NEWTON POLYHEDRA AND ESTIMATION OF OSCILLATING INTEGRALS

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The aim of this paper is to calculate the principal term of the asymptotic expansion of an oscillating integral in a neighborhood of a singular critical point of the phase in terms of Newton's diagram of the phase expansion in a Taylor series in a neighborhood of this critical point. The main result of this paper consists in the fact that this principal term is determined by the point of intersection of Newton's diagram with the diagonal of the coordinate octant (under the conditions formulated below). Under these conditions the arithmetic progressions to which the indices of all the terms of the asymptotic expansion belong will depend only on Newton's diagram of the phase function. In this paper we specify the form of these progressions in terms of Newton's diagram. The obtained formulas confirm the hypothesis of V. I. Arnol'd that all appropriate discrete invariants of an analytic function can be expressed in a simple manner in terms of Newton's diagram for almost all the functions with a given Newton diagram (see [11 and 13]). We calculate below the indices of the principal terms of the asymptotic expansion for all the phase functions classified in [16] (in two cases our theorems yield an inequality for the index of the principal term). We present an example that refutes the hypothesis of semicontinuity of the index of the principal term of the asymptotic expansion.

§ 0. Introduction

0.1. Newton's Diagram. Let $\mathbf{N} \subset \mathbf{R} \subset \mathbf{R}_+$ be the sets of all nonnegative integers, all nonnegative real numbers, and all real numbers respectively. Let $K \subset \mathbf{N}^k$.

Definition. Newton's polyhedron of a set K is defined by the convex hull in $\mathbf{R}_+^k$ of the set $\bigcup\limits_{n\in K} (n + \mathbf{R}_+^k)$.

Definition. Newton's diagram of a set K is defined by the union of all compact faces of Newton's polyhedron of K.

Newton's polyhedron is denoted by $\Gamma_+(K)$ and Newton's diagram by $\Gamma(K)$.

Let $f = \sum\limits_{n\in \mathbf{N}^k} a_n x^n$, $a_n \in \mathbf{C}$. Let us write $\operatorname{supp} f = \{n \in \mathbf{N}^k \mid a_n \neq 0\}$.

Definition. Newton's polyhedron of a series f (or Newton's diagram) is defined by Newton's polyhedron (Newton's diagram) of the set supp f.

Newton's polyhedron (Newton's diagram) of the series f is denoted by $\Gamma_+(f)$ (and $\Gamma(f)$ respectively).

Definition. The principal part of a series f is defined by the polynomial $r = \sum\limits_{n\in\Gamma(f)} a_n x^n$.

For any closed face $\gamma \subset \Gamma(f)$ we shall denote by f_γ the polynomial $\sum\limits_{n\in\gamma} a_n x^n$.

Definition. The principal part of a series f is said to be nonsingular if for any closed face $\gamma \subset \Gamma(f)$ the polynomials $x_1 \frac{\partial f_\gamma}{\partial x_1}, \ldots, x_k \frac{\partial f_\gamma}{\partial x_k}$ do not vanish simultaneously at $\{x \in \mathbf{R}^k \mid x_1 \ldots x_k \neq 0\}$.

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It easily follows from Sard's lemma that the set of singular principal parts is a semi-algebraic subvariety in the variety of all principal parts corresponding to a given Newton diagram.

0.2. Oscillating Integrals. Let $C(\mathbf{R}^k)$ be a set of infinitely differentiable functions on $\mathbf{R}^k$ with compact supports. Let $f: \mathbf{R}^k \to \mathbf{R}$ be an infinitely differentiable function.

Definition. An oscillating integral with a phase f is defined by

$$I(\tau, \varphi) = \int_{\mathbf{R}^k} e^{i\tau f(x)} \varphi(x)\, dx,$$

where τ is a real parameter and $\varphi \in C(\mathbf{R}^k)$.

It will be assumed throughout in this article that f is an analytic function at the origin of coordinates.

If the support of a function φ is concentrated in a sufficiently small neighborhood of a zero point, then the oscillating integral will have an asymptotic expansion for $\tau \to +\infty$:

$$I(\tau, \varphi) \approx e^{i\tau f(0)} \sum_p \sum_{n=0}^{k-1} a_{p,n}(\varphi)\, \tau^p (\ln \tau)^n, \tag{0.3}$$

where p runs through finitely many arithmetic progressions not dependent on φ that are constructed from negative rational numbers (see, for example, [1]).

Definition. The oscillation index of the function f at zero is defined by a number $\beta(f)$ which is the maximal of the numbers p having the following property: For any neighborhood of zero in $\mathbf{R}^k$ there exists a $\varphi \in C(\mathbf{R}^k)$ with a support in this neighborhood such that in the asymptotic expansion (0.3) for $I(\tau, \varphi)$ there exists an n with $a_{p,n}(\varphi) \neq 0$.

Everywhere below it is assumed that $f(0) = 0$, $df|_0 = 0$.

0.4. Formulation of Principal Result. Let us specify a coordinate system in $\mathbf{R}^k$ and denote by $\hat{f}$ the Taylor series of the function f at zero in this coordinate system. Let us denote by t_0 the parameter of the intersection of the straight line $x_1 = \ldots = x_k = t$, $t \in \mathbf{R}$, with the boundary of the Newton polyhedron $\Gamma_+(f)$. This number will be called the distance between Newton's polyhedron and the origin.

THEOREM. Suppose that the principal part of the series $\hat{f}$ is nonsingular. Then:

1. There exists a method (described in Sec. 2.17) of calculation, on the basis of Newton's polyhedron of a subset in $\mathbf{N}^k$, of finitely many arithmetic progressions constructed from negative rational numbers. These arithmetic progressions calculated on the basis of Newton's polyhedron $\Gamma_+(f)$ have the following property: If the support $\varphi \in C(\mathbf{R}^k)$ is sufficiently small and the coefficient $I(\tau, \varphi)$ in the asymptotic expansion (0.3) for the integral $a_{p,n}(\varphi)$ does not vanish, then p will be a term of one of the arithmetic progressions calculated by us.

2. If the distance to Newton's polyhedron is not larger than 1, then the oscillation index $\beta(f)$ at zero will not exceed $-(t_0)^{-1}$.

3. If the distance to Newton's polyhedron is strictly larger than 1, then the oscillation index $\beta(f)$ at zero will be equal to $-(t_0)^{-1}$.

4. If the distance to Newton's polynomial is strictly larger than 1 and the point $(t_0, \ldots, t_0)$ lies at the intersection of $l(k-1)$-dimensional faces of the Newton polyhedron $\Gamma_+(\hat{f})$, then for any nonnegative $\varphi \in C(\mathbf{R}^k)$ with $\varphi(0) \neq 0$ and a support lying in a sufficiently small neighborhood of zero in $\mathbf{R}^k$, we shall have in the expansion (0.3) of the integral $I(\tau, \varphi)$ a coefficient $a_{\beta(f),\bar{l}-1}(\varphi) \neq 0$, where $\bar{l} = \min(l, k)$. Moreover, for any $\varphi \in C(\mathbf{R}^k)$, with a support lying in a sufficiently small neighborhood of zero, we have in the expansion (0.3) for the integral $I(\tau, \varphi)$ a coefficient $a_{\beta(f),\bar{l}+n}(\varphi) = 0$ for $n \in \mathbf{N}$.

The hypothesis that the principal term of the asymptotic expansion is determined by the distance to Newton's polyhedron has been formulated by V. I. Arnol'd.

Remarks. In §5 we present an example of a function f of five variables for which the principal part of the series $\hat{f}$ is nonsingular, $t_0 < 1$, and the oscillation index $\beta(f)$ is strictly smaller than $-(t_0)$.

The principal part of the series $\hat{f}$ cannot always be considered as nonsingular. In Sec. 2.18 we prove a theorem which asserts that in the case $t_0 > 1$ and a principal part of the series $\hat{f}$ that is not necessarily nonsingular, the oscillation index $\beta(f)$ is not smaller than $-(t_0)^{-1}$. The case of a not necessarily nonsingular principal part can be examined in detail for functions of two variables.

0.5. Adapted Coordinate Systems. Let $f: \mathbf{R}^k \to \mathbf{R}$ be the same as above, let $y = (y_1, \ldots, y_k)$ be a local analytic coordinate system at zero in $\mathbf{R}^k$, let $\hat{f}_y$ be Taylor's series of the function f at zero in the coordinates y, and let t_y be the distance from the origin to Newton's polyhedron $\Gamma_+(\hat{f}_y)$. Let us write $t(f) = \sup_y t_y$ with respect to all local analytic coordinate systems at zero. The number $t(f)$ is called the height of the function f.

Definition. A local analytic coordinate system y at zero is said to be adapted to f if $t_y = t(f)$.

0.6. THEOREM. Let $f: \mathbf{R}^2 \to \mathbf{R}$ be a function that is analytic at zero, $f(0) = 0$, $df|_0 = 0$, $d^2f|_0$ is singular, and the germ at zero of the set $\{x \in \mathbf{R}^2 \mid f(x) = 0\}$ does not have multiple components. Then:

1. There exist coordinate systems that are adapted to f.

2. The oscillation index $\beta(f)$ of the function f is equal to $-(t(f))^{-1}$.

3. For any nonnegative $\varphi \in C(\mathbf{R}^2)$ with $\varphi(0) \neq 0$ and a support that lies in a sufficiently small neighborhood of zero, the quantity $\max\{p \mid \exists n \mid a_{p,n}(\varphi) \neq 0\}$ in the expansion (0.3) of the integral $I(\tau, \varphi)$ will be equal to the oscillation index of the function f.

4. If there exists a coordinate system y that is adapted to f and such that the point (t_y, t_y) (in a standard coordinate system in which the Newton polyhedron is constructed) lies at the intersection of two faces of Newton's polyhedron $\Gamma_+(\hat{f}_y)$, then for any φ occurring in Assertion 3 of the theorem we have $a_{\beta(f),1}(\varphi) \neq 0$. If such a coordinate system does not exist, then for any $\varphi \in C(\mathbf{R}^2)$ with a support that lies in a sufficiently small neighborhood of zero, we have $a_{\beta(f),1}(\varphi) = 0$.

Remarks. By allowing for a slightly more complicated proof, it is possible to drop the assumption that there are no multiple components in the germ of the set $\{x \in \mathbf{R}^2 \mid f(x) = 0\}$. On the other hand, the case in which such components occur will have codimension infinity. In §5 we present an example of a function $f: \mathbf{R}^3 \to \mathbf{R}$ for which the oscillation index is strictly larger than $-(t(f))^{-1}$. In Sec. 3.15 we describe an algorithm of determination of adapted coordinate systems for $f: \mathbf{R}^2 \to \mathbf{R}$.

The following two propositions can be used for recognizing adapted coordinate systems.

Let $f: \mathbf{R}^2 \to \mathbf{R}$ be the same as in Theorem 0.6, let y be a local analytic coordinate system at zero, and let γ be one of the closed compact faces of Newton's polyhedron $\Gamma_+(\hat{f}_y)$. The straight line on which γ lies can be specified by the equation $a_1(\gamma)x_1 + a_2(\gamma)x_2 = m(\gamma)$, where a_1, a_2, and m are natural numbers, with a_1 and a_2 being relatively prime.

0.7. Proposition. y is a coordinate system adapted to f if one of the following conditions holds:

1. The point (t_y, t_y) lies at the intersection of two faces of the Newton polyhedron $\Gamma_+(\hat{f}_y)$.

2. The point (t_y, t_y) lies on a closed compact face γ of the Newton polyhedron $\Gamma_+(\hat{f}_y)$, and both numbers $a_1(\gamma)$ and $a_2(\gamma)$ are larger than 1.

Now suppose that the point (t_y, t_y) lies on a closed compact face γ of the Newton polyhedron $\Gamma_+(\hat{f}_y)$ and let $a_1(\gamma) = 1$. Let $\hat{f}_y = \sum_{n \in \mathbf{N}^2} a_n y^n$. Let us denote by $f_{y,\gamma}$ the polynomial $\sum_{n \in \gamma} a_n y^n$. By virtue of our condition, the polynomial $f_{y,\gamma}$ can be expressed in the form $y_1^{\iota(\gamma)} \cdot P_s(y_2/y_1^{a_2(\gamma)})$, where P_s is a polynomial of degree s of one variable, $s \leqslant m(\gamma)$.

0.8. Proposition. If the polynomial P_s does not have a real root of multiplicity larger than $m(\gamma)(1 + a_2(\gamma))^{-1}$, then y will be a coordinate system adapted to f.

0.9. Constancy of Oscillation Indices for Functions of Two Variables along the Stratum μ = const. Proposition. Let $f_t: \mathbf{R}^2 \to \mathbf{R}$ be a family of functions that are infinitely differentiably dependent on a parameter $t \in [0,1]$, and which are analytic at zero in R^2 for any t. We shall assume that Milnor's number

$$\mu_t = \dim_{\mathbf{C}} \mathbf{C}\langle\langle x_1, x_2\rangle\rangle \Big/ \left(\frac{\partial \hat{f}_t}{\partial x_1}, \frac{\partial \hat{f}_t}{\partial x_2}\right)$$

of the function f_t at zero does not change when t varies. Then the oscillation index of the function f does not change when t varies.

This assertion has been formulated by Arnol'd [3] as a hypothesis for functions with any number of variables.

0.10. Generalized Functions $f_\pm^\tau$. Let $f: \mathbf{R}^k \to \mathbf{R}$ be a function that is analytic at zero, $f(0) = 0$, $df|_0 = 0$. Let us write

$$f_+(x) = \begin{cases} f(x) & \text{for} \quad f(x) \geqslant 0, \\ 0 & \text{for} \quad f(x) < 0, \end{cases} \qquad f_-(x) = \begin{cases} 0 & \text{for} \quad f(x) \geqslant 0, \\ -f(x) & \text{for} \quad f(x) < 0. \end{cases}$$

Let $\varphi \in C(\mathbf{R}^k)$. Let us consider the integrals

$$I_+(\tau, \varphi) = \int_{\mathbf{R}^k} (f_+(x))^\tau \varphi(x)\, dx, \quad I_-(\tau, \varphi) = \int_{\mathbf{R}^k} (f_-(x))^\tau \varphi(x)\, dx,$$

where $\tau \in \mathbf{C}$, $\operatorname{Re} \tau > 0$, and I_+ and I_- are analytic functions of the parameter τ. According to the theorems of Bernshtein–Gel'fand [4] and of Atiyah [5], and assuming that the support of the function φ is concentrated in a sufficiently small neighborhood of zero, it is possible to analytically continue I_+ and I_- on C as meromorphic functions of the parameter τ, and their poles belong to finitely many arithmetic progressions that do not depend on φ and that are constructed from negative rational numbers.

THEOREM. Suppose that the Taylor series $\hat{f}$ of the function f has a nonsingular principal part. Then the arithmetic progressions of numbers containing the poles of $I_+(\tau, \varphi)$ and $I_-(\tau, \varphi)$ can be calculated on the basis of Newton's polyhedron $\Gamma_+(\hat{f})$ in the manner described in Sec. 2.17.

0.11. Examples. V. I. Arnol'd has posed the following questions.

Let $f: \mathbf{R}^k \to \mathbf{R}$ be a smooth function, let $F: \mathbf{R}^k \times \mathbf{R}^l \to \mathbf{R}$ be its deformation (i.e., F is a smooth function and $F(\cdot, 0) = f$), and let β be the oscillation index of the function f at zero.

Question 1. For any positive ε, does an oscillating integral with a function $F(\cdot, \lambda)$ admit a bound

$$\left|\int_{\mathbf{R}^k} e^{i\tau F(x, \lambda)} \varphi(x, \lambda)\, dx\right| \leqslant C(\varphi, \varepsilon)\, \tau^{\beta+\varepsilon}$$

for any smooth φ with a support in a sufficiently small neighborhood of the origin in $\mathbf{R}^k \times \mathbf{R}^l$?

Question 2. Is the oscillation index semicontinuous in the sense that the oscillation index of the function $F(x, \lambda_0)$ at the point x_0 does not exceed β for any (x_0, λ_0) lying in a sufficiently small neighborhood of the origin in $\mathbf{R}^k \times \mathbf{R}^l$?

Let $f_1: \mathbf{R}^k \to \mathbf{R}$ and $f_2: \mathbf{R}^k \to \mathbf{R}$ be analytic functions at the origin. Let $f_1^{\mathbf{C}}: \mathbf{C}^k \to \mathbf{C}$, $f_2^{\mathbf{C}}: \mathbf{C}^k \to \mathbf{C}$ be "complexifications" of the functions f_1 and f_2, i.e., $f_i^{\mathbf{C}}$ is an analytic function that has at zero the same Taylor series as f_i. Suppose there exists an analytic change of coordinates $y = g(x)$ in C^k that preserves the origin and such that $f_1^{\mathbf{C}}(g(x)) = f_2^{\mathbf{C}}(x)$.

Question 3. Are the oscillation indices of the functions f_1 and f_2 at zero equal?

Let us note that if the change of coordinates $y = g(x)$ is specified by functions with real Taylor coefficients, we shall evidently obtain a positive answer to question 3.

The questions 1 and 2 were formulated in [3] and [16], whereas question 3 was formulated in [3]. The example presented in §5 yields a negative answer to all three questions.

0.12. Our analysis will proceed as follows. The aim of §1 is to prove Proposition 1.4 in which we summed up the entire analytic part of the paper. In §2 we prove Theorems

0.4 and 0.10. In §3 we prove all the results relating to functions of two variables. In §4 we present the results of applying our theorems in the calculation of the oscillation indices of the functions classified in [16]. Some of the results presented in §4 were formulated in [3]. In §5 we present examples that yield negative answers to the questions 1-3 of Sec. 0.11.

In conclusion, the author expresses his gratitude to V. I. Arnol'd for posing the problem, and to V. N. Karpushkin and A. G. Khovanskii for numerous useful discussions.

§1. Resolution of Singularities and Oscillating Integrals

The aim of this section is to prove Proposition 1.4.

Let $f\colon \mathbf{R}^k \to \mathbf{R}$ be a function that is analytic at the origin, $f(0) = 0$, $df|_0 = 0$. Let Y be a nonsingular real analytic k-dimensional manifold, and $\pi\colon Y \to \mathbf{R}^k$ a proper analytic mapping such that at each point of the set $S = \pi^{-1}(0)$ there exist local coordinates $y_1, \ldots, y_k$ at which

$$f \circ \pi (y_1, \ldots, y_k) = \pm y_1^{n_1} \ldots y_k^{n_k}. \tag{1.1}$$

(1.2) The Jacobian J_π of the mapping π has the form

$$J_\pi(y_1, \ldots, y_k) = y_1^{m_1} \ldots y_k^{m_k} \bar{J}_\pi(y_1, \ldots, y_k), \text{ where } \bar{J}_\pi(0, \ldots, 0) \neq 0.$$

(1.3) In a neighborhood of zero in $\mathbf{R}^k$, π is an analytic isomorphism outside a proper analytic subset in $\mathbf{R}^k$.

Let us denote by $\{(n, m)\}_Y$ a set of pairs (n_i, m_i) with $n_i > 0$ and $(n_i, m_i) \neq (1, 0)$ encountered in such notations for $y \in S$. Let us write $\beta_Y = \min\{-(m+1)/n \mid (n, m) \in \{(n, m)\}_Y\}$. (If the set $\{(n, m)\}_Y$ is empty we shall write $\beta_Y = -\infty$). The set $\{(n, m)\}_Y$ will be called an array of multiplicities of the resolution (Y, π). The number β_Y will be called the weight of the resolution (Y, π).

Let $\varphi \in C(\mathbf{R}^k)$, and $I(\tau, \varphi)$ and $I_\pm(\tau, \varphi)$ are functions defined in the Introduction.

Proposition 1.4. 1. If the support of φ is concentrated in a sufficiently small neighborhood of zero, then $I_+(\tau, \varphi)$ and $I_-(\tau, \varphi)$ can be analytically continued on C as meromorphic functions of τ, and their poles belong to the terms of arithmetic progressions one of which consists of negative integers, whereas the others are parametrized by the elements of an array of multiplicities $\{(n, m)\}_Y$ of the resolution (Y, π). To the pair $(n, m) \in \{(n, m)\}_Y$ there corresponds the arithmetic progression $-(m + 1)/n, -(m + 2)/n, \ldots$.

2. We shall assume that for any point $y \in S$ and any local coordinate system $y_1, \ldots, y_k$ centered at y that satisfies (1.1) and (1.2), there do not exist two pairs equal to (1, 0) among the pairs (n_i, m_i) $(i = 1, \ldots, k)$ in the expansions (1.1) and (1.2). We shall also assume that the weight β_Y of the resolution (Y, π) is not larger than -1. Let $1, \ldots, \bar{j}$ be all the natural numbers strictly smaller than the number $-\beta_Y$. Hence, if the support of φ is concentrated in a sufficiently small neighborhood of zero, then $I_+(\tau, \varphi)$ and $I_-(\tau, \varphi)$ will have at the points $\tau = -1, \ldots, -\bar{j}$ poles of multiplicity not higher than 1. If a_j^+ (or a_j^-) is a residue of $I^+(\tau, \varphi)$ (a residue of $I_-(\tau, \varphi)$) at the point $\tau = -j$, where $j = 1, \ldots, \bar{j}$, then $a_j^+ = (-1)^{j-1} a_j^-$. On the set $\operatorname{Re} \tau > \beta_Y$ the functions $I_+(\tau, \varphi)$ and $I_-(\tau, \varphi)$ do not have other poles.

3. Let $\beta_Y > -1$. Let us write $\bar{j} = \max\{j \mid$ there exists a $y \in S$ and a local coordinate system $y_1, \ldots, y_k$ at the point y that has the properties (1.1)-(1.2) and such that in (1.1) and (1.2), j numbers among $(m_1 + 1)/n_1, (m_2 + 1)/n_2, \ldots, (m_k + 1)/n_k$ are equal to $-\beta_Y\}$. Hence, if φ has a support that is concentrated in a sufficiently small neighborhood of zero, $\varphi(0) \neq 0$, and φ is nonnegative, then:

a) The functions $I_+(\tau, \varphi)$ and $I_-(\tau, \varphi)$ have for $\tau = \beta_Y$ a pole of order not higher than $\bar{j}$;

b) the sum of coefficients of $1/(\tau - \beta_Y)^{\bar{j}}$ in the Laurent expansion of the functions $I_+(\tau, \varphi)$ and $I_-(\tau, \varphi)$ at the point $\tau = \beta_Y$ is nonzero.

4. If the assumptions of Part 2 of the proposition are satisfied and the support of φ is concentrated in a sufficiently small neighborhood of zero, then p in the expansion (0.3) of the integral $I(\tau, \varphi)$ will run through a set of numbers belonging to the arithmetic pro-

gressions described in Part 1 of the proposition and from which we dropped all the integers strictly larger than β_Y.

5. If $\beta_Y > -1$ and the support of φ is concentrated in a sufficiently small neighborhood of zero, then:

a) In the expansion (0.3) of the integral $I(\tau, \varphi)$, the quantity p runs through the arithmetic progressions described in Part 1 of the proposition;

b) the oscillation index $\beta(f)$ of the function f at zero is equal to the weight β_Y of the resolution (Y, π);

c) if $\varphi(0) \neq 0$ and φ is nonnegative, then $a_{\beta(f), \bar{j}-1} \neq 0$ in the expansion (0.3) of the integral $I(\tau, \varphi)$; here $\bar{j}$ is the quantity defined in Part 3 of the proposition.

The remainder of this section is devoted to the proof of Proposition 1.4.

It is easy to see that the following objects of study exist:

(1.5) A neighborhood U of zero in R^k;

(1.6) a neighborhood V of the set S in Y;

(1.7) finitely many infinitely differentiable finite functions $\{\varphi_\alpha: Y \to \mathbf{R}\}$ in Y.

They have the following properties:

(1.8) f is analytic in U and it does not have on U critical values other than zero;

(1.9) $V \subset \pi^{-1}(U)$;

(1.10) the functions $\{\varphi_\alpha\}$ are nonnegative and $\sum_\alpha \varphi_\alpha|_V \equiv 1$;

(1.11) for any φ_α there exists an open set W containing the support of φ_α, and local coordinates on W such that (1.1) and (1.2) are satisfied on W.

Let us specify these objects. In this section, the assumption that the support of φ is sufficiently small will signify that the support of φ lies in U. In this section we shall assume that the support of φ is sufficiently small.

Proof of Part 1 of Proposition 1.4. For Re $\tau > 0$ we have

$$I_\pm(\tau, \varphi) = \int_{\mathbf{R}^k} f_\pm^\tau \varphi\, dx = \int_Y (f \circ \pi)_\pm^\tau (\varphi \circ \pi) |J_\pi|\, dy = \sum_\alpha \int_Y (f \circ \pi)_\pm^\tau (\varphi \circ \pi) \varphi_\alpha |J_\pi|\, dy, \tag{1.12}$$

where dy is a volume element in Y and J_π is the Jacobian of the transition from dx to dy. We shall prove that in the last sum each term can be analytically continued on C as a meromorphic function of τ with poles belonging to the terms of the arithmetic progressions mentioned in Part 1.

Indeed, by virtue of (1.11) we have for any α in some coordinate system the formula

$$\int_Y (f \circ \pi)_\pm^\tau (\varphi \circ \pi) \varphi_\alpha |J_\pi|\, dy = \int_W (\delta y_1^{n_1} \dots y_k^{n_k})_\pm^\tau (\varphi \circ \pi) \varphi_\alpha |\bar{J}_\pi y_1^{m_1} \dots y_k^{m_k}|\, dy_1 \dots dy_k, \tag{1.13}$$

where δ is equal to 1 or to -1.

The last integral is a finite sum of integrals

$$\int_W \left(\prod_{i=1}^k (y_i)_{\delta(i)}^{\tau n_i + m_i} \right) (\varphi \circ \pi) \varphi_\alpha |\bar{J}_\pi|\, dy_1 \dots dy_k,$$

where $\delta(i)$ is equal to + or $-$, depending on i. Now the assertion to be proved follows directly from the Lemma 1.14 formulated below.

LEMMA 1.14. Let $\psi(y_1, \dots, y_k, \mu)$ be a finite infinitely differentiable function on R^k that is a meromorphic function of the parameter $\mu \in \mathbf{C}^l$. Then the function

$$I(\tau_1, \dots, \tau_k, \mu) = \int_{\mathbf{R}^k} \left(\prod_{i=1}^k (y_i)_{\delta(i)}^{\tau_i} \right) \psi(y_1, \dots, y_k, \mu)\, dy_1 \dots dy_k,$$

where $\delta(i)$ is equal to + or − (depending on i), can be analytically continued at all the values of $\tau_1, \ldots, \tau_k$ and μ as a meromorphic function, and all its poles other than those already possessed by the function ψ can lie only on hyperplanes of the form $\tau_i + s = 0$, where the s are natural numbers.

Lemma 1.14 can be proved in the same way as Lemma 2 in [4].

Proof of Part 2 of Proposition 1.4. By virtue of (1.12), it suffices to prove Part 2 for the integral in the left-hand side of (1.13) with any α. By virtue of our conditions, the right-hand side of (1.13) contains not more than one subscript i for which $n_i = 1$ and $m_i = 0$. If there is no such i, then it follows from the conditions of Part 2 and of Lemma 1.14 that this integral does not have poles on the set $\operatorname{Re} \tau > \beta_Y$ for f_+ or f_-.

Now suppose that such a subscript exists. For definiteness, let $n_1 = 1$ and $m_1 = 0$. Thus, let us prove Part 2 for the integrals

$$\int_W (\delta y_1 y_2^{n_2} \ldots y_k^{n_k})_\pm^\tau (\varphi \circ \pi) \varphi_\alpha \,|\, \bar{J}_\pi y_2^{m_2} \ldots y_k^{m_k} \,|\, dy_1 \ldots dy_k,$$

where δ is equal to 1 and −1.

For this purpose it suffices to prove Part 2 for the integrals

$$\int_W (y_1)_\pm^\tau (y_2)_{\delta(2)}^{n_2\tau+m_2} \ldots (y_k)_{\delta(k)}^{n_k\tau+m_k} (\varphi \circ \pi) \varphi_\alpha \,|\, \bar{J}_\pi \,|\, dy_1 \ldots dy_k, \tag{1.15}$$

where $\delta(i)$ is equal to + or −.

Let l be a natural number strictly smaller than $-\beta_Y$. From the explicit formulas of regularization of the integral $\int_0^\infty x^\tau \psi(x)\, dx$ (see [6]) it then easily follows that:

(1.16) The integral (1.15) has for $(y_1)_+^\tau$, as well as for $(y_1)_-^\tau$, a pole of order not higher than the first at the point $\tau = -l$.

(1.17) The residue of the integral (1.15) for $(y_1)_+^\tau$ at the point $\tau = -l$ is equal to

$$\frac{1}{(l-1)!} \int_{\substack{y \in W \\ y_1 = 0}} (y_2)_{\delta(2)}^{-n_2 l+m_2} \ldots (y_k)_{\delta(k)}^{-n_k l+m_k} \frac{\partial^{(l-1)}}{\partial y_1^{(l-1)}} [(\varphi \circ \pi) \varphi_\alpha \,|\, \bar{J}_\pi \,|]\, dy_2 \ldots dy_k.$$

(1.18) The residue of the integral (1.15) for $(y_1)_-^\tau$ at the point $\tau = -l$ is equal to

$$\frac{(-1)^{(l-1)}}{(l-1)!} \int_{\substack{y \in W \\ y_1 = 0}} (y_2)_{\delta(2)}^{-n_2 l+m_2} \ldots (y_k)_{\delta(k)}^{-n_k l+m_k} \frac{\partial^{(l-1)}}{\partial y_1^{(l-1)}} [(\varphi \circ \pi) \varphi_\alpha \,|\, \bar{J}_\pi \,|]\, dy_2 \ldots dy_k.$$

This completes the proof of Part 2 of Proposition 1.4.

Proof of Part 3 of Proposition 1.4. Let us consider the integral (1.13) with any α. If the number of subscripts i in the right-hand side of (1.13) such that $-(m_i + 1)/n_i = \beta_Y$, is smaller than the number $\bar{j}$ in Part 3, then it is easy to see that the integral (1.13) has for this α at the point $\tau = \beta_Y$ a pole of order strictly smaller than $\bar{j}$. Now let α be such that the number of subscripts i such that $-(m_i + 1)/n_i = \beta_Y$, is equal to $\bar{j}$. For definiteness, let these subscripts be $-1, \ldots, \bar{j}$. The integral (1.13) is a finite sum of integrals

$$\int_W (y_1)_{\delta(1)}^{\tau n_1+m_1} \ldots (y_k)_{\delta(k)}^{\tau n_k+m_k} (\varphi \circ \pi) \varphi_\alpha \,|\, \bar{J}_\pi \,|\, dy_1 \ldots dy_k. \tag{1.19}$$

From the formulas of regularization of the integral $\int_0^\infty x^\tau \psi(x)\, dx$ it follows that:

(1.20) The integral (1.19) has at the point $\tau = \beta_Y$ a pole of order not higher than $\bar{j}$.

(1.21) The coefficient of $1/(\tau - \beta_Y)^{\bar{j}}$ in the Laurent expansion of the integral (1.19) is equal to

$$\int\limits_{\substack{y\in W\\ y_1=\ldots=y_{\bar{j}}=0}} (y_{\bar{j}+1})_{\delta(\bar{j}+1)}^{\beta_Y n_{\bar{j}+1}+m_{\bar{j}+1}} \ldots (y_k)_{\delta(k)}^{\beta_Y n_k+m_k} (\varphi\circ\pi)\varphi_\alpha |\bar{J}_\pi|\, dy_{\bar{j}+1}\ldots dy_k.$$

This proves Part 3 of Proposition 1.4.

For proving Parts 4 and 5, we shall use the following theorem of I. M. Gel'fand and Z. Ya. Shapiro. Let ω be a $(k-1)$-dimensional differential form that satisfies the relation

$$df \wedge \omega = dx_1 \wedge \ldots \wedge dx_k. \tag{1.22}$$

A form ω that satisfies such a relation exists in a neighborhood of points at which $df \neq 0$. The condition (1.22) invariantly specifies a restriction of the form ω to a nonsingular part of any level line of the function f.

Let us write $K(f, \varphi, c) = \int\limits_{f=c} \varphi\cdot\omega$.

<u>THEOREM (see [6], p. 407).</u> If a function $I_+(\tau, \varphi)$ has poles at the points $-\tau_1, -\tau_2, \ldots, -\tau_l, \ldots (\tau_1 < \tau_2 < \ldots \tau_l < \ldots)$ and if m_l is the multiplicity of a pole at $-\tau_l$, then we have an asymptotic expansion

$$K(f, \varphi, c) \approx \sum_{l=1}^{\infty}\sum_{m=1}^{m_l} a_{l,m} c^{\tau_l-1} (\ln c)^{m-1} \quad \text{for} \quad c \to +0, \tag{1.23}$$

the coefficient $a_{l,m}$ is equal to the coefficient of $1/(\tau + \tau_l)^m$ in the Laurent expansion for $I_+(\tau, \varphi)$ at $\tau = -\tau_l$ multiplied by $(-1)^{m-1}/(m-1)!$

<u>Proof of Part 4 of Proposition 1.4.</u> We have

$$I(\tau, \varphi) = \int_{-\infty}^{\infty} e^{i\tau c} K(f, \varphi, c)\, dc = \int_{0}^{\infty} e^{i\tau c} K(f, \varphi, c)\, dc + \int_{-\infty}^{0} e^{i\tau c} K(f, \varphi, c)\, dc. \tag{1.24}$$

Let

$$K(f, \varphi, c) \approx \sum_{l=1}^{\infty}\sum_{m=1}^{m_l} a^+_{l,m} c^{\tau_l-1} (\ln c)^{m-1} \qquad \text{for} \quad c \to +0, \tag{1.25}$$

$$K(f, \varphi, c) \approx \sum_{l=1}^{\infty}\sum_{m=1}^{m_l} a^-_{l,m} (-c)^{\tau_l-1} (\ln(-c))^{m-1} \quad \text{for} \quad c \to -0 \tag{1.26}$$

be asymptotic expansions for K.

We shall use the following well-known formulas. Let $\theta = C(\mathbf{R}^1)$ and $\theta \equiv 1$ in a neighborhood of zero. For $\tau \to +\infty$ we then have the following asymptotic expansions:

$$\int_{0}^{\infty} e^{i\tau c} c^\alpha (\ln c)^q \theta(c)\, dc \approx \frac{d^q}{d\alpha^q} \frac{\Gamma(\alpha+1)}{(-i\tau)^{\alpha+1}} \left(\text{where } \arg(-i\tau) = -\frac{\pi}{2}\right), \tag{1.27}$$

$$\int_{-\infty}^{0} e^{i\tau c} (-c)^\alpha (\ln(-c))^q \theta(c)\, dc \approx \frac{d^q}{d\alpha^q} \frac{\Gamma(\alpha+1)}{(i\tau)^{\alpha+1}} \quad \left(\text{where } \arg(i\tau) = \frac{\pi}{2}\right). \tag{1.28}$$

According to [7], the expansions (1.25) and (1.26) can be termwise differentiated with respect to c as many times as desired. Then the asymptotic expansion of the integral $I(\tau, \varphi)$ for $\tau \to +\infty$ can be obtained by termwise application of the formulas (1.27) and (1.28) to the expansions (1.25) and (1.26). Since by virtue of our condition, we have for $l = 1, \ldots, \bar{j}$ the relations $\tau_l = 1$, $m_l = 1$, $a^+_{l,1} = (-1)^{l-1} a^-_{l,1}$, $\tau_{\bar{j}+1} = -\beta_Y$, it follows that the asymptotic terms of K for $c \to +0$ and of K for $c \to -0$ cancel out for monomials τ^{-l}. This completes the proof of Part 4 of Proposition 1.4.

<u>Proof of Part 5 of Proposition 1.4.</u> By virtue of Part 3 of Proposition 1.4 and of the Gel'fand–Shapiro theorem we have $\tau_1 = -\beta_Y$, $m_1 = \bar{j}$, and $a^+_{1,\bar{j}}$ and $a^-_{1,\bar{j}}$ have the same sign, and $a^+_{1,\bar{j}} + a^-_{1,\bar{j}} \neq 0$. On the other hand, the principal term of the asymptotic expansion (1.27) (or

(1.28)) will be $d_+\frac{(\ln\tau)^q}{\tau^{\alpha+1}}$ (or $d_-\frac{(\ln\tau)^q}{\tau^{\alpha+1}}$), where $d_\pm=\Gamma(\alpha+1)e^{\pm\frac{\pi i}{2}(\alpha+1)}$. For $0 < \alpha < 1$ we have $\mathrm{Re}\,d_+ = \mathrm{Re}\,d_- \neq 0$. The coefficient $a_{\beta\gamma,\bar{j}}$ in the expansion (0.3) is equal to $a^+_{1,\bar{j}}d_+ + a^-_{1,\bar{j}}d_-$, and by virtue of $\mathrm{Re}\,d_+ = \mathrm{Re}\,d_- \neq 0$ it does not vanish. The other assertions of Part 5 are obvious.

§2. Proof of Theorems 0.4 and 0.10

In this section we prove the Theorems 0.4 and 0.10. For a given Newton polyhedron Γ we construct a manifold Y(Γ) and its projection $\pi : Y(\Gamma)\to \mathbf{R}^k$ that will satisfy the conditions (1.1)-(1.3) for almost all the functions f with a given Newton polyhedron. For such a Newton polyhedron we calculate an array of multiplicities $\{(n, m)\}_{Y(\Gamma)}$ of the resolution (Y(Γ), π), and thus, we can formulate the assertions of Proposition 1.4 in terms of the geometrical characteristics of Newton's polyhedron. This will precisely be the proof of Theorems 0.4 and 0.10.

The manifold Y(Γ) will be constructed as follows. For a given Newton diagram we determine a partition into convex cones of the positive octant in a space conjugate to $\mathbf{R}^k$. After that we refine this partition. With the aid of the theory developed in [8] we construct on the basis of this new partition a k-dimensional nonsingular complex manifold X(Γ) and its projection onto C^k. The real part of the manifold X(Γ) and its restriction to its projection will be the sought Y(Γ) and π.

The procedure of construction (on the basis of a Newton polyhedron Γ) of a manifold X(Γ) described below is a local modification of Khovanskii's method of assigning a compact complex nonsingular toroidal manifold to an integer-valued compact convex polyhedron in R^k.

Partition of Positive Octant into Convex Cones. Let $K\subset \mathbf{N}^k$. We shall assume that K has the following property. Any series $f\in \mathbf{C}\langle\langle x_1, \ldots, x_k\rangle\rangle$ for which the Newton polyhedron $\Gamma_+(f)$ coincides with the Newton polyhedron $\Gamma_+(K)$ belongs to the square of a maximal ideal in $\mathrm{C}\langle\langle x_1, \ldots, x_k\rangle\rangle$.

With respect to the polyhedron $\Gamma_+(K)$ we determine a partition into convex cones of the positive octant in a space $\mathbf{R}^{k*}$ which is the conjugate of $\mathbf{R}^k$.

Let $x_1, \ldots, x_k$ be standard coordinates in R^k, and let $a_1, \ldots, a_k$ be conjugate coordinates in R^{k*}. For $a\in \mathbf{R}^{k*}$ with $a_i \geqslant 0$, $i=1,\ldots,k$, let us write

$$m(a) = \max\{m \mid (a, x) \geqslant m\ \forall x\in\Gamma_+(K)\}.$$

Let us note that $m(a)\geqslant 0$. Two vectors $a, a'\in \mathbf{R}^{k*}$ with $a_i\geqslant 0, a'_i\geqslant 0, i=1,\ldots,k$, are said to be equivalent if $\{x\in\Gamma_+(K) \mid (a, x) = m(a)\} = \{x\in\Gamma_+(K)\mid (a', x) = m(a')\}$. It is easy to see that

(2.1) Any equivalence class is a convex cone with its vertex at zero that is specified by finitely many linear equations and strictly linear inequalities with rational coefficients.

The closures of equivalence classes specify a partition Σ_0 of the positive cone $\{a\in R^{k*} \mid a_i\geqslant 0,\ i=1,\ldots,k\}$ into closed convex cones that have the properties (2.2) and (2.3).

(2.2) If σ_1 is the face of a cone $\sigma\in\Sigma_0$, then $\sigma_1\in\Sigma_0$.

(2.3) For any $\sigma_1, \sigma_2\in\Sigma_0$, the quantity $\sigma_1\cap\sigma_2$ will be a face of both σ_1 and σ_2.

Theorem 1.1 of [8] (p. 32) gives an explicit description of an algorithm that makes it possible to construct on the basis of Σ_0 a partition Σ of the cone $\{a\in\mathbf{R}^{k*}\mid a_i\geqslant 0\}$ into finitely many closed convex cones with their vertex at zero such that:

(2.4) Any cone belonging to Σ lies in one of the cones in Σ_0.

(2.5) Any cone belonging to Σ is specified by finitely many linear equalities and linear inequalities with rational coefficients.

(2.6) If σ_1 is the face of a cone $\sigma\in\Sigma$, then $\sigma_1\in\Sigma$.

(2.7) For any $\sigma_1, \sigma_2\in\Sigma$, the quantity $\sigma_1\cap\sigma_2$ will be the face of both the cone σ_1 and the cone σ_2.

(2.8) The structure† of any cone belonging to Σ can be completed to the base of an integer-valued mesh in R^{k*}.

Let us construct and specify a Σ with the following properties.

The Manifold $X(\Gamma)$. Let $\sigma \in \Sigma$, $\dim \sigma = k$, $a^1(\sigma), \ldots, a^k(\sigma)$ be the structure of a cone σ that has been ordered once and for all. With each such σ we shall associate a copy of C^k denoted by $C^k(\sigma)$. Let us denote by $\pi(\sigma)$: $C^k(\sigma) \to C^k$ a mapping defined by the formulas

$$x_i = y_1^{a_i^1(\sigma)} \ldots y_k^{a_i^k(\sigma)},$$

where $x_1, \ldots, x_k$ are coordinates in C^k, $y_1, \ldots, y_k$ are coordinates in $C^k(\sigma)$, and $a_1^j(\sigma), \ldots, a_k^j(\sigma)$ are the coordinates of the vector $a^j(\sigma)$. We shall identify any two copies $C^k(\sigma)$ and $C^k(\sigma')$ with respect to a rational mapping: $\pi^{-1}(\sigma')\circ\pi(\sigma): C^k(\sigma) \to C^k(\sigma')$ (i.e., $x \in C^k(\sigma)$ and $x' \in C(\sigma')$ will coalesce if $\pi^{-1}(\sigma')\circ\pi(\sigma): x \mapsto x'$). The thus-obtained set will be denoted by $X(\Gamma)$.

From the properties (2.5)-(2.8) of the partition Σ it follows by virtue of Theorems 6, 7, and 8 of [8, pp. 24-26] that:

(2.9) $X(\Gamma)$ is a nonsingular k-dimensional algebraic complex manifold.

(2.10) The mapping $\pi: X(\Gamma) \to C^k$ defined on each $C^k(\sigma)$ by $\pi(\sigma): C^k(\sigma) \to C^k$ is a proper mapping onto C^k.

The transition functions between local maps of the manifold $X(\Gamma)$ are real on real parts of the manifold $X(\Gamma)$ which will be denoted by $Y(\Gamma)$. The restriction of the projection π to $Y(\Gamma)$ is also denoted by π. We have:

(2.11) $Y(\Gamma)$ is a nonsingular k-dimensional real algebraic manifold.

(2.12) $\pi: Y(\Gamma) \to R^k$ is a proper mapping onto R^k.

We shall use the following lemmas concerning $Y(\Gamma)$.

LEMMA 2.13. Let $f(x_1, \ldots, x_k)$ be a convergent power series with real coefficients (f specifies an analytic function in a neighborhood of zero in R^k) and let $\Gamma_+(f) = \Gamma_+(K)$. Let $\sigma \in \Sigma$, $\dim \sigma = k$. Then:

1. $f\circ\pi(\sigma)[y_1, \ldots, y_k] = y_1^{m(a^1(\sigma))} \ldots y_k^{m(a^k(\sigma))} f_\sigma(y_1, \ldots, y_k)$, where $y_1, \ldots, y_k$ are coordinates in $C^k(\sigma)$, $f_\sigma(0, \ldots, 0) \neq 0$.

2. The Jacobian of the mapping $\pi(\sigma)$ is equal to $y_1^{A_1} \ldots y_k^{A_k} \cdot C$, where $A_i = \left(\sum_{j=1}^{k} a_j^i(\sigma)\right) - 1$, C = const.

3. A set of points in R^k in which π is not an isomorphism is a union of coordinate planes.

Assertions 2 and 3 follow directly from the formulas for π, Assertion 1 follows from the formulas for π and the equivalence of all the vectors in a cone σ (this signifies that Newton's diagram of the series $f\circ\pi(\sigma)$ in coordinates $y_1, \ldots, y_k$ is a point).

LEMMA 2.14. Let $f(x_1, \ldots, x_k)$ be a convergent power series with real coefficients $\Gamma_+(f) = \Gamma_+(K)$, and the principal part of the series f is nonsingular. Then the manifold $Y(\Gamma)$ and the projection $\pi: Y(\Gamma) \to R^k$, together with the analytic function defined by the series f, will satisfy the conditions (1.1), (1.2), and (1.3).

Lemma 2.14 easily follows from the conditions of nonsingularity, the proper character of the mapping π, as well as Lemma 2.13 and Lemma 2.15 that follows.

LEMMA 2.15. Let $\sigma \in \Sigma$, $\dim \sigma = k$, $I \subset \{1, \ldots, k\}$, $T_I = \{y \in C^k(\sigma) \mid y_i = 0 \ \forall i \in I,\ y_1, \ldots, y_k$ real$\}$. Then

1. The following conditions (a) and (b) are equivalent:

†The structure of a convex rational cone σ is defined as a set of primitive integer vectors in the faces of σ of dimension 1 (see [8]).

a) For any convergent power series $f(x_1, \ldots, x_k)$ with real coefficients and with $\Gamma_+(f) = \Gamma_+(K)$, a function $f_{\sigma,I}$ that depends on y_i, $i \notin I$, and is defined on T_I, and equal on T_I to the function f_σ occurring in Part 1 of Lemma 1.13, will be a polynomial;

b) $\pi(\sigma)[T_I] = 0$.

2. If the principal part of the series f is nonsingular and $\pi(\sigma)[T_I] = 0$, then the set $\{y \in T_I \mid f_\sigma(y) = 0\}$ (the function f_σ being defined in Part 1 of Lemma 2.13) will be nonsingular, i.e., the gradient of the restriction of the function f_σ to T_I will be nonvanishing at the points of this set.

The proof follows from the definition of the projection $\pi(\sigma)$.

2.16. Array of Multiplicities $\{(n, m)\}_{Y(\Gamma)}$ of the Resolution $(Y(\Gamma), \pi)$. (The definition of the array $\{(n, m)\}_Y$ can be found prior to Proposition 1.4). Let us describe the pairs belonging to $\{(n, m)\}_{Y(\Gamma)}$.

The pairs in $\{(n, m)\}_{Y(\Gamma)}$ are parametrized by one-dimensional cones in Σ with the following property: If $\sigma \in \Sigma$, $\dim \sigma = 1$, $a^1(\sigma)$ being a unique vector forming the structure of the cone σ, then: a) $m(a^1(\sigma)) > 0$, b) if $m(a^1(\sigma)) = 1$, then $\sum_{j=1}^{k} a_j^1(\sigma) \neq 1$, where $a_1^1(\sigma), \ldots, a_k^1(\sigma)$ are the coordinates of the vector $a^1(\sigma)$.

To such a cone σ there corresponds in $\{(n, m)\}_{Y(\Gamma)}$ the pair $\left(m(a^1(\sigma)), \left(\sum_{j=1}^{k} a_j^1(\sigma) - 1\right)\right)$ (see Lemma 2.13).

The number $-(m + 1)/n$ corresponding to this pair and used in the definition of the weight $\beta_{Y(\Gamma)}$ of the resolution $(Y(\Gamma), \pi)$ (see its definition prior to Proposition 1.4) is equal to $\left(-\sum_{j=1}^{k} a_j^1(\sigma)\right) \Big/ m(a^1(\sigma))$ and it has the following geometrical meaning. Let t_1 be the parameter of the point of intersection of the straight line $x_1 = \ldots = x_k = t$, $t \in \mathbf{R}$ with the hyperplane $(a^1(\sigma), x) = m(a^1(\sigma))$; then $\left(-\sum_{j=1}^{k} a_j^1(\sigma)\right) \Big/ m(a^1(\sigma)) = -(t_1)^{-1}$.

2.17. Proof of Theorems 0.4 and 0.10. Let $K = \operatorname{supp} \hat{f}$. With respect to K let us construct a manifold $Y(\Gamma)$ and its mapping $\pi\colon Y(\Gamma) \to \mathbf{R}^k$. If the principal part of the series f is nonsingular, then by virtue of Lemma 2.14 the mapping $\pi\colon Y(\Gamma) \to \mathbf{R}^k$ together with f will satisfy the conditions of Proposition 1.4.

From Propositions 1.4 and Sec. 2.16 we obtain the following:

Method of Calculation of Arithmetic Progressions That Satisfy the Conclusions of Theorem 0.10 and Part 1 of Theorem 0.4. With respect to $\Gamma_+(K)$ let us define a partition Σ_0 that has the properties (2.1)-(2.3). With the aid of the algorithm described in Theorem 11 of [8] (p. 32), let us construct a partition Σ with the properties (2.4)-(2.8). In this case, one of the sought progressions will consist of the negative integers, whereas the other progressions are parametrized by one-dimensional cones belonging to Σ that have the following property: If $\sigma \in \Sigma$, $\dim \sigma = 1$, $a^1(\sigma)$ being the only vector forming the structure of the cone σ, then: a) $m(a^1(\sigma)) > 0$; b) if $m(a^1(\sigma)) = 1$, then $\sum_{j=1}^{k} a_j^1(\sigma) \neq 1$, where $a_1^1(\sigma), \ldots, a_k^1(\sigma)$ are the coordinates of the vector $a^1(\sigma)$.

To such a cone σ there corresponds the arithmetic progression

$$\left(-\sum_{j=1}^{k} a_j^1(\sigma)\right) \Big/ m(a^1(\sigma)), \quad \left(-1-\sum_{j=1}^{k} a_j^1(\sigma)\right) \Big/ m(a^1(\sigma)), \ldots$$

If f has a complex isolated singularity at zero, i.e., if

$$\dim_{\mathbf{C}} \mathbf{C}\langle\langle x_1, \ldots, x_k\rangle\rangle \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}\right) < \infty,$$

we possess a more constructive method of calculating arithmetic progressions that satisfy the conclusion of Part 1 of Theorem 0.4. (Let us note that the case of a singularity that is not isolated will have codimension infinity.) It follows from Malgrange's theorem [1] that if the coefficient $a_{p,n}(\varphi) \not\equiv 0$ in the expansion (0.3) of the integral $I(\tau, \varphi)$, then $e^{2\pi i p}$ will be a root of the characteristic polynomial of monodromy in $(k-1)$-dimensional homologies of a fiber of Milnor's fibering associated with f (the corresponding definition can be also found in [1]). A method of calculation of this polynomial on the basis of Newton's polyhedron (which is more constructive than the above method of description of arithmetic progressions) will be presented in our next article.

Proof of Part 2 of Theorem 0.4. Let t_0 be the distance from the origin to Newton's polyhedron $\Gamma_+(\hat{f})$. By virtue of our condition we have $t_0 \leqslant 1$. We shall consider two cases.

First Case. Suppose that the series $\hat{f}(x_1, \ldots, x_k)$ is divisible by one of the variables $x_1, \ldots, x_k$. This signifies that $t_0 \geqslant 1$, i.e., $t_0 = 1$. In this case the assertion of Part 2 of the theorem follows from the description 2.16 of the array of multiplicities $\{(n, m)\}_{Y(\Gamma)}$, the geometrical meaning of the numbers $-(m+1)/n$ for the pairs (n, m) (also explained in this section), and Proposition 1.4 (in this case the weight $\beta_{Y(\Gamma)}$ of the resolution $(Y(\Gamma), \pi)$ is not larger than -1).

Second Case. The series $\hat{f}(x_1, \ldots, x_k)$ is not divisible by any of the variables $x_1, \ldots, x_k$. In this case it follows from Part 2 of Lemma 2.15 that the condition of Part 2 of Proposition 1.4 is satisfied (concerning the absence of a point y and of a local coordinate system $y_1, \ldots, y_k$). It follows from 2.16 that the number $-(t_0)^{-1}$ is equal to the number $\beta_{Y(\Gamma)}$ in Proposition 1.4. Now Part 2 of Theorem 0.4 follows from Part 4 of Proposition 1.4.

Proof of Parts 3 and 4 of Theorem 0.4. It follows from 2.16 and from the condition $t_0 > 1$ that the number $\beta_{Y(\Gamma)}$ is equal to $-(t_0)^{-1}$ and that $\beta_{Y(\Gamma)} > -1$. Part 3 of Theorem 0.4 follows from Part 5 of Proposition 1.4. For proving Part 4, it suffices to prove that the number l in Part 4 of Theorem 0.4 is equal to the number $\bar{J}$ in Part 3 of Proposition 1.4, and then use Part 5 of Proposition 1.4. But the relation $l = \bar{J}$ evidently follows from the geometrical meaning of the numbers $-(m+1)/n$ for $(n, m) \in \{(n, m)\}_{Y(\Gamma)}$ described in 2.16, and from the definition of the partition Σ_0.

2.18. THEOREM. Let $f\colon \mathbf{R}^k \to \mathbf{R}$ be a function that is analytic at the origin, $df|_0 = 0$, $\hat{f}$ is its Taylor series, t_0 is the distance from the origin to Newton's polyhedron $\Gamma_+(\hat{f})$, $t_0 > 1$, and $\beta(f)$ is the oscillation index of the function f at zero. Then $\beta(f) \geqslant -(t_0)^{-1}$.

Proof. For $\Gamma = \Gamma_+(\hat{f})$ let us construct $\pi\colon Y \to Y(\Gamma) \to \mathbf{R}^k$ as described above. By virtue of [2] there exist $Y, \pi'\colon Y \to Y(\Gamma)$ such that the pair $(Y, \pi\circ\pi')$ together with f has the properties (1.1)-(1.3). By virtue of $t_0 > 1$, the weight of the resolution $(Y, \pi\circ\pi')$ is not smaller than $-t_0^{-1}$, and the assertion of the theorem follows from Part 5 of Proposition 1.4.

§3. Two-Dimensional Case

In this section we shall prove at first the Theorem 0.6, and then also the other assertions formulated in the Introduction, for functions of two variables. The proof of Theorem 0.6, just as the proof of Theorem 0.4 in §2, consists in considering the resolution of the germ at zero of a zero level line of a function, and then calculate for this resolution the data occurring in the condition of Proposition 1.4, this being followed by the use of Proposition 1.4.

Thus, it is easy to see that Theorem 0.6 follows from Proposition 3.1 formulated and proved below, as well as Parts 3 and 5 of Proposition 1.4, and the obvious fact that the height $t(f)$ of a function f of two variables that has at zero a critical point with a critical value 0 will not be smaller than 1; here $t(f) = 1$ only if f has a nonsingular second differential at zero.

Let us formulate Proposition 3.1. Let f be the function occurring in Theorem 0.6, let Y be a nonsingular real two-dimensional analytic manifold, and let $\pi\colon Y \to \mathbf{R}^2$ be a proper analytic mapping in $\mathbf{R}^2$ that has together with f the properties (1.1)-(1.3). Let (Y, π) be a minimal pair that also has these properties, i.e., for any (Y', π') that have together with f the properties (1.1)-(1.3), there exists a proper analytic mapping $\psi\colon Y' \to Y$ such that $\pi' = \pi\psi$.

Proposition 3.1. 1. $\beta_Y = -(t(f))^{-1}$, where β_Y is the weight of the resolution (Y, π) defined in §1.

2. There exists a local analytic coordinate system y at zero in R^2 such that $-t_y^{-1} = \beta_Y$, where t_y (defined in Subsection 0.5) is the distance from the origin to Newton's polyhedron.

3. a) If there exists a coordinate system y adapted to f and such that the point (t_y, t_y) (in a standard coordinate system) lies at the intersection of two faces of Newton's polyhedron $\Gamma_+(\hat{f}_y)$, then the number $\bar{j}$ defined on the basis of f, Y, and π in Part 3 of Proposition 1.4 will be equal to 2. b) If such a coordinate system does not exist, then $\bar{j} = 1$.

Proposition 3.1 will be proved in Subsections 3.7-3.14. It is proved with the aid of Lemma 3.2.

For formulating this lemma we shall use the function $m: \{a \in \mathbf{R}^{k*} \mid a_1, \ldots, a_k \geqslant 0\} \to \mathbf{R}_+$ defined in §2 for a subset K in N^k.

Let $\pi_1: \mathbf{C}^k \to \mathbf{C}^k$ be an analytic mapping such that the Jacobian $J(x_1, \ldots, x_k)$ of the mapping π_1 is equal to $x_1^{m_1} \ldots x_k^{m_k} \bar{J}(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are the coordinates in $\mathbf{C}^k$, $\bar{J}(0, \ldots, 0) \neq 0$. Let $\pi_2: \mathbf{C}^k \to \mathbf{C}^k$ be a mapping defined by the formulas

$$x_i \circ \pi_2 = x_1^{a_i^1} \ldots x_k^{a_i^k}, \text{ where } a_i^j \in \mathbf{N},\ \det(a_j^i) = \pm 1.$$

Let us consider a function $g(x_1, \ldots, x_k)$ that is analytic at zero in C^k. Let us write $f(x_1, \ldots, x_k) = x_1^{n_1} \ldots x_k^{n_k} g(x_1, \ldots, x_k)$, where $n_1, \ldots, n_k \in \mathbf{N}$. Let us also write $a^i = (a_1^i, \ldots, a_k^i)$, $m = (m_1, \ldots, m_k)$, $n = (n_1, \ldots, n_k)$. Let $m(a^i)$ be a number defined for the vector a^i with respect to the subset $\operatorname{supp}(\hat{g}(x_1, \ldots, x_k)) \subset \mathbf{N}^k$ in §2. Let us denote by t_0 the parameter of the point of intersection of the straight line $\{x \in \mathbf{R}^k \mid x_i = t(m_i + 1) - n_i,\ t \in \mathbf{R},\ i = 1, \ldots, k\}$ and the hyperplane $\left\{x \in \mathbf{R}^k \,\middle|\, \sum_{j=1}^{k} a_j^i x_j = m(a^i)\right\}$. By $\beta(i)$ let us denote the number $-(m(i) + 1)/n(i)$, where $m(i)$ is the multiplicity of a zero of the Jacobian of the mapping $\pi_1 \circ \pi_2$ onto the hyperplane $\{x \in \mathbf{R}^k \mid x_i = 0\}$, $n(i)$ being the maximum degree of the variable x_i that divides $f \circ \pi_2$.

LEMMA 3.2. 1. $m(i) + 1 = (m, a^i) + \sum_{j=1}^{k} a_j^i$.

2. $n(i) = (n, a^i) + m(a^i)$.

3. $\beta(i) = -(t_0)^{-1}$.

4. Let $k = 2$ and $m_1 = n_1 = 0$. We shall assume that n is a finite maximum degree of the variable x_1 that divides $g(x_1, 0)$, and that $n < n_2/(m_2 + 1)$. Then: a) $-\beta(i) > (m_2 + 1)/n_2$; b) if γ is a compact face of Newton's polyhedron $\Gamma_+(\hat{g})$, that lies on the straight line $a_1x_1 + a_2x_2 = m$, then the length of its projection onto the x_1 axis will be strictly smaller than the value of the parameter of the point of intersection of the straight line $x_1 = t,\ x_2 = (m_2 + 1)t - n_2,\ t \in \mathbf{R}$ with the line $a_1x_1 + a_2x_2 = m$.

Parts 1-3 of Lemma 3.2 are obvious. The proof of Part 4 follows from Fig. 1.

Now let us define the concepts to be used in the proof of Proposition 3.1.

Classes of Components. Let Y be the manifold mentioned in Proposition 3.1. The manifold Y can be obtained as follows. We carry out a σ-process at zero in R^2, and at each point of the coalescent $\mathbf{RP}^1$ at which condition (1.1) is not satisfied for our function f and the obtained manifold we again carry out a σ-process, etc., until the condition (1.1) is satisfied at all the points of the coalescent $\mathbf{RP}^1$. It is evident that the manifold Y and the projection $\pi: Y \to \mathbf{R}^2$ obtained by this procedure will not depend on the order in which the σ-processes are carried out.

For each irreducible component $X \subset \pi^{-1}(0)$ there exists a unique sequence $X_1, X_2, \ldots, X_l$ of irreducible components in $\pi^{-1}(0)$ such that X_1 is obtained during the first σ-process at zero in R^2, and X_i is obtained during the σ-process at the point of the component X_{i-1}, $X_l = X$. Let us denote by $Y_i(X_i)$ the manifold obtained in R^2 after i σ-processes carried out in this order. The sequence of components $X_1, \ldots, X_l$ will be called a sequence of components preceding the component X, whereas X_{l-1} is called a component preceding the component X.

Let us define the concept of irreducible component of p-th class in $\pi^{-1}(0)$, where p is a natural number. Each component will belong to one of the classes. At first let us define the components of first class. Let $X_1, X_2, \ldots, X_l$ be a sequence of irreducible components in $\pi^{-1}(0)$ that precede the component X_l. A component X_l is said to be a component of first class if there exists a local coordinate system $\bar{y}_1, \bar{y}_2$ at zero in R^2 that has the following property.

3.3. Property of Coordinate System $\bar{y}_1, \bar{y}_2$. For obtaining the manifold Y, we shall at first carry out a σ-process that yields X_1, then a σ-process that yields X_2, etc., until we obtain X_l; after that we carry out all the remaining σ-processes in any order. After the first σ-process the neighborhood of the component X_1 in the obtained manifold $Y_1(X_1)$ will be covered by two specified coordinate maps, i.e., in the first map the coordinates will be $\bar{y}_1$ and $\bar{y}_2/\bar{y}_1$, whereas in the second they are $\bar{y}_1/\bar{y}_2$ and $\bar{y}_2$. Suppose we have carried out σ-processes that yielded the components $X_1, \ldots, X_{i-1}$ $(i \leqslant l-1)$, and on the thus-obtained manifold Y_{i-1} (X_{i-1}), the neighborhood of the component X_{i-1} is covered by two specified coordinate maps. In this case the point at which we carry out the σ-process for obtaining X_i must have the coordinates (0, 0) in one of the specified maps with coordinates y_1 and y_2. Let us carry out this σ-process, and let us cover by two coordinate maps the neighborhood of the component X_i in the thus-obtained manifold $Y_i(X_i)$; the first of these maps uses the coordinates y_1 and y_2/y_1, whereas the second uses the coordinates y_1/y_2 and y_2

It is evident that if X_l is a component of first class, then X_{l-1} will be likewise. Let us also note that if a neighborhood of the component X_l in Y_l (X_l) is covered by two specified maps and if y_1 and y_2 are the coordinates in one of them, then X_l will be determined in this map by the vanishing of one of the coordinates.

Let us note that if f has a nonsingular principal part in a coordinate system x_1x_2, then all the irreducible components in $\pi^{-1}(0)$ will belong to the first class, and x_1 and x_2 can be taken as the coordinates $\bar{y}_1$ and $\bar{y}_2$.

Let us assume that we have defined the concept of irreducible components in $\pi^{-1}(0)$ of 1st, 2nd, . . ., p-th class, and that if X is a component of j-th class $(j \leqslant p)$, then the component preceding X will belong to a class not higher than the j-th. Let us define the concept of irreducible component in $\pi^{-1}(0)$ of (p + 1)-th class. Let $X_1, \ldots, X_l$ be a sequence of irreducible components in $\pi^{-1}(0)$ that precede the component X_l. A component X_l is said to be a component of (p + 1)-th class if:

(3.4) X_l is not a component of 1st, 2nd, . . ., p-th class.

(3.5) The sequence $X_1, \ldots, X_l$ contains a component X_s $(s < l)$ and a local coordinate system $\bar{y}_1\bar{y}_2$ in a neighborhood of a point $y^s \in Y_s(X_s)$ at which the σ-process for obtaining X_{s+1} is carried out; they have the following properties:

1. X_s is a component of p-th class.

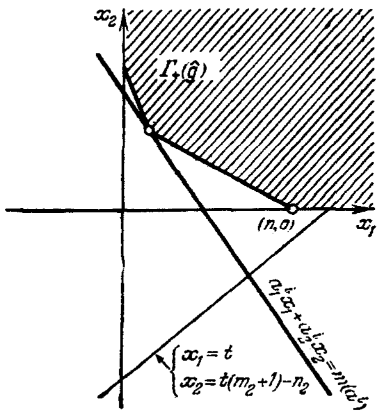

Fig. 1

2. X_{s+1} is not a component of the 1st, 2nd, . . ., p-th class.

3. In the local $\bar{y}_1\bar{y}_2$ coordinate system, y^s has the coordinates (0, 0), X_s is specified by the equation $\bar{y}_2 = 0$, and the Jacobian of the natural projection of the manifold $Y_s(X_s)$ onto R^2 is equal (to within the sign) to $\bar{y}_2^m$ for some m.

4. Let us cover the neighborhood of the component X_{s+1} in Y_{s+1} (X_{s+1}) by two specified coordinate maps, so that in the first map we have the coordinates $\bar{y}_1$ and $\bar{y}_2/\bar{y}_1$, whereas in the second we have the coordinates $-\bar{y}_1/\bar{y}_2$ and $\bar{y}_2$. After carrying out σ-processes we obtain $X_1, \ldots, X_{i-1}$ $(s < i - 1 < l)$ and the neighborhood, in Y_{i-1} (X_{i-1}), of the component X_{i-1} will be covered by two specified maps. In this case the point at which we carry out a σ-process for obtaining X_i must have the coordinates (0, 0) in one of the specified maps with coordinates y_1 and y_2. Let us carry out this σ-process and cover the neighborhood of the component X_i in $Y_i(X_i)$ by two maps, so that in the first map we have the coordinates y_1 and y_2/y_1, and in the second the coordinates y_1/y_2 and y_2.

Let us note that if the neighborhood of the component X_l in $Y_l(X_l)$ is covered by two specified maps, and if y_1 and y_2 are the coordinates in one of them, then X_l will be determined in this map by the vanishing of one of the coordinates.

It is easy to see that a certain class is assigned to each irreducible component belonging to $\pi^{-1}(0)$.

Definition. Let $X_1, \ldots, X_l$ be a sequence of irreducible components in $\pi^{-1}(0)$ that precede the component X_l, and let X_l be a component of first class. A coordinate system y_1y_2 that has together with X_l the property 3.3 is called a coordinate system adapted to X_l.

LEMMA 3.6. 1. If $\bar{y}_1\bar{y}_2$ is a coordinate system adapted to X_l, and φ_1 and φ_2 are nonvanishing analytic functions in a neighborhood of zero in $\mathbf{R}^2$, then $\bar{y}_1\varphi_1, \bar{y}_2\varphi_2$ will likewise be a coordinate system adapted to X_l.

2. If y_1y_2 is a local analytic coordinate system at zero in $\mathbf{R}^2$ and X_l is a component of first class, then for an appropriate analytic function φ of one variable, $\varphi(0) = 0$, either the functions $y_1, y_2 + \varphi(y_1)$, or the functions $y_1 + \varphi(y_2), y_2$ will form a coordinate system adapted to X_l.

Part 1 of Lemma 3.6 is obvious and Part 2 easily follows from Part 1.

3.7. Proof of Proposition 3.1. By virtue of the absence of multiple components in the germ of the set $\{x \in \mathbf{R}^2 \mid f(x) = 0\}$, it is possible to describe an array of multiplicities of the resolution (Y, π) (for the definition, see §1) as follows. For any irreducible component $X \subset \pi^{-1}(0)$ we shall set m(X) equal to the multiplicity of a zero of the Jacobian of the mapping $\pi\colon Y \to \mathbf{R}^2$ on the component X, whereas n(X) is set equal to the multiplicity of a zero of the function $f \circ \pi$ on the component X. Then $\{(n, m)\}_Y = \{(n(X), m(X)) \mid X$ will be an irreducible component of the set $\pi^{-1}(0)\}$.

It is easy to see that to prove Part 1 of Proposition 3.1 it suffices to prove that:

(3.8) For any component X of (p + 1)-th class (p > 0), there exists a component X' of p-th class such that $(m(X) + 1)/n(X) > (m(X') + 1)/n(X')$.

(3.9) If X is a component of first class, then $(m(X) + 1)/n(X) \geqslant (t(f))^{-1}$.

(3.10) There exists an irreducible component of first class $X \subset \pi^{-1}(0)$ such that $(m(X)+1)/n(X) = (t(f))^{-1}$.

Let us prove (3.8) and (3.9). Let $X_1, \ldots, X_l$ be a sequence of irreducible components in $\pi^{-1}(0)$ that precede the component X_l, and let X_l be a component of (p + 1)-th class (p > 0). Let X_s, y^s, and $(\bar{y}_1, \bar{y}_2)$ be (respectively) a component of p-th class, a point, and a coordinate system that together with X_l satisfy the conditions (3.4)-(3.5). By virtue of (3.5), the neighborhood of the component X_l in $Y_l(X_l)$ is covered by two specified maps. If y_1 and y_2 are the coordinates of one of them, then it is easy to see that:

(3.11) The restriction of a natural projection $\pi_{l,s}'\colon Y_l(X_l) \to Y_s(X_s)$ to this map specifies a mapping of this map in a neighborhood of the point y^s in $Y_s(X_s)$ in which the coordinates are $\bar{y}_1$ and $\bar{y}_2$.

$$(3.12) \qquad \bar{y}_1 \circ \pi_{l,s}' = y_1^{a_1^1} y_2^{a_1^2}, \ \bar{y}_2 \circ \pi_{l,s}' = y_1^{a_2^1} y_2^{a_2^2}, \text{where } a_j^i \in \mathbf{N} \text{ and } \det(a_j^i) = \pm 1.$$

Let the function $g(\bar{y}_1, \bar{y}_2)$ be defined in a neighborhood of the point y^s in $Y_s(X_s)$ by the equation $f \circ \pi_s (\bar{y}_1, \bar{y}_2) = \bar{y}_2^{n(X_s)} g(\bar{y}_1, \bar{y}_2)$, where $\pi_s \colon Y_s(X_s) \to \mathbf{R}^2$ is a natural projection onto $\mathbf{R}^2$. Let n be the maximum degree of the variable $\bar{y}_1$ that divides $g(\bar{y}_1, 0)$. Let us prove that

$$n(X_s) / (m(X_s) + 1) > n. \tag{3.13}$$

Let us note that this inequality suffices for proving (3.8). Indeed, by applying Part 4 of Lemma 3.2 to $\bar{y}_2^{n(X_s)} g(\bar{y}_1, \bar{y}_2)$ and to the projection $\pi_{l,s} \colon Y_l(X_l) \to Y_s(X_s)$, we obtain $(m(X_s) + 1) / n(X_s) < (m(X_l) + 1) / n(X_l)$.

We shall prove (3.13) by induction on p. At the same time we shall prove (3.9).

Let p = 1; then X_s will be a component of first class. Let $\bar{y}_1\bar{y}_2$ be a coordinate system adapted to X_s. The neighborhood of the component X_s in $Y(X_s)$ will be covered by two specified maps. If y_1 and y_2 are the coordinates of one of them, then $\bar{y}_1 \circ \pi_s = y_1^{a_1^1} y_2^{a_1^2}$, $\bar{y}_2 \circ \pi_s = y_1^{a_2^1} y_2^{a_2^2}$, where $a_j^i \in \mathbf{N}$ and $\det(a_j^i) = \pm 1$. X_s is specified in this map by the vanishing of one of the coordinates. For definiteness, let this coordinate by y_2. Let us consider Newton's diagram of the function f in the coordinate system $\bar{y}_1\bar{y}_2$. It then follows from Lemma 3.2 that $(m(X_s) + 1)/n(X_s) = (t_1)^{-1}$, where t_1 is the parameter of the point of intersection of the straight line $x_1 = x_2 = t\ (t \in \mathbf{R}$, x_1 and x_2 being standard coordinates in $\mathbf{R}^2$ in terms of which Newton's polyhedron is constructed) with the straight line $a_1^2 x_1 + a_2^2 x_2 = m((a_1^2, a_2^2))$, where $m((a_1^2, a_2^2))$ is a number determined with the aid of the diagram of the function f and the vector (a_1^2, a_2^2) in §2.

Since $\det(a_j^i) = \pm 1$, it follows that a_1^2 and a_2^2 are relatively prime. Since X_{s+1} is not a component of first class, it follows that $a_1^2 > 1, a_2^2 > 1$. Now let us note that n is not larger than the number of integer points minus one on the segment along which the straight line $a_1^2 x_1 + a_2^2 x_2 = m((a_1^2, a_2^2))$ intersects the Newton polyhedron of the function f. But since a_1^2 and a_2^2 are relatively prime and since $a_1^2 > 1, a_2^2 > 1$, this number will be strictly smaller than t_1.

Let us note that by virtue of Lemma 3.2 the number $(t_1)^{-1}$ is not larger than $(t_{(\bar{y}_1, \bar{y}_2)})^{-1}$ where $t_{(\bar{y}_1, \bar{y}_2)}$ is the distance from the origin to Newton's polyhedron $\Gamma_+ (\hat{f}_{(\bar{y}_1, \bar{y}_2)})$, and this signifies that (3.9) is valid.

Let p > 1. Let X_r, y^r, and $(\bar{y}_1, \bar{y}_2)$ be (respectively) the component, the point, and the local coordinate system in a neighborhood of the point y^r in $Y_r(X_r)$ whose existence proves that X_s is a component of p-th class (see 3.5). The neighborhood of the component X_s in $Y_s(X_s)$ is covered by two specified maps. If y_1 and y_2 are the coordinates of one of them, then

$$\bar{y}_1 \circ \pi_{s,r} = y_1^{a_1^1} y_2^{a_1^2}, \quad \bar{y}_2 \circ \pi_{s,r} = y_1^{a_2^1} y_2^{a_2^2}, \tag{3.14}$$

where

$$a_j^i \in \mathbf{N} \text{ and } \det(a_j^i) = \pm 1.$$

X_s is determined by the vanishing of one of the coordinates y_1 and y_2. For definiteness, let it be the coordinate y_2. Suppose that the function $h(\bar{y}_1, \bar{y}_2)$ is defined in a neighborhood of the point y^r by the relation $f \circ \pi_r (\bar{y}_1, \bar{y}_2) = \bar{y}_2^{(nX_r)} h(\bar{y}_1, \bar{y}_2)$.

Let us consider Newton's diagram of the function $h(\bar{y}_1, \bar{y}_2)$. Let n' be the maximum degree of the variable $\bar{y}_1$ that divides $h(\bar{y}_1, 0)$. We have already proved that $n' < n(X_r) / (m(X_r) + 1)$. From this inequality, from Part 4 of Lemma 3.2, and from the formula (3.14) and the fact that n is not larger than the length of the projection (onto the x_1 axis) of the segment along which the straight line $a_1^2 x_1 + a_2^2 x_2 = m((a_1^2, a_2^2))$ intersects Newton's polyhedron of the function $h(\bar{y}_1, \bar{y}_2)$, we obtain (3.13).

Let us prove (3.10) by assuming the contrary. Suppose that (3.10) is not valid. It then follows from (3.8)-(3.9) that $\beta_Y < -(t(f))^{-1}$. Moreover, it is easy to see that $\beta_Y > -1$. We obtain a contradiction if we can prove that there exists a Y' and a $\pi' \colon Y' \to \mathbf{R}^2$ that satisfy together with f the conditions (1.1)-(1.3) for which there exists in $(\pi')^{-1}(0)$ a component X such that $-\beta_Y > (m(X)+1)/n(X) \geqslant (t(f))^{-1}$. Indeed, by virtue of Part 5 of Proposition 1.4 applied to the pair (Y, π), the index of the first term of the asymptotic expansion will be equal to β_Y, whereas the application of Proposition 1.4 to (Y', π') yields an index of the first term of the asymptotic expansion not smaller than $-(m(X) + 1)/n(X)$.

Let us construct (Y', π') with the required properties. By virtue of the definition of the number t(f) there exists a local analytic coordinate system y at zero in R^2 such that $-\beta_Y^{-1} < t_y \leqslant t(f)$ (t_y being the distance to Newton's polyhedron $\Gamma_+(\hat{f}_y)$). Let us consider Newton's polyhedron $\Gamma = \Gamma_+(\hat{f}_y)$. Let $Y(\Gamma)$ and π_Γ: $Y(\Gamma) \to \mathbf{R}^2$ be the manifold and the projection constructed with respect to Γ in §2. It is then easy to see that in $\pi_\Gamma^{-1}(0)$ there exists a component X such that $(m(X) + 1)/n(X) = (t_y)^{-1}$. If f does not yet have the form (1.1)-(1.2) at all the points in $\pi_\Gamma^{-1}(0)$, it will be necessary to carry out corresponding sequences of σ-processes at these points and obtain Y', π': $Y' \to \mathbf{R}^2$. Thus, we have proved (3.10), and hence also Part 1 of Proposition 3.1.

Let us prove Part 2 of Proposition 3.1. By virtue of (3.10) there exists a component of first class such that $(m(X) + 1)/n(X) = (t(f))^{-1}$. Let y be a coordinate system adapted to X. It is easy to see that y is adapted to f.

Let us prove Part 3a of Proposition 3.1. By virtue of Part 5 of Proposition 1.4, the presence of two intersecting components X_1 and X_2 with $(m(X_1) + 1)/n(X_1) = (m(X_2) + 1)/n(X_2) = (t(f))^{-1}$ in a preimage of a zero of any resolution π': $Y' \to \mathbf{R}^2$ that satisfies (1.1)-(1.3) implies the appearance of terms of the form $\tau^{-(t(f))^{-1}} \log \tau$ in the asymptotic expansion of the integral $I(\tau, \varphi)$. For proving Part 3a of Proposition 3.1, it therefore suffices to prove the existence of such a resolution π': $Y' \to \mathbf{R}^2$. Let us take the coordinate system $y = (y_1, y_2)$ whose existence is indicated in Part 3a. Let us consider the Newton polyhedron Γ of the function f in this coordinate system. Let $Y(\Gamma)$ and π_Γ: $Y(\Gamma) \to \mathbf{R}^2$ be the manifold and the projection constructed with respect to Γ in §2. It is then easy to see (by virtue of Lemma 2.3) that in $\pi^{-1}(0)$ we have two intersecting components X_1 and X_2 with $(m(X_1) + 1)/n(X_1) = (m(X_2) + 1)/n(X_2) = (t(f))^{-1}$, and $f \circ \pi_\Gamma$ will have the form (1.1)-(1.2) in a neighborhood of their intersection point. But if f does not yet have the form (1.1)-(1.2) at all the points of $\pi_\Gamma^{-1}(0)$, it will be necessary to carry out corresponding sequences of σ-processes at such points and obtain Y' and π': $Y' \to \mathbf{R}^2$. Then X_1 and X_2 will not cease to intersect. This completes the proof of the Part 3a. Part 3b easily follows from Part 2 of Lemma 3.6.

3.15. Propositions 0.7 and 0.8 follow from Proposition 3.16.

Proposition 3.16. Let f satisfy the conditions of Theorem 0.6, and let $y = (y_1, y_2)$ be a local analytic coordinate system at zero in R^2. Then the functions $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ defined for y_1 and y_2 by the algorithm described below will be coordinate functions of a coordinate system adapted to f.

Algorithm of Determination of Functions $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$. At first we shall determine the transition from y_1, y_2 to z_1, z_2, where either $z_1 = y_1$, $z_2 = y_2$, or $z_1 = y_2$, $z_2 = y_1$. After that we have $f_1 = z_1$ and $f_2 = z_2 + a_1 z_1 + a_2 z_1^2 + \ldots + a_n z_1^n$, where n and $a_1, a_2, \ldots, a_n$ can be determined by the rule described below.

Transition from y_1, y_2 to z_1, z_2. Let us consider the Newton polyhedron $\Gamma_+(\hat{f}_y)$. Let x_1 and x_2 be standard coordinates used for constructing $\Gamma_+(\hat{f}_y)$. If there exists a point $(x_1^0, x_2^0) \in \Gamma_+(\hat{f}_y)$, such that $x_1^0 + x_2^0 \leqslant 2t_y$ and $x_1^0 \neq x_2^0$, we shall write $z_1 = y_1$ and $z_2 = y_2$ in the case $x_1^0 < x_2^0$, and $z_1 = y_2$ and $z_2 = y_1$ in the case $x_1^0 > x_2^0$. If there is no such point we shall write $f_1 = y_1$, $f_2 = y_2$.

Determination of the Numbers $a_1, \ldots, a_n$ and n if z_1 and z_2 Have Been Determined. The numbers $a_1, \ldots, a_n$ are determined by induction.

Determination of a_1. In the coordinate system $z_1 z_2$ let us express $f(z_1, z_2)$ in the form $f = f_{d_1}^1 + f_{d_1+1}^1 + \ldots + f_i^1 + \ldots$, where the f_i^1 are homogeneous terms of degree i of Taylor's series, $f_{d_1}^1 \neq 0$. $f_{d_1}^1(1, \lambda)$ being a polynomial in λ. (It follows from the definition of the functions z_1 and z_2 that the degree of this polynomial is not smaller than $d_1/2$). If $f_{d_1}^1(1, \lambda)$ does not have a real root λ_0 of multiplicity strictly larger than $d_1/2$, then n = 1 and $a_1 = 0$. But if such a λ_0 exists, we shall write $a_1 = \lambda_0$ and seek a_2.

Determination of the Numbers a_l if the Numbers $a_1, \ldots, a_{l-1}$ Have Been Determined. Let us consider a coordinate system $z_2' = z_2 + a_1 z_1 + \ldots + a_{l-1} z_1^{l-1}$, $z_1' = z_1$. Let us assign to z_2'

the weight l, and to z_1^l the weight 1. Let us express $f(z_1^l, z_2^l)$ in these coordinates in the form $f = f_{d_l}^l + f_{d_l+1}^l + \ldots$, where the f_i^l are homogeneous (in the weights just mentioned) polynomials of degree i, $f_{d_l}^l \neq 0$. If $\deg_\lambda f_{d_l}^l (1, \lambda) \leqslant d_l / (l + 1)$, we shall write $n = l - 1$, and the algorithm is terminated. Let $\deg_\lambda f_{d_l}^l (1, \lambda) > d_l / (l + 1)$. If there does not exist a real root λ_0 of the polynomial $f_{d_l}^l (1, \lambda)$ of multiplicity strictly larger than $d_l/(l + 1)$, then $n = l - 1$, and the algorithm will be terminated. If such a root exists, we shall write $a_l = \lambda_0$, and then seek a_{l+1}.

Proposition 3.16 easily follows from Part 2 of Lemma 3.6 and (3.8)-(3.10).

3.17. Proof of Proposition 0.9. Let us denote by $f_t^c: \mathbf{C}^2 \to \mathbf{C}$ a complexification of the function f_t. By virtue of the theorem of Lê and Ramanujam [9], the germs at zero of the sets $\{x \in \mathbf{C}^2 \mid f_t^c(x) = 0\}$ are topologically equivalent for any $t \in [0, 1]$. According to the results of Zariski and Hironaka [10, 15], this signifies that the number of irreducible components in a germ at zero of the set $\{x \in \mathbf{C}^2 \mid f_t^c(x) = 0\}$, and of their characteristic Puiseux exponents and their linkage factors does not change when t varies. According to [12] and [14] this signifies that there exists a family of nonsingular two-dimensional analytic manifolds Y_t that smoothly depend on a parameter $t \in [0, 1]$, and a family of mappings $\pi_t: Y_t \to \mathbf{R}^2$ on $\mathbf{R}^2$ that smoothly depend on $t \in [0, 1]$, and such that:

a) For a given $t \in [0, 1]$, the pair (Y_t, π_t) together with f_t satisfies the conditions (1.1)-(1.3).

b) The number of irreducible components in $\pi_t^{-1}(0)$ does not change with t, and each of these components smoothly depends on t; if X_t is an irreducible component in $\pi_t^{-1}(0)$, then the multiplicity of a zero of the Jacobian of the mapping π_t on X_t and the multiplicity of a zero of the function $f_t \circ \pi_t$ on X_t will not change with t.

On the other hand it is easy to see that $\beta_Y > -1$. Now Proposition 0.9 follows from Part 5 of Proposition 1.4.

§4. Calculation of Principal Term of Asymptotic Expansion for Functions Classified in [16]

At present we possess a far-reaching classification of the first cases of singularity of critical points of functions. In [16] we can find a survey of this theory. In this section we present the results of calculations of the principal terms of the asymptotic expansion of oscillating integrals whose phases are functions classified in [16]. Some of the results presented below were formulated in [3].

Each of the functions classified in [16] has its literal notation. For example, $S^{\#}_{k,2q-1}$ denotes

$$x^2z + yz^2 + zy^{2k+1} + bx^2y^{2k+q} + axy^{3k+q+1},$$

where $a = a_0 + \ldots + a_{k-2}y^{k-2}$, $b = b_0 + \ldots + b_{2k-1}y^{2k-1}$, $k > 1$, $q > 0$, $b_0 \neq 0$. We shall use these notations. The calculation results are listed in Tables 1-5. The functions listed in a table will be called by their names given in [16]. The upper row of a table lists the literal notation of the functions, and the lower row the indices of singularity at a zero of these functions.

Definition. Let $f: \mathbf{R}^k \to \mathbf{R}$ be an analytic function, and $\beta(f)$ its oscillation index at zero. The singularity index at a zero of a function f is defined by the number $\beta(f) + k/2$.

The reason for this definition can be found in [3].

The singularity indices of tabulated functions will be calculated in terms of their oscillation indices. The oscillation indices will be calculated by the formulas obtained in Theorems 0.4 and 0.6 with the use of the fact that all the functions listed in [16] have a nonsingular principal part, or (in the case of functions of two variables) they are written in an adapted coordinate system. This assertion can be easily verified in each particular case. See Tables 1-5.

§5. Examples

At first we shall present an example that gives a negative answer to the questions 1-3 of Sec. 0.11.

TABLE 1. Simple Singularities

A_k	D_k	E_6	E_7	E_8
$\frac{k-1}{2k+2}$	$\frac{k-2}{2k-2}$	$\frac{5}{12}$	$\frac{4}{9}$	$\frac{7}{15}$

TABLE 2. Unimodal Singularities

P_8, T_{pqr}	X_9, J_{10}	E_{12}	E_{13}	E_{14}, Q_{10}	Z_{12}	Z_{13}, Q_{11}	W_{12}	W_{13}, S_{11}	Q_{12}	S_{12}	U_{12}
$\leqslant \frac{1}{2}$	$\frac{1}{2}$	$\frac{11}{21}$	$\frac{8}{15}$	$\frac{13}{24}$	$\frac{6}{11}$	$\frac{5}{9}$	$\frac{11}{20}$	$\frac{9}{16}$	$\frac{17}{30}$	$\frac{15}{26}$	$\frac{7}{12}$

TABLE 3. Bimodal Singularities

$J_{3,0}$ $J_{3,p}$	$Z_{1,0}$ E_{19} $Z_{1,p}$	$W_{1,0}$ $W^{\#}_{1,2q-1}$ $Q_{2,0}$ Z_{17} $W_{1,p}$ $W^{\#}_{1,2q}$ $Q_{2,p}$	$S_{1,0}$ $S^{\#}_{1,2q-1}$ W_{17} Q_{17} $S_{1,p}$ $S^{\#}_{1,2q}$
$\frac{5}{9}$	$\frac{4}{7}$	$\frac{7}{12}$	$\frac{3}{5}$

S_{17}	$U_{1,0}$ $U_{1,2q-1}$ $U_{1,2q}$	E_{18}	E_{20}	Z_{18}	Z_{19}	W_{18}	Q_{16}	Q_{18}	S_{16}	U_{16}
$\frac{5}{8}$	$\frac{11}{18}$	$\frac{17}{30}$	$\frac{13}{24}$	$\frac{10}{17}$	$\frac{16}{27}$	$\frac{17}{28}$	$\frac{25}{42}$	$\frac{29}{48}$	$\frac{21}{34}$	$\frac{19}{30}$

TABLE 4. Singularities of Corank 2 with a Nonzero Four-Jet

$J_{k,0}$ $J_{k,i}$	E_{6k}	E_{6k+1}	E_{6k+2}	$X_{k,0}$ $Y^k_{r,s}$ $X_{k,p}$
$\frac{2k-1}{3k}$	$\frac{6k-1}{9k+3}$	$\frac{4k}{6k+3}$	$\frac{6k+1}{9k+6}$	$\frac{3k-1}{4k}$

$Z^k_{i,0}$ $Z^k_{12k+6i-1}$ Z^k_{12k+6i} $Z^k_{12k+6i+1}$ $Z^k_{i,p}$ } for $k>2$	$Z^2_{i,0}$ $Z^2_{i,p}$
$\frac{3k-1}{4k}$	$\frac{2i+5}{3i+8}$

Z^2_{23+6i}	Z^2_{24+6i}	Z^2_{25+6i}	$Z_{i,0}$ $Z_{i,p}$	Z_{6i+11}	Z_{6i+12}	Z_{6i+13}
$\frac{6i+17}{9i+27}$	$\frac{4i+12}{6i+19}$	$\frac{6i+19}{9i+30}$	$\frac{2i+2}{3i+4}$	$\frac{6i+8}{9i+15}$	$\frac{4i+6}{6i+11}$	$\frac{6i+10}{9i+18}$

W_{12k}	W_{12k+1}	$W_{k,0}$ $W^{\#}_{k,2q-1}$ $W_{k,i}$ $W^{\#}_{k,2q}$	W_{12k+5}	W_{12k+6}
$\frac{12k-1}{16k+4}$	$\frac{9k}{12k+4}$	$\frac{12k+2}{16k+8}$	$\frac{9k+3}{12k+8}$	$\frac{12k+5}{16k+12}$

TABLE 5. Singularities of Corank 3 with a Reduced Three-Jet and a Three-Jet x^2y

$Q_{k,0}$ $Q_{k,i}$	Q_{6k+4}	Q_{6k+5}	Q_{6k+6}	S_{12k-1}	S_{12k}
$\frac{4k-1}{6k}$	$\frac{12k+1}{18k+6}$	$\frac{8k+2}{12k+6}$	$\frac{12k+5}{18k+12}$	$\frac{12k-3}{16k}$	$\frac{18k-3}{24k+2}$

$S_{k,0}$ $S^{\#}_{k,2q-1}$ $S_{k,i}$ $S^{\#}_{k,2q}$	S_{12k+4}	S_{12k+5}	U_{12k}	$U_{k,2q}$ $U_{k,2q-1}$	U_{12k+4}	$V_{1,0}$ $V^{\#}_{1,2q-1}$ $V_{1,p}$ $V^{\#}_{1,2q}$
$\frac{6k}{8k+2}$	$\frac{18k+3}{24k+10}$	$\frac{12k+3}{16k+8}$	$\frac{15k-1}{18k+6}$	$\frac{10k+1}{12k+6}$	$\frac{15k+4}{18k+12}$	$\frac{5}{8}$

Example 1. Let us write $F(x_1, x_2, x_3, \lambda) = (\lambda x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + x_2^{4p} + x_3^{4p}$, where λ is a real parameter and p a sufficiently large natural number. Let β_λ be the oscillation index of the function $F(\cdot, \lambda)$ at zero. F has the following properties:

(5.1) $F(\cdot, \lambda)$ has for any λ an isolated singularity at zero;

(5.2) $\beta_0 = -5/8$;

(5.3) $\beta_\lambda = -3/4$ for $\lambda > 0$;

(5.4) $\beta_\lambda > -(1/2 + \gamma(p))$ for $\lambda < 0$ and $\lim_{p\to\infty} \gamma(p) = 0$;

(5.5) there exists a neighborhood U of zero in R^3 and a neighborhood V of zero in R such that the oscillation index of the function $F(\cdot, \lambda), \lambda \in V$ at any of its critical points $x^0 \in U, x^0 \neq 0$ is smaller than -1.

It is easy to see that the properties (5.2) and (5.4) give a negative answer to question 2 of Sec. 0.11, and the properties (5.2), (5.4), and (5.5) give a negative answer to question 1 of Part 0.11. Finally, let us note that the change of variables $x_1' = ix_1, x_2' = x_2, x_3' = x_3$ carries $F(\cdot, \lambda)$ into $F(\cdot, -\lambda)$. Thus, we obtain a negative answer to question 3 of Part 0.11.

Property (5.1) can be verified by a direct calculation. The properties (5.2) and (5.3) of the function F follow from Part 3 of Theorem 0.4 by virtue of the nonsingularity of the principal part of the function $F(\cdot, \lambda)$ for the given λ.

For proving the property (5.5), it suffices to note that at critical points $x^0 = (x_1^0, x_2^0, x_3^0)$ of the function $F(\cdot, \lambda_0)$ that are near zero, we have $x_2^0 = x_3^0 = 0$ for small λ_0. As is easy to see, this implies

$$\frac{\partial^2 f}{\partial x_2 \partial x_3}(x_0, \lambda_0) = 0, \quad \frac{\partial^2 f}{\partial x_2^2}(x_0, \lambda_0) \neq 0; \quad \frac{\partial^2 F}{\partial x_3^2}(x_0, \lambda_0) \neq 0,$$

i.e., the rank of the second differential is not smaller than two. It follows from Lemma 4.1 of [17] that in a certain coordinate system (u_1, u_2, u_3) in a neighborhood of the point x^0, the function $F(x, \lambda_0)$ can be reduced to the form $\varphi(u_1) + u_2^2 + u_3^2$. This function has a nonsingular principal part. Thus, the property (5.5) follows from Part 2 of Theorem 0.4.

Proof of Property (5.4). We shall prove that there exists a pair (Y, π) that satisfies together with $F(\cdot, \lambda)$ ($\lambda < 0$ being fixed) the conditions (1.1)-(1.3) and has the following property. There exists an irreducible component $X \subset \pi^{-1}(0)$ such that $(m(X) + 1)/n(X) = 1/2 + \gamma(p)$, where m(X) is the multiplicity of a zero on X of the Jacobian of the mapping π, n(X) is the multiplicity of a zero on X of the function $F(\cdot, \lambda)\circ\pi$ and $\lim_{p\to\infty} \gamma(p) = 0$.

By virtue of Part 5 of Proposition 1.4, the existence of such a pair (Y, π) implies the property (5.4).

By a change of variables $u_1 = g(x_1)$, $u_2 = x_2$, $u_3 = x_3$ let us reduce the function $F(x_1, x_2, x_3, \lambda)$ to the form $(-u_1^2+u_2^2+u_3^2)^2 + \varphi(u_1) + u_2^{4p} + u_3^{4p}$, where $\varphi = \sum_{i=4p}^{\infty} a_i u_1^i$ is a convergent power series, with $a_{4p} \neq 0$. Let us prove the existence of a pair (Y, π) for such a function f.

Let us carry out a σ-process $\pi\colon Y_1 \to \mathbf{R}^3$ at zero in $\mathbf{R}^3$. The neighborhood of a coalescent RP^2 will be covered by three maps whose coordinates t_1, t_2, t_3 are expressed by $u_1, u_2/u_1, u_3/u_1$; $u_1/u_2, u_2, u_3/u_2$; $u_1/u_3, u_2/u_3, u_3$. In these coordinates the function $f\circ\pi_1$ will have the form $t_1^4[(-1+t_2^2+t_3^2)^2 + t_1^{4p-4}\varphi_1(t_1, t_2; t_3)]$, $t_2^4[(-t_1^2+1+t_3^2)^2 + t_2^{4p-4}\varphi_2(t_1, t_2, t_3)]$, $t_3^4[(-t_1^2+t_2^2+1)^2 + t_3^{4p-4}\varphi_3(t_1, t_2, t_3)]$, where φ_1, φ_2, and φ_3 are analytic functions. The multiplicity of a zero of the Jacobian of the mapping π_1 on a coalescent RP^2 is equal to two. A proper preimage of the set $\{u \in \mathbf{R}^3 | f(u) = 0\}$ intersects a coalescent RP^2 along a set Z_1 specified in each of the maps by the equations $-1+t_2^2+t_3^2 = 0$, $t_1 = 0$; $-t_1^2+1+t_3^2 = 0$, $t_2 = 0$; $-t_1^2+t_2^2+1 = 0$, $t_3 = 0$.

It is easy to see that in a neighborhood of any point of this set the function can be reduced by a change of coordinates to the form

$$v_1^4(v_2^2 + v_1^{4p-4}\psi_1(v_1, v_2, v_3)), \tag{5.6}$$

where ψ_1 is an analytic function, and the coalescent RP^2 is specified by the equation $v_1 = 0$.

Let us carry out a σ-process $\pi_2\colon Y_2 \to Y_1$ centered at Z_1. The multiplicity of a zero of the Jacobian of the mapping $\pi_1\circ\pi_2$ on the newly coalescent submanifold X_1 is equal to three. Let us denote by Z_2 the set along which X_1 intersects with a proper preimage under a mapping $\pi_1\circ\pi_2$ of the set $\{u \in \mathbf{R}^3 | f(u) = 0\}$. It is easy to see from (5.6) that in a neighborhood of any point of the set Z_2 the function $f\circ\pi_1\circ\pi_2$ can be reduced by a change of coordinates to the form

$$v_1^6(v_2^2 + v_1^{4p-6}\psi_2(v_1, v_2, v_3)), \tag{5.7}$$

where ψ_2 is an analytic function, and X_1 is specified by the equation $v_1 = 0$.

Next let us carry out (2p-4) successive σ-processes as follows. We denote the i-th σ-process by $\pi_{i+2}\colon Y_{i+2} \to Y_{i+1}$, the manifold coalescent during the i-th σ-process by X_{i+1}, and the intersection of X_{i+1} with a proper preimage of the set $\{u \in \mathbf{R}^3 | f(u) = 0\}$ by Z_{i+2}; the σ-process π_{i+2} is centered at Z_{i+1}.

It is easy to prove by induction that Z_{i+2} $(i = 0, \ldots, 2p-4)$ is a nonsingular manifold. The multiplicity of a zero of the Jacobian of the mapping $\pi_1\circ\pi_2\ldots\circ\pi_{i+2}$ on X_{i+1} is equal to $3 + i$. In a neighborhood of any point of Z_{i+2} the function $f\circ\pi_1\circ\pi_2\circ\ldots\circ\pi_{i+2}$ can be reduced by a change of coordinates to the form

$$v_1^{6+2i}(v_2^2 + v_1^{4p-6-2i}\psi_{i+2}(v_1, v_2, v_3)), \tag{5.8}$$

where ψ_{i+2} is an analytic function and X_{i+1} is specified by the equation $v_1 = 0$.

For $i = 2p - 4$ we conclude from (5.8) that the multiplicity of a zero on X_{2p-3} of the function $f\circ\pi_1\circ\ldots\circ\pi_{2p-2}$ is equal to $4p - 2$. On Y_{2p-2} we therefore have for the component X_{2p-3} the relation $(m(X_{2p-3})+1)/n(X_{2p-3}) = 2p/(4p-2)$. It follows from Hironaka's theorem [2] that there exists a real analytic nonsingular three-dimensional manifold Y and a proper analytic mapping $\pi_0\colon Y \to Y_{2p-2}$ such that the pair $(Y, \pi_1\circ\pi_2\circ\ldots\circ\pi_{2p-2}\circ\pi_0)$ together with f satisfies the conditions (1.1)-(1.3). For an irreducible component X in $[\pi_1\circ\pi_2\circ\ldots\circ\pi_{2p-2}\circ\pi_0]^{-1}(0)$ that is a proper preimage of the component X_{2p-3} under the mapping π_0, we have $(m(X) + 1)/n(X) = 2p/(4p - 2)$. Thus, we have proved the property (5.4).

Now let us present an example of a polynomial f of five variables whose principal part is nonsingular, and with a distance t_0 from the origin of coordinates to its Newton polyhedron (defined in 0.4) smaller than 1, and an oscillation index $\beta(f)$ at zero that is strictly smaller than $-(t_0)^{-1}$.

Example 2. Let $f = x_4^2 + x_1^{4p} + x_2^{4p} + x_3^{4p} + (x_4 - (x_1^2 + x_1^4 + x_2^2 + x_3^2))x_5$, $f_- = x_4^2 + x_1^{4p} + x_2^{4p} + x_3^{4p} + (x_4 - (-x_1^2 + x_1^4 + x_2^2 + x_3^2))x_5$. f and f_- have the following properties:

(5.9) f and f_- have a nonsingular principal part;

(5.10) f and f_- have the same Newton polyhedra;

(5.11) the oscillation index of the polynomial f at zero is equal to $-(3/4 + 1)$;

(5.12) the oscillation index of the polynomial f_- at zero is not smaller than $-(1/2 + 1 + \gamma(p))$, where $\lim_{p\to\infty} \gamma(p) = 0$;

(5.13) the distance t_0 from the origin to the Newton polyhedron $\Gamma_+(f)$ is smaller than 1.

From the properties (5.9)-(5.13) follow the sought properties of the polynomial f.

The property (5.10) is obvious. The properties (5.9) and (5.13) can be verified by direct simple calculations. Let us prove the property (5.11), whereas the property (5.12) can be proved in a similar way.

By effecting a change of variables $v = x_5 + (x_4 + (x_1^2 + x_1^4 + x_2^2 + x_3^2))$, we obtain $f = (x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + x_2^{4p} + x_3^{4p} + (x_4 - (x_1^2 + x_1^4 + x_2^2 + x_3^2))\, v$. By effecting a change of variables $u = x_4 - (x_1^2 + x_1^4 + x_2^2 + x_3^2)$ and also $v = s + t$, $u = s - t$, we obtain $f = (x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + x_2^{4p} + x_3^{4p} + s^2 - t^2$. Thus, we have obtained $f = F(x_1, x_2, x_3, 1) + s^2 - t^2$, where F is the function occurring in Example 1. It is easy to see that $\beta(f) = \beta(F(\bullet, 1)) - 1/2 - 1/2$. Now the property (5.11) follows from the property (5.3) of the function F.

<u>Example 3.</u> Let us write $f = (-x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + x_2^{4p} + x_3^{4p}$. Let $\beta(f)$ be the oscillation index of the function f at zero. Then:

(5.14) $\beta(f) \geqslant -\left(\frac{1}{2} + \gamma(p)\right)$, where $\lim_{p\to\infty} \gamma(p) = 0$;

(5.15) $\beta(f) > -(t(f))^{-1}$, where $t(f)$ is the height of the function f.

The proof of this assertion follows from property (5.4) of the function F in Example 1.

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