leqslant \exp\left\{-\beta\left(\frac{a}{2}|G| - \frac{|\tilde{\mu}||G|_{hor}}{2}\right)\right\}. \tag{5.9}$$

An analogous estimate is true, of course, if the number of horizontal sides of the contour G is replaced with the number of vertical sides $|G|_{ver}$. Since $\min(|G|_{hor}, |G|_{ver}) \leq 1/2|G|$, it follows that (5.5) derives from (5.9) and the analogous inequality with $|G|_{ver}$.

Then reasoning fully analogous to that performed in Section 4 demonstrates that (see (4.11)) for a Gibbsian distribution with the boundary conditions $x_0(t)$ we have

$$\begin{aligned} &P\{\xi(t_i)=1\} \leqslant \gamma < \frac{1}{2}, \quad \text{if} \quad t_i \text{ is even } t_i \in V,\\ &P\{\xi(t_i)=0\} \leqslant \gamma < \frac{1}{2}, \quad \text{if} \quad t_i \text{ is odd } t_i \in V, \end{aligned} \tag{5.10}$$

and analogous inequalities for the Gibbsian distribution with the boundary conditions $x_1(t)$. From this it follows that two noncoinciding Gibbsian distributions exist. These two distributions make the transition into one another when the field is shifted by 1. Thus, these two distributions do not differ from each other in their macroscopic characteristics. The ranges of values of the parameters (β, μ) for which uniqueness and nonuniqueness of the Gibbsian distribution, respectively, have been proven in this case are shown in Fig. 8. The line separating the uniqueness and nonuniqueness regions must pass somewhere between them; this line can also be described as the line formed by the branching points of the operator Q introduced in (2.4). It is natural to assume that at those branching points analyticity of the macroscopic field characteristics is violated; however, this remains unproved. With respect to the complete description of all Gibbsian distributions we can state a hypothesis analogous to the one cited in Section 4. If we adopt this hypothesis, then we find that in the investigated example the translation-invariant distribution is unique.

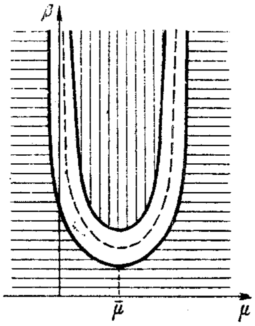

Fig. 8. Uniqueness and nonuniqueness regions in the Ising model with repulsion. Their hypothetical boundary is shown by the dashed line.

6. We shall now study any potential U(t), $t \in T^{\nu}$, such that

$$U(t)=\infty,\quad |t|=1,\quad \sum_{|t|>1}|U(t)||t|<\infty. \tag{6.1}$$

We find that for such a potential there exist at least two different Gibbsian distributions when

$$\mu/2\nu > C,\ \beta > \frac{1}{2}\left(\frac{\mu}{2\nu}-C\right)^{-1}s_{\nu}, \tag{6.2}$$

where

$$C=\frac{1}{\nu}\max_{j=1,\dots,\nu}\sum_{t\in T^{\nu}}|t^{j}||U(t)|,\quad t=(t^{1},\dots,t^{\nu}).$$

For the particular case U(t) = 0, $|t|\neq 1$, the analogous result was proved by means of a more complex method by Moscow State University student Sukhov in his thesis written under the supervision of Ya. G. Sinai.

The derivation of this result is analogous to the conclusion drawn in Section 5. In particular, we adopt the definitions of the boundaries and the functions $x_0(t)$, $x_1(t)$ introduced there. We call the situation $(x_1,\dots,x_{|V|})$ allowed if no two shaded squares have common sides. It is obvious that in view of (6.1) and (1.2) the probabilities of nonallowed situations are equal to zero. Assume for simplicity that $|V|=l^2$, where l is an even number. It is clear that the number of shaded cells in one column of the square V is equal to $l/2-g/2$ for an allowed situation, where g is the number of horizontal boundary sides in the column. The analogous postulate is also valid for rows, to that for the allowed situation $(x_1,\dots,x_{|V|})$

$$q_V(x_1,\dots,x_{|V|}/x(t))=\frac{\exp\left\{-\beta\left(\frac{\mu}{4}\tilde{\Gamma}(x_1,\dots,x_{|V|}/x(t))+\frac{1}{2}\sum_{i=1}^{|V|}\sum_{j=1,i\neq j}^{|V|}x_ix_jU(t_i-t_j)+\sum_{i=1}^{|V|}\sum_{t\in T^{\nu}\setminus V}x_ix(t)U(t_i-t)\right)\right\}}{\sum_{x_1\in X,\dots,x_{|V|}\in X}\exp\left\{-\beta\left(\frac{\mu}{4}\tilde{\Gamma}(x_1,\dots,x_{|V|}/x(t))+\frac{1}{2}\sum_{i=1}^{|V|}\sum_{j=1,i\neq j}^{|V|}x_ix_jU(t_i-t_j)+\sum_{i=1}^{|V|}\sum_{t\in T^{\nu}\setminus V}x_ix(t)U(t_i-t)\right)\right\}}. \tag{6.3}$$

LEMMA 3. For a Gibbsian distribution in the square V with boundary conditions of the form (5.4) the probability is given by the formula

$$\Pr\{G\}\leqslant\exp\left\{-\beta\left(\frac{\mu}{4}-C\right)|G|\right\}. \tag{6.4}$$

The proof of that lemma is performed according to the same technique as that used in the proof of Lemmas 1 and 2 on the basis of the transformation $\tilde{T}_G$ introduced in (5.7). Here it should be noted that the transformation $\tilde{T}_G$ converts an allowed situation into another allowed situation, and also that for the allowed situations we have

$$\left|\frac{1}{2}\sum_{i=1}^{|V|}\sum_{j=1,i\neq j}^{|V|}x_ix_jU(t_i-t_j)+\sum_{i=1}^{|V|}\sum_{t\in T^{\nu}\setminus V}x_ix(t)U(t_i-t)\right.$$
$$\left.-\frac{1}{2}\sum_{i=1}^{|V|}\sum_{j=1,j\neq i}^{|V|}\tilde{x}_i\tilde{x}_jU(t_i-t_j)-\sum_{i=1}^{|V|}\sum_{t\in T^{\nu}\setminus V}\tilde{x}_ix(t)U(t_i-t)\right|\leqslant\sum_{\substack{t\text{ — is inside the contour } G,\\ \tilde{t}\text{ — is outside the contour } G,\\ |t-\tilde{t}|>1}}|U(t-\tilde{t})|. \tag{6.5}$$

Then we connect the points t and $\tilde{t}$, where $t=(t^1,t^2)$, $\tilde{t}=(\tilde{t}^1,\tilde{t}^2)$, by means of a corner consisting of a horizontal segment with a length $|t^1-\tilde{t}^1|$ and a vertical segment of length $|t^2-\tilde{t}^2|$ (it is assumed that the first coordinate axis is horizontal). If t is inside the contour G and $\tilde{t}$ is outside it, then this corner intersects one of the sides of the contour G; furthermore, for fixed $s=t-\tilde{t}=(s^1,s^2)$ the given vertical side of the contour G is intersected by no more than $|s^1|$ corners corresponding to different t, $\tilde{t}$ pairs. We have an analogous result for the horizontal sides of the contour G. From this we obtain

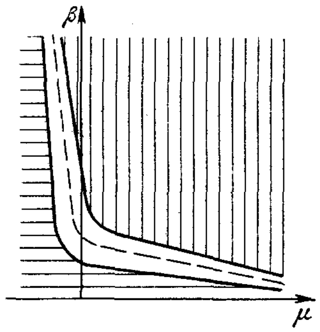

Fig. 9. Regions of uniqueness and nonuniqueness in the case of a potential of the type (6.1). Their hypothetical boundary is shown by the dashed line.

$$\sum_{\substack{t \text{—is inside the contour } G, \\ \tilde{t} \text{—is outside the contour } G, \\ |t-\tilde{t}| = 1}} |U(t-\tilde{t})|$$

$$\leqslant \frac{1}{2}\Big(|G|_{\text{hor}} \sum_{s \in T^{\nu}, |s| = 1} |s_1||U(s)| + |G|_{\text{ver}} \sum_{s \in T^{\nu}, |s| = 1} |s_2||U(s)|\Big) \leqslant C|G|. \qquad (6.6)$$

Now the postulate of the lemma derives from (6.3).

Then it is necessary to repeat what was said in the last part of Section 5, except that the reference to Fig. 8 must be replaced with a reference to Fig. 9. In such a model the phase transition evidently exists for all β.

The method developed in Sections 4, 5, and 6 makes it possible to estimate the region of nonuniqueness of the Gibbsian distribution for other potentials as well.

LITERATURE CITED

1. R. L. Dobrushin, "Gibbsian random fields for lattice systems with paired interactions," Funktsional'. Analiz i Ego Prilozhen., 2, No. 4, 31-43 (1968).
2. R. L. Dobrushin, "Description of a random field by means of conditional probabilities and the conditions governing its regularity," Teor. Veroyatn. i Ee Primen., 13, No. 2, 201-229 (1968).
3. R. A. Minlos, "Regularity of the limiting Gibbsian distribution," Funktsional'. Analiz i Ego Prilozhen., 1, No. 3, 40-53 (1967).
4. R. A. Minlos and Ya. G. Sinai, "New results on phase transitions of the first kind in models of a lattice gas," Trudy Mosk. Matem. Obshchestva, 11, 213-242 (1967).
5. R. Griffiths, "Peierl's proof of spontaneous magnetization in two-dimensional Ising ferromagnet," Phys. Rev., A136, 437-438 (1964).
6. R. L. Dobrushin, "Existence of a phase transition in the two-dimensional and three-dimensional Ising models," Teor. Veroyatn. i Ee Primen., 10, No. 2, 209-230 (1965).
7. K. Huang, Statistical Mechanics, N. Y.-London, John Wiley (1963).
8. R. L. Dobrushin, Existence of Phase Transition in Models of a Lattice Gas, Proc. Fifth Berkeley Symp. on Math. Stat. and Probabilities, Vol. 3, Univ. of Calif. Press (1967), pp. 73-87.
9. D. Ruelle, Statistical Mechanics of a One-Dimensional Lattice Gas, Centre de Physique Theorique de l'Ecole Polytechnique, Preprint (1968).
10. R. E. Peierls, "On Ising's ferromagnet model," Proc. Camb. Phil. Soc., 32, 477 (1936).