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ORDINAL PREFERENCE REPRESENTATIONS

ABSTRACT. Ordinal preferences have several advantages over the traditional cardinal expressions of preference. Three different representations of ordinal preferences useful in multi-participant modelling are presented, and their features compared. One approach is the *payoff* representation that is based on an ordinal normal form game. A second representation of ordinal preferences is the *preference vector*, based on the option form of the game. The option form consists of a list of players, with each player followed by the options under its control. The third representation of ordinal preferences is the *preference tree*. A preference tree is an implied binary tree that captures the information of preference vector in a more compact manner by making use of its lexicographic structure. The preference tree offers considerable compactness and computational efficiency over the other two approaches.

Keywords: Game theory, ordinal games, preferences, normal form, options, binary trees, computation.

INTRODUCTION

Much of the formal study of decision making relies on the idea that human preferences can be expressed in some numeric manner. This has been a very fruitful approach, and useful techniques and results have been attained. However, an alternative procedure is to emphasise the ordinal aspects of preference. The available alternatives for ordinal representations of preference are limited, and each has disadvantages. The main purpose of this paper is to present a new method of representing ordinal preferences, called a *preference tree*, that has substantial benefits in certain applications.

The research reported in this paper results from taking a non-traditional perspective on the analysis of multi-party relationships. Most game theory is normative in nature, striving to inform parties how they should behave in general. This is useful in broad contexts, and also gives rise to principles that can then be used to guide behaviour in specific situations. A different approach is to develop a descriptive model, involving a specific problem with the details of action and preference appropriate to it. Decision analysis is often

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descriptive rather than normative, as exemplified by the use of very specific information in a decision tree model. In game theory, very often parameters are used rather than specific values, emphasising the normative paradigm. There are good reasons why game theory is usually not specific and descriptive, which are discussed later in this paper. The use of ordinal preferences is one way of dealing with these reasons.

Part of this research was motivated by the development of a new version of the computer program *DecisionMaker: the Conflict Analysis Program* (Waterloo Engineering Software, 1992). The *DecisionMaker* program uses ordinal models of preference to provide advice to a decision maker involved in multi-party relationships. Many of the results in this paper are oriented towards needs generated by the program design.

PREFERENCE REPRESENTATIONS

Human preferences can be difficult to model accurately. Whenever a choice is made, each person evaluates preference over alternatives, even if it is simply choosing to do something over not doing it. It is a familiar process, but hard to capture in a formal manner.

One way of dealing with preference is to use a *countable proxy*. For example, dollars (francs, yen, etc.) are a countable proxy of preference because it is assumed that everyone values a higher number of dollars more than a lower number. Dollars can be added, subtracted, multiplied, divided, integrated and differentiated, making it possible to evaluate tradeoffs explicitly and fairly, to optimise some objective function, and to compare value among parties. Often operations research models optimise some commodity other than money, such as space, inventory, distance or time. Some qualities are not readily convertible to dollars, such as human lives or environmental damage, but can also be counted.

There are several problems with countable proxies, many of which are dealt with by the concept of utility. Utility is a numerical representation of preference among alternatives which is not tied into a single measure of value. If a person prefers alternative A over alternative B, then by definition alternative A has higher utility. Utility serves many useful purposes because it can also be added, subtracted, multiplied, divided, integrated and differentiated. Once posited, utility permits the same kinds of mathematical analyses as done for countable proxies, without concern for the problems associated with incompletely modelling the decision maker's preferences. Countable proxies and utility can collectively be called *cardinal representations of preference.*

Ordinal representations of preference do not express preferences on some sort of real-valued scale, but rather emphasise the order or position of something. For example, simply the information that alternative A is preferred to alternative B is retained without mapping this preference to a cardinal scale, or even assuming that such a mapping is possible. Of course a cardinal representation also captures ordinal information; the relative position on the scale can simply be ignored. Nonetheless an *explicitly* ordinal representation offers useful advantages in some contexts.

For example, it is relatively easy to obtain ordinal information from a person. Asked if they prefer coffee or tea (and given as much detailed information as to the brand, etc., as necessary), anyone can provide a preference. However, if they are asked to map their preference to some cardinal scale, they would find it very difficult. Such ordinal preferences are always meaningful because they make no assumptions beyond the information provided by the decision maker. Although often assumed for both practical and mathematical reasons, transitivity is not even required, so that someone could prefer coffee to tea, tea to coke and coke to coffee.

It is worth noting that, in decision analysis, there are few motives for a decision maker to hide his true preferences from an analyst. The decision maker truly wants to make the best decision, and would normally recognise that this can only be achieved by full disclosure of his interests. However, in a decision problem involving several parties, there is considerable motivation for the parties to mask their true interests. This is one strong reason it has been difficult for game theory models to be specific and descriptive – the information was simply unavailable. With ordinal preferences, however, the model can be descriptive without being specific. This is a considerable advantage.

THREE ORDINAL PREFERENCE REPRESENTATIONS

For modelling multiple participant decision making ordinally, several different concepts need to be represented. These are:

- the participants, known as *players* or *decision makers*;
- actions the participants can take, known as *strategies* or *options*;
- states that can result from actions being taken by each participant, known as *outcomes*;
- the rank of each outcome for each participant.

There are three ordinal preference representations that are useful for multi-party decision making. They are the *Payoff* representation, used in the normal form of the game (Van Neumann and Morgenstern, 1943; ordinal preferences for a normal form game were first used extensively in Rappoport *et al.*, 1976), the *Preference Vector* representation (Howard, 1971), and the *Preference Tree* representation (Fraser, 1989). In the Payoff Representation, rank is given as an integer, where a higher number means a higher rank. In a Preference Vector representation, higher ranked outcomes appear to the left of lower ranked ones in an ordered set. Preference rankings for a Preference Tree are indicated by the position of outcomes in a binary tree.

In order to discuss the three ordinal preference representations, the following definitions are useful.

DEFINITION 1 (*Decision Makers* [*players*]). The set of decision makers is given by

$$N = \{1, 2, \ldots, i, \ldots, n\}\ ; \quad n \geqslant 2$$

DEFINITION 2 (*Options*). Each decision maker i possesses a set of options given by

$$O_i = \{o_{1i}, o_{2i}, \ldots, o_{ji}, o_{m_i i}\}\ ; \quad m_i \geqslant 1\ \forall i \in N$$

DEFINITION 3 (*Strategy*). Any subset of options that can be taken from the set O_i by i is a strategy, denoted s_{ri}. Where $P(O_i)$ is the power set (set of all subsets) of O_i:

$$S_i = \{s_{1i}, s_{2i}, \ldots, s_{ri}, \ldots, s_{R_i i}\}$$
$$s_{ri} \in P(O_i)$$

where $R_i = 2^{m_i}$

Certain combinations of options will not be possible for underlying physical or logical reasons. For example, a decision maker may have the two options of 'going up' and 'going down'. Doing both at the same time is not physically feasible.

DEFINITION 4 (*Feasible Strategy*). A strategy is feasible if the option combination forming it is feasible. A feasible strategy for decision maker i is denoted s^*_{ri}, and the set of feasible strategies for i is S^*_i.

DEFINITION 5 (*Outcome*). An outcome q is formed by selecting one strategy for each decision maker:

$$q = \{s_{a1}, s_{b2}, \ldots, s_{ri}, \ldots, s_{tn}\}$$

where $s_{a1} \in S_1, s_{b2} \in S_2, \ldots, s_{ri} \in S_i, \ldots, s_{tn} \in S_n$.

The set of all outcomes Q is the Cartesian product of the strategy sets:

$$Q = \{S_1 \times S_2 \times \cdots \times S_i \times \cdots \times S_n\}$$

DEFINITION 6 (*Feasible Outcome*). An outcome q^* is feasible if it is formed of feasible strategies:

$$q^* = \{s^*_{a1}, s^*_{b2}, \ldots, s^*_{ri}, \ldots, s^*_{tn}\}$$

where $s^*_{a1} \in S^*_1, s^*_{b2} \in S^*_2, \ldots, s^*_{ri} \in S^*_i, \ldots, s^*_{tn} \in S^*_n$.

The set of all feasible outcomes Q^* is the Cartesian product of the feasible strategy sets:

$$Q^* = \{S^*_1 \times S^*_2 \times \cdots \times S^*_i \times \cdots \times S^*_n\}$$

Payoff Representation

Mexico, Canada and the US have recently negotiated the North American Free Trade Agreement (NAFTA). There were many issues under discussion, but two important ones concerned the liberalization in Mexico of the energy and financial sectors to foreign investment. Figure 1 is a 'normal' form illustration of a simple model of the NAFTA negotiations which uses a payoff representation of preferences. Mexico can open either its energy sector or financial sector, or both, to foreign investment in this model. The US can invest in Mexico, or it can use political pressure which would be particularly effective in the financial sector. This model is not intended to illustrate accurately the positions and interests of the negotiating teams, but rather is primarily a pedagogical example. For a comprehensive model of the NAFTA negotiations, see the paper by Fraser and Garcia (1993).

The players in Figure 1, Mexico and the US, each control a dimension; Mexico controls the rows, the US the columns. Canada, if it were in the model, would control the planes, etc. The available strategies, labelled as to their meaning, are the rows, columns, etc. The outcomes are the cells of the resulting matrix. Each element of the matrix is a set of integer rankings, ordered by player, so that the first number is the rank of the outcome for the row player, the second for the column player, etc., where a higher number means a more

		US			
		Invest and Political Pressure	Invest and no Political Pressure	Do not Invest and Political Pressure	Do Not Invest and no Political Pressure
Mexico	Open Energy and Financial	9,15	13,16	5,13	1,14
	Open Energy only	10,12	14,11	6,10	2,9
	Open Financial only	11,5	15,6	7,7	3,8
	Close Energy and Financial	12,2	16,1	8,4	4,2

Fig. 1. Payoff representation of NAFTA negotiations.

preferred ranking. Thus in the NAFTA model of Figure 1, the upper right outcome is the situation where Mexico opens both the energy and financial sectors to investment while the US invests and applies political pressure. It is the eighth best outcome for Mexico and the second best for the US.

Note that, in the payoff representation, the payoffs are not cardinal, even though they have the appearance of being so. An outcome with a payoff of 4 is not twice as preferred as an outcome with a payoff of 2. Similarly, the difference in preference between outcomes of payoffs 3 and 4 has no relationship to the difference in preference between outcomes of payoff 2 and 3. The payoffs are simply used as numbers to note the ordinal relationship among the outcomes.

Preference Vector Representation

A preference vector representation of the NAFTA negotiations is illustrated in Table I. Each player is listed in a column. Under each player in the list are the options available to the player. As can be seen in Definitions 2 and 3, an option is different from a strategy in that options are not necessarily mutually exclusive. This results fewer options being listed over the corresponding list of strategies. Thus

TABLE I

Preference vector representation of prisoners' dilemma.

Preferences for Mexico																
Mexico																
(1) Open Energy	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
(2) Open Financial	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
US																
(3) Invest	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
(4) Pol. Pressure	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
Preferences for US																
Mexico																
(1) Open Energy	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
(2) Open Financial	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
US																
(3) Invest	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
(4) Pol. Pressure	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0

there are only two options for each player, 'Open Energy' and 'Open Banking' for Mexico, and 'Invest' and 'Political Pressure' for the US.

Outcomes are indicated in the preference vector representation by a column of 1s and 0s, where a 1 indicates that the corresponding option is taken and a 0 indicates that it is not. Outcomes are ordered from most preferred on the left to least preferred on the right for each player.

For example, the most preferred outcome for Mexico is where it opens neither sector to investment (a 0 opposite these two options) while the US invests (a 1 opposite this option) and does not apply political pressure (a 0 opposite this option). This corresponds to the fourth row, second column of Figure 1, and also happens to be the worst outcome for the US. Note that all outcomes are feasible, i.e., $Q = Q^*$. Thus, in a preference vector an outcome is represented as a column of 1s and 0s, rather than as a position in a matrix. The rank is given by the position in a vector, rather than an integer number.

Preference Tree Representation

In order to explain the preference tree representation, several more definitions are necessary.

DEFINITION 7 (*Binary tree*; *Subtree*; *Descendent*; *Ancestor*; *Leaf*). Binary tree T is an ordered triple (L, t, R). L and R are binary trees of $l > 0$ and $r > 0$ nodes, respectively, and t is a node called the root of T. L is the left subtree of T, and R is the right subtree of T. S is a subtree of T $[S \subseteq \text{sub}(T)]$ iff S is T or S is $L(T)$ or S is $R(T)$ or $S \subseteq \text{sub}(L(T))$ or $S \subseteq \text{sub}(R(T))$. If $S \subseteq \text{sub}(T)$, then $t(S)$ is a descendent of $t(T)$ and $t(T)$ is an ancestor of $t(S)$. If $l(S) = 0$ or $r(S) = 0$, then $L(S)$ or $R(S)$, respectively, is a leaf of T.

DEFINITION 8 (*Preference Statement*). A preference statement P is

- an option o_{ji}, or
- o_{ji} AND P', or
- o_{ji} OR P', or
- NOT P'

where P' is a preference statement. The symbols &, / and − are used for AND, OR and NOT respectively.

A preference statement thus expresses a logical relationship among options that can be used to bifurcate a set of outcomes. For example, given the set of outcomes

$$Q^* = \begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{matrix}$$

and the preference statement $P = (3\ \&\ -2)$, the set splits into

$$P(Q^*) = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

and

$$\text{NOT}\ P(Q^*) = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

which can be written

$$\begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix} \left| \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{matrix} \right.$$

the left group including outcomes that match P and the right group those that do not.

DEFINITION 9 (*Preference tree*). A preference tree ${}_pT$ is a binary tree with each node $t_k = t(S_k)\ \forall S_k \subseteq \text{sub}({}_pT)$ labelled according to some preference statement P_k.

By associating the root of ${}_pT$ with the set Q^*, each node in ${}_pT$ partially orders a subset of Q^*, and in the limit completely orders Q^*. Specifically, if each node t_k is associated with $Q_k^* \subseteq Q^*$, then for the corresponding subtree S_i, $L(S_k)$ is associated with $P_k(Q_k^*)$ and $R(S_k)$ is associated with $-P_k(Q_k^*)$.

DEFINITION 10 (*Simple preference tree*). A simple preference tree is one in which all preference statements consist of a single option, or the negation of a single option.

A preference tree representation of Mexico's preferences in the NAFTA model is illustrated in Figure 2, along with the preference vector implied from the order of leaves of the tree. Note that it is a simple preference tree, because every preference statement consists of a single option. The option numbers at each node are only meaningful when associated with a list of decision makers and options. This list is the same as with the preference vector approach, i.e.:

Mexico
(1) Open Energy
(2) Open Financial
US
(3) Invest
(4) Political Pressure

This preference tree fully orders the outcomes, because there is only one outcome associated with each leaf. The preference tree for the US is in Figure 3. It is also a simple preference tree that fully orders the outcomes.

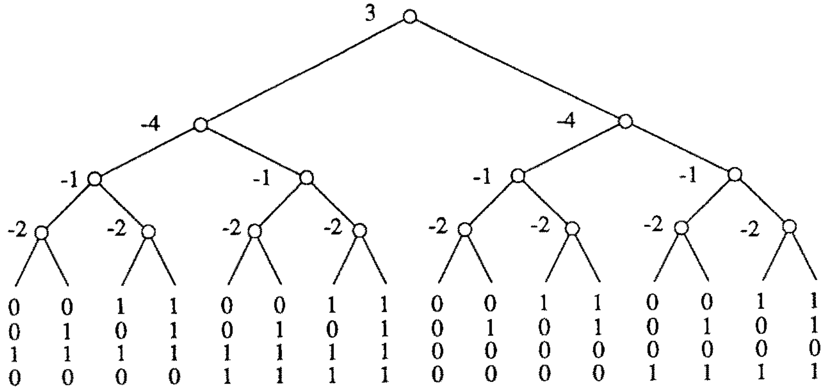

Fig. 2. Preference tree for Mexico in NAFTA – purely lexicographic preference tree.

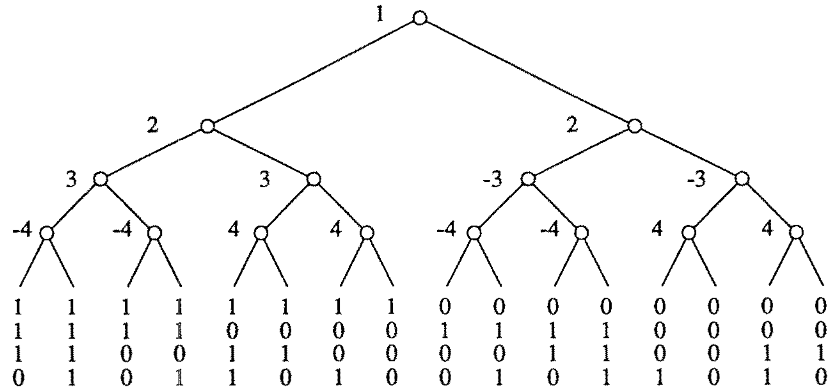

Fig. 3. Preference tree for the US in NAFTA – conditional lexicographic preference tree.

A preference tree is not particularly compact in full form, but the natural patterns in human preferences permit a much more efficient presentation. People tend to have preferences that are *lexicographic* on options or issues. Lexicographic means ordered like words in a dictionary, where the ‘a’ words precede the ‘b’ words, and then within the group of ‘a’ words the ‘aa’ words precede the ‘ab’ words, etc. The preferences of Mexico are lexicographic because all outcomes where the US invests are preferred to those where it does not. Of secondary importance is US political pressure, which Mexico does not want applied, independently of whether the US invests. Similarly, Mexico does not want to open its energy sector to investment, and least in important it does not want to open its financial sector. This lexicographic principle can be expressed more formally:

DEFINITION 11 (*Purely lexicographic preference tree*). A purely lexicographic preference tree T is a preference tree in which $L(S) = R(S)$ for all $S \subseteq \text{sub}(T)$.

As can be seen, Mexico’s preference tree in Figure 2 is purely lexicographic, while the US preference tree in Figure 3 is not.

In the case of a purely lexicographic preference tree a short notation can be used to represent the entire tree. Since all the subtree pairs are identical, the entire tree can be deduced from only the preference statements associated with the rightmost nodes at each level. Such a list of preference statements is called an *abbreviated preference tree.* From Figure 1, Mexico's preference tree in abbreviated form is:

3
−4
−1
−2

This abbreviated form is also convenient for use by a computer because all nodes of the binary tree need not be retained. Note that the number of outcomes in a preference vector is 2^m, where $m = \Sigma\, m_i$ is the total number of options, while the number of elements of an abbreviated preference tree is only m. This makes an abbreviated preference tree very compact.

Intuitively, each preference tree contains two kinds of information, *importance* and *desirability*. If an option is more important to a player, it appears higher in the preference tree. The fact that investment by the US is more important for Mexico than political pressure is indicated by option 3 being higher in the preference tree for Mexico than option 4. The minus sign in front of option 4 indicates that Mexico does not want the US to apply political pressure. It is worth mentioning that preference tree information can be extracted from a decision maker directly by asking which outcomes are important, and whether they are desired or not.

Clearly not all preferences follow purely lexicographic structure. There are, however, simple variations from a purely lexicographic structure. One of these variations is preference trees with conditional preferences statements.

DEFINITION 12 (*Conditional Preference Statement*). A conditional preference statement P^c is
- P' IF P'', or
- P' IFF P''

where P' and P'' are preference statements.

A conditional preference statement is like a preference statement except that the existence or form of the left hand side (LHS) depends on the truth of the right hand side (RHS). For example, consider node t_k associated with $Q_k^* \subseteq Q^*$ and with the conditional preference statement P' IFF P''. $P''(Q_k^*) = Q_k^* \Rightarrow P''$ is true, and the conditional preference statement should be interpreted as P', while $P''(Q_k^*) = \emptyset \Rightarrow -P''$ is true, and the conditional preference statement should be interpreted as $-P'$. It is assumed that the RHS condition has been specified in some ancestor of t_k, and so the case where $P''(Q_k^*)$ is neither Q_k^* nor $\emptyset$ does not arise. In the case of P' IF P'', if P'' were false ($-P''$ were true), no preference statement is applied to Q_k^*.

Conditional preference statements are not particularly useful in a fully written preference tree because for each instance the conditional statement could be replaced by the LHS or the negation of the LHS depending on status of the RHS. However, they are valuable in permitting preference trees that include conditional preferences to be expressed as an abbreviated preference tree.

For example, in the US preference tree of Figure 3, it can be seen that the desirability of investing is dependent on whether Mexico opens the energy sector for investment. If the energy sector is open, the US wants to invest, while if the energy sector is closed, the US does not want to invest. Similarly, if Mexico does not open the financial sector, the US prefers to apply political pressure. The preference tree of Figure 3 could be written as illustrated in Figure 4, and then abbreviated as:

1
2
3 IFF 1
4 IFF -2

It may be the case that preferences are neither purely nor conditionally lexicographic. It is still possible for useful efficiency to be attained by recognising any lexicographic patterns in the tree.

DEFINITION 13 (*Inconsistently lexicographic preference tree*). A preference tree is inconsistently lexicographic iff $\exists S_k : L(S_k) = R(S_k)$.

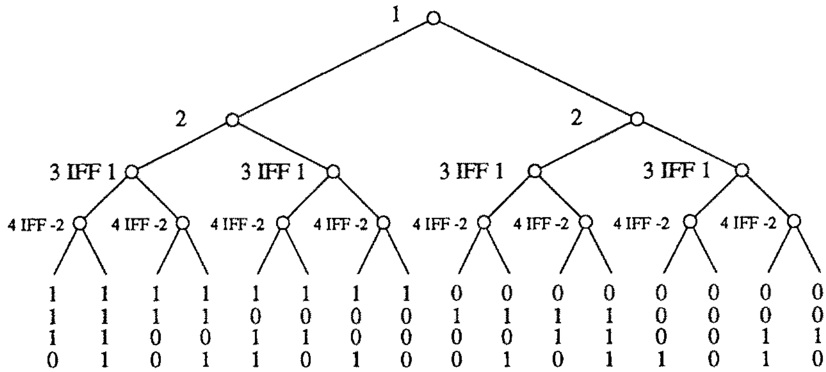

Fig. 4. Preference tree for the US in NAFTA with explicit conditional preference statements.

In an inconsistently lexicographic preference tree at least one subtree is purely lexicographic. This is useful to know because then all results for purely lexicographic preference trees can then be applied to the purely lexicographic subtree or subtrees, including the idea of abbreviated preference trees.

For example, it could be that for Mexico, if the US does not invest, the most important option is to open the financial sector (to help to obtain funds from other countries). In other words, the order of importance of the remaining options depends on whether option 3 is taken. This is illustrated in Figure 5. Note that there is still considerable lexicographic structure here. The abbreviated form of this preference tree is given by writing the rightmost branches of the distinct subtrees. For the node at which distinct subtrees split, the preference statement for that node heads the left hand subtree, while its complement heads the right hand subtree. Thus the preference tree of Figure 5 is abbreviated as:

3 *	−3
−4	−1
−1	−4
−2	−2

Finally, any ordering can be accommodated through a non-simple

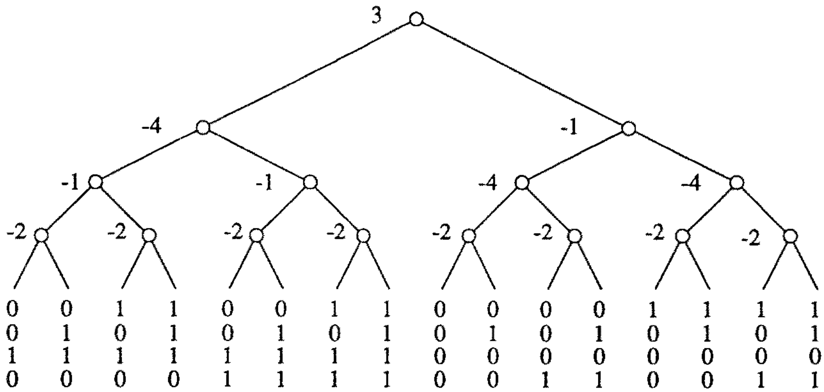

Fig. 5. Preference tree for inconsistently lexicographic preferences.

preference tree. Recall from Definition 8 that a preference statement can be any logical combination of options. Consider the preference vector in Figure 6. The vector is apparently not lexicographic on options. However, appropriate use of a compound preference statement it can be expressed as the preference tree illustrated. In this case the abbreviated notation would be:

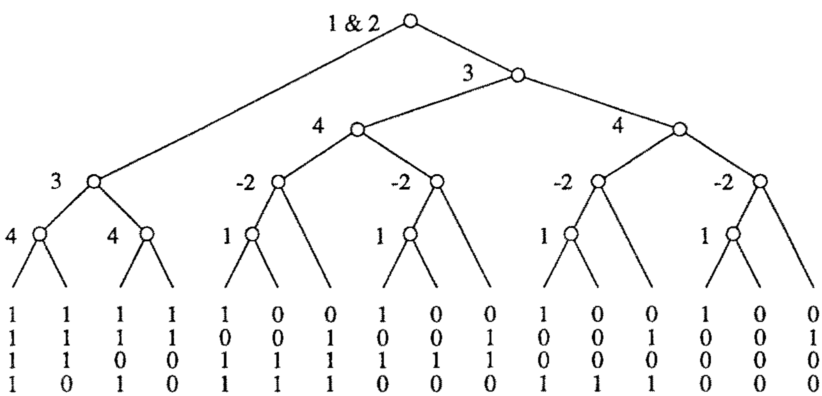

Fig. 6. Non-simple preference tree.

1 & 2
3
4
−2
1

Note that the options 1 and 2 have to appear twice each since the statement 1 & 2 does not give sufficient information to order the outcomes in the branch headed by the complement of 1 & 2, −1/−2.

Since a compound preference statement can, in the limit, isolate each individual outcome, it is clear that any ordering of outcomes can be representing by a preference tree. It should be reiterated, however, that preferences do tend to be strongly lexicographic.

SOLUTION CONCEPTS

There is only a limited range of ways to analyze game models where preferences are expressed ordinally. One technique is to identify outcomes that exhibit properties that are assumed to have some relevance to the behaviour of people, or groups of people, in the real world. There are several different approaches to distinguishing such outcomes, which can be collectively labelled solution concepts. For example, a classic solution concept is Nash Rationality (Nash, 1951).

An outcome is Rational (R) for a player if it is the best outcome (or one of the best, if there are more than one best) that he can attain unilaterally, given that the other players' strategies are fixed. For the payoff representation of preferences, this is the same as saying that an outcome is R for the row player if it maximizes his payoff in the row, or an outcome is R for the column player if it maximizes his payoff in the column. An outcome that is simultaneously R for every player is called group R.

Traditionally in game theory, players choose strategies and an outcome is formed. A variation on this approach is to consider the cells of the normal form matrix to represent different states of the world that can be changed by the players changing their strategies. This approach gives rise to many other sensible solution concepts. One of these is Fraser–Hipel Sequential (FHQ) stability (Fraser and Hipel, 1984).

An outcome is FHQ for a player if it is *R*, or if for any unilateral improvement available to a player, there is a credible sanction available by another player that will result in an outcome less preferred by the first player than the original outcome. A sanction is credible if it is itself a unilateral improvement by the sanctioning player. A more strict definition of FHQ is available in Fraser and Hipel (1984).

Both the *R* and FHQ solution concepts have been employed in the *DecisionMaker* computer program. *DecisionMaker* provides a modelling and analysis tool for management problems involving multiple parties.

A FAST FHQ ALGORITHM

One of the advantages of the preference tree representation is that it permits the use of efficient algorithms for determining outcomes that exhibit a solution concept. In particular, the calculations for FHQ can be done very much more quickly using a preference tree rather than a preference vector or payoff representation.

The reason for this is that the preference tree representation captures in an abbreviated fashion all those similar characteristics of a preference structure. For example, consider the preference tree for Mexico (Figure 2) given in abbreviated form as:

$$\begin{array}{r} 3 \\ -4 \\ -1 \\ -2 \end{array}$$

Mexico always wants to not take either option 1 or 2, independently of the choices of the US (since it is purely lexicographic). The four individual *R* outcomes can be immediately deduced as:

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}$$

In other words, a decision maker's individually *R* outcomes can be read simply as the outcomes in which preference statements involving the players options are true. Group *R* outcomes can be immediately identified as all outcomes that simultaneously satisfy the preference statements involving each players option in the player's own preference tree. For example, the abbreviated preference tree for the US in the NAFTA model is:

1
2
3 IFF 1
4 IFF −2

A group *R* outcome must then satisfy:

−1
−2

(from Mexico's preference tree) and

3 IFF 1
4 IFF −2

(From the US preference tree) giving the single group *R* outcome

0
0
0
0

The algorithm for calculating FHQ stability is too complicated to detail in this paper, but is available elsewhere (Fraser, 1989), and in a future publication. Broadly speaking, it determines for each option 'stability conditions' by examining the circumstances under which the player owning the option would not unilaterally change it. Options in a lower lexicographic rank cannot affect a player's decision. Options in a higher rank may affect it when the sign of the option in the player's tree is different from the sign in the owner's tree. The set of mutually consistent stability conditions form the group FHQ outcomes.

For example, consider the situation where Mexico was considering opening the energy sector for investment. Mexico has a unilateral improvement from all outcomes in which the energy sector is open as indicated by the −1 in its abbreviated preference tree. By examination of the US abbreviated preference tree it can be seen that as long as Mexico maintains an open energy sector the US will invest. However, should Mexico close the energy sector to investment, the US would prefer to not invest, as indicated by the preference statement '3 IFF 1'. Since US investment, option 3, is more important to Mexico than whether the energy sector is open or closed, option 1, as indicated by their relative position in Mexico's preference tree, and the signs of the two outcomes are only the same if option 1 is taken, this would constrain Mexico's ability to close investments in the energy sector. This would then create a 'stability condition' with respect to option 1.

The *R* and FHQ solution concepts have been partly implemented in the *DecisionMaker* computer program. The implemented algorithm has been proven for purely lexicographic preferences (Fraser, 1989). The extension to conditional lexicographic preferences has demonstrated and implemented.

THE ARMENIAN–AZERBAIJANI CONFLICT OVER NAGORNO-KARABAKH

As a comparison among the three preference representations, consider the model of the Armenian–Azerbaijani conflict over Nagorno-Karabakh, illustrated in Table II (Fraser *et al.*, 1990). The details of the case are not important here but rather the structure and size.

If this model were in the normal form using a payoff representation of ordinal preferences, it would require a six-dimensional matrix of size $8 \times 2 \times 2 \times 2 \times 4 \times 4$. Moreover, each cell of the matrix would contain a 6-tuple of payoffs, one for each player. Clearly this would not be useful model.

If the preference vector representation were used, 6 vectors of 1024 entries of 10 binary digits each would be required. Although this could theoretically be presented in two dimensions, it would be difficult to record. Certainly a comprehensive analysis would be impossible.

A preference tree approach requires the indication of 6 trees of 10

TABLE II

The Armenian–Azerbaijani conflict over Nagorno-Karabakh.

Preferences Trees (owned options in bold):

*DM*1	*DM*2	*DM*3	*DM*4	*DM*5	*DM*6
2	3	1	1	−1	−1
−5	**4**	4	5	2	8
7	−10	−8	4	**8**	−3
−8	1	**5**	−8	**7**	**9**
−1	−8	3	3	−10	7
−10	−9	−10	−7	9	2
−9	7	−9	−2	−5	**10**
−4	−5	−2	−10	−4	−5
6	6	7	−9	3	−4
3	2	6	**6**	6	6

*DM*1: Moscow
*DM*2: Armenian Government
*DM*3: Armenian People
*DM*4: Armenian Dissidents
*DM*5: Azerbaijani
*DM*6: Azerbaijani People

elements each. This is very easily presented, as seen in Table II. Moreover, it is very easy to analyze. By observation, it can be seen that there is a single group *R* outcome (since all trees are purely lexicographic) which in preference vector notation is

$$\begin{matrix}0\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{matrix}$$

Comparable efficiency is demonstrated for calculating FHQ stability.

Using the *DecisionMaker* program with a preference vector-based analysis, the time to analyze the case fully was 3 minutes and 43 seconds. Using a preference tree-based approach, this was reduced to less than 0.5 seconds. Both comparisons were done on an IBM PS/2 with 16 MHz clock. A pencil and paper analysis of the preference tree form of this case took 15 minutes, and would be unmeasurably long with either a payoff or preference vector representation.

OTHER APPLICATION AREAS OF THE PREFERENCE TREE REPRESENTATION

The preference tree representation of human preferences offers potential in areas other than the modeling and analysis of game models. Two applications have been explored.

Different decision makers will, in general, have different preferences. However, when decision makers share goals, their preferences will also have similarities. Unfortunately, if outcomes are ranked, it is difficult to perceive or measure these similarities. The preference tree approach represents interests at a fairly high level, since it is in terms of options, not outcomes. This makes it easier to both recognise common interests, and to measure them in a formal way. Fraser and Hipel (1989) and Meister *et al.* (1991) have developed formal 'metrics' that ascribe a number (between 0 and 1) to a pair of preference trees which is purported to be a measure of the tree's similarity. Such a metric can be used, for example, to assess which decision makers of a set are the most similar in preference, and thus by implication the most likely to join in coalition.

Another application of preference trees is in the selection of a best alternative from several, where there are many criteria to consider. The decision maker can recognise a lexicographic ranking of criteria, thus forming a preference tree. An algorithm has been developed (Meister and Fraser, 1991) which can then compare the alternatives pairwise, and determine the 'best', possibly with some further information from the decision maker. The advantage of the preference tree approach is that it is ordinal, so that no weightings of alternatives need to be made. This can be contrasted with approaches such as AHP

(Saaty, 1980) or Electre (Roy, 1985) which do require detailed weightings. In addition to the data acquisition cost of using weightings, theoretical problems arise such as rank reversal (Dyer, 1990).

CONCLUSIONS

There are considerable advantages to the use of explicitly ordinal preferences over the conventional cardinal approach. The data acquisition cost is considerably reduced, the information used is more likely to be meaningful, paradoxes and theoretical controversies are avoided, and concentration can be placed on the fundamental structure of the problem rather than on perhaps meaningless numerical observations.

Explicitly ordinal preferences are particularly appropriate for multiple participant decision problems, since it is quite unlikely that detailed preference information is known for all involved parties. Table III summarizes the comparison among the three ordinal preference

TABLE III

Comparison of ordinal preference representations.

Feature	Representation		
	Payoff	Preference Vector	Preference Tree
Ability to handle large models	– poor, requires many dimensions	– fair, but gets unwieldy	– excellent
Speed of computation	– poor	– very limited; combinatorically challenging	– excellent; permits more powerful algorithms
Graphical presentation	– excellent for small models; very easy to interpret – awkward for large number of strategies – impossible for more than 2 players	– fair for small models – good for medium models – very difficult for large models – individual outcomes are apparent	– very compact – fair for small models – excellent for medium to large models – individual outcomes are not apparent
Familiarity	– similar to cardinal representation – in wide usage	– not commonly used but easy to pick up	– new and takes training
Ease of eliciting information	– can be awkward	– awkward but can use lexicographic information effectively	– easy due to lexicographic structure of preferences
Ease of updating	– very difficult	– difficult	– very easy

representations discussed in this paper. The appropriate choice of representation depends on the size of the model and its use and derivation. Certainly the preference tree representation is best for any larger model. The preference tree approach also offers value in other application areas, such as coalitions and multiple criteria decision making.

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