VERSAL DEFORMATIONS OF A SINGULAR POINT OF A VECTOR FIELD ON THE PLANE IN THE CASE OF ZERO EIGENVALUES

R. I. Bogdanov

One further considers systems of the form

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2) \tag{1}$$

and families of systems (1), depending on two parameters ε_1 and ε_2,

$$\dot{x}_1 = f_1(x_1, x_2, \varepsilon_1, \varepsilon_2), \quad \dot{x}_2 = f_2(x_1, x_2, \varepsilon_1, \varepsilon_2), \tag{2}$$

where f_i are functions of class $C^\infty (i = 1, 2)$.

For $\varepsilon_1 = \varepsilon_2 = 0$ system (1) of family (2) has a singular point $x_1 = x_2 = 0$, for which the matrix of the linear part of the system has eigenvalues $\lambda_1 = \lambda_2 = 0$ and is equivalent to a Jordan cell. One studies the disposition of the trajectories of the system in a fixed neighborhood of the point $x_1 = x_2 = 0$ for small ε_1, ε_2 in the case of a family in "general position" (a nondegenerate family).

LEMMA. For $\varepsilon_1 = \varepsilon_2 = 0$, by a change of coordinates in the phase space of class $C^\infty(x)$, system (1) can be reduced to the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = q_{11}x_1^2 + q_{12}x_1x_2 + q_{22}x_2^2 + o(\|x\|^2),$$

where q_{11}/q_{12} is an invariant of the system relative to the group of changes of variables in phase space which leave the coordinate origin fixed (if $q_{12} = 0$, then the invariant is equal to ∞).

We denote by $\mathfrak{A}$ the set of systems such that $q_{11} \cdot q_{12} \neq 0$.

Definition. A family (2) is called nondegenerate at the point $\varepsilon_1 = \varepsilon_2 = 0$ if system (1) for $\varepsilon_1 = \varepsilon_2 = 0$ lies in $\mathfrak{A}$ and the map $(x_1, x_2, \varepsilon_1, \varepsilon_2) \mapsto (f_1(x_1, x_2, \varepsilon_1, \varepsilon_2), f_2(x_1, x_2, \varepsilon_1, \varepsilon_2))$ is transversal* to $\mathfrak{A}$ at the point $\varepsilon_1 = \varepsilon_2 = 0$.

One has the following theorems.

THEOREM 1. The phase curves of nondegenerate system (2), with the help of a homeomorphism depending continuously on the parameters of a sufficiently small neighborhood of the point $x_1 = x_2 = 0$ in the phase space, and changes of parameters $\eta_1(\varepsilon_1, \varepsilon_2)$, $\eta_2(\varepsilon_1, \varepsilon_2)$ (with nonzero Jacobian at the point $\varepsilon_1 = \varepsilon_2 = 0$) are carried into the phase curves of one of the following families:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \eta_1 + \eta_2 x_1 + x_1^2 \pm x_1x_2, \tag{2^α}$$

where α is equal to the sign in front of the monomial x_1x_2 (+ or −).

*Here transversality is in the space of k-jets of germs of vector fields of class $C^\infty(x)$ (see [2]). If one denotes by π_k the natural projection of the space of germs of vector fields into the space of k-jets, then one has the following lemma.

LEMMA. $\pi_k \mathfrak{A}$ is a smooth semialgebraic submanifold of the space of k-jets and has the homotopy type of the nonconnected sum of four circles.

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 9, No. 2, p. 63, April–June, 1975. Original article submitted November 11, 1974.

THEOREM 2. A family (2) which depends continuously on the parameters and has for parameter values ε_1^0, ε_2^0 a singular point (x_1^0, x_2^0) with two nonzero eigenvalues of the matrix of the linear part can be deformed by an arbitrarily small amount so that for parameter values close to ε_1^0, ε_2^0 in a sufficiently small fixed neighborhood of the point (x_1^0, x_2^0) either there will be no singular point with twofold zero eigenvalue of the matrix of the linear part or there arise a finite number, but then in a neighborhood of the corresponding values of the parameters and phase variables the deformed family will be nondegenerate.

The phase curves of the family (2^+) are mentioned in [1] (see p. 176). For the proof of Theorem 2, see "Trudy Seminara Imeni I. G. Petrovskogo," No. 2, Moscow, Izd-vo MGU.

The author thanks V. I. Arnol'd for posing the problem and his constant attention to the work and also Yu. S. Il'yashenko and A. F. Filippov for looking at the manuscript and making a series of remarks about improving the text of the proof.

LITERATURE CITED

1. V. I. Arnol'd, Usp. Matem. Nauk, 27, No. 5, 119-185 (1972).
2. R. Thom and G. Levine, Singularities of Differentiable Maps [Russian translation], Mir, Moscow (1968), pp. 10-79.