DYNAMIC SYSTEMS WITH HIERARCHICAL CONTROL STRUCTURE

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We shall consider two-level dynamic systems with discrete time in which the upper level (the center) can use various modes of control. One such mode that to a large extent uses information about the actions of a lower-level element and the strategy of punishment has been studied in [1]. But the impossibility of simultaneously punishing all these elements which must be overcome by the center with the aid of a bluff [2] constitutes a shortcoming when such a control is used in systems with arbitrarily many elements at the lower level.

In this paper we shall consider some other modes of control by the center that do not explicitly use information about the actions of the lower-level elements (subsystems). For the static case, such a control in two-level systems has been studied, for example, in [3].

1. Hierarchical Systems with One-Step Planning in the Subsystems

Let us consider the simplest case, when the lower-level elements are planning for only one step ahead in time. The planning period T is represented at the center by a fixed integer.

The overall dynamic equation of the system is

$$x_{t+1}=f_t(x_t, u_t, v_t^1, \ldots, v_t^n),\quad t=\overline{0, T-1}, \tag{1.1}$$

where x_0 is fixed, x_t is a phase state vector of dimension k, u_t is the control performed by the center (a vector of dimension m), v_t^i is the control of the i-th element of the lower level (subsystem) (a vector of dimension r_i), $i=\overline{1, n}$.

The (integral-terminal) performance criterion of the center is

$$I(\bar{x}, \bar{u}, \bar{v})=\sum_{t=0}^{T-1} g_t(x_t, u_t, v_t^1, \ldots, v_t^n)+g_T(x_T), \tag{1.2}$$

$$\bar{x}=(x_0, \ldots, x_T),\quad \bar{u}=(u_0, \ldots, u_{T-1}),$$

$$\bar{v}=(\bar{v}_0, \ldots, \bar{v}_{T-1})=(v_0^1, \ldots, v_0^n, \ldots, v_{T-1}^1, \ldots, v_{T-1}^n),$$

whereas the performance criteria of the subsystems at the instant t are $h_t^i(x_t, u_t, v_t^i)$, $t=\overline{0, T-1}$, $i=\overline{1, n}$.

The control space of the center at the instant t can depend on the running state x_t of the system, and it is specified by the mapping $U_t(x_t)$; the control spaces of the subsystems at the instant t depend on the system state x_t and the center control u_t; they are specified by the mappings $V_t^i(x_t, u_t)$. The center selects its control u_t in the knowledge of the state x_t, but without knowing the choice of the subsystems; the center communicates its control to the subsystems, and after that the subsystems select their controls v_t^i, with x_t and u_t being known to them. Moreover, the center must ensure the fulfillment of the phase constraints

$$x_t \in X_t,\quad t=\overline{1, T}. \tag{1.3}$$

The fulfillment of the conditions (1.3) is strongly dependent on the controls selected by the subsystems, but the subsystems are not obliged to observe these conditions. It is only assumed that the behavior of the subsystems tends to optimize (maximize) the functions h_t^i. The center is an optimizing-coordinating element; i.e., it tends to optimize (maximize) the functional (1.2) under the joint constraints (1.3) and (1.1). Its possibilities to control the subsystems are based on the dependence of the control spaces and the performance criteria of the latter on the control realized by the center. If we adopt the hypotheses of behavior and possession of knowledge stated in [3], then at the instant t the subsystem controls v_t^i and the new system state x_{t+1} will be undetermined uncontrollable factors for the center with the following domains of feasible values:

$$\Omega_t^i(x_t, u_t)=\{v_t^i \mid v_t^i \in V_t^i(x_t, u_t),\ h_t^i(x_t, u_t, v_t^i) \geqslant h_t^i(x_t, u_t, z)\quad \forall z \in V_t^i(x_t, u_t)\},\quad i=\overline{1, n},$$

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$$A_t(x_t, u_t) = \{x_{t+1} \mid x_{t+1} = f_t(x_t, u_t, v_t^1, \ldots, v_t^n)\ \forall v_t^i \in \Omega_t^i(x_t, u_t), \quad i = \overline{1, n}\}.$$

We shall assume that the sets

$$U_t^0(x_t) = \{u_t \mid u_t \in U_t(x_t), \quad A_t(x_t, u_t) \subseteq X_{t+1}\} \neq \varnothing \quad \forall x_t \in X_t;$$

i.e., for any allowed state of the system there exists a center control that carries the system into an allowed state for any optimal controls of the subsystems. Let us introduce also the sets

$$B_t(x_t, u_t) = \{(x_{t+1}, v_t^1, \ldots, v_t^n) \mid v_t^i \in \Omega_t^i(x_t, u_t),$$
$$i = \overline{1, n}, \ x_{t+1} = f_t(x_t, u_t, v_t^1, \ldots, v_t^n)\},$$
$$\Omega_t(x_t, u_t) = \prod_{i=1}^{n} \Omega_t^i(x_t, u_t), \quad t = \overline{0, T-1}.$$

Then the problem of finding an optimal guaranteed center control (of program type) and the maximum guaranteed result will take the form

$$I_0 = [\sup_{u_t \in U_t^0(x_t)} \ \inf_{(x_{t+1}, \bar{v}_t) \in B_t(x_t, u_t)}]_{t=0}^{T-1} I(\bar{x}, \bar{u}, \bar{v}). \tag{1.4}$$

Remark 1. Existence of an optimal control requires only that the suprema in (1.4) could be reached. A sufficient condition of reachability of the infima and suprema in (1.4) is continuity of the mappings $U_t^0(x_t)$ and $B_t(x_t, u_t)$ in Hausdorff's metric [4].

Remark 2. The uncontrollable factors must not necessarily take their worst values; therefore the actual result for the center can be also larger than I_0. For utilizing the favorable course of a process, the center must use synthesis-type controls, but the maximum guaranteed result will also in this case be equal to I_0, and the problems will be similar.

Remark 3. Problems of the form (1.4) have been in fact considered in [5], but the undetermined uncontrollable factors were not related in that paper to the actions of the subsystems. In [5] we have also proposed a method of solution of this problem in the form of a combined method of penalty functions and dynamic programming; therefore we shall not consider it here.

2. Hierarchical Systems with a Single Control Action of the Center during the Entire Planning Period

Let us consider the case that the planning schedules of the subsystems coincide with the planning schedule of the center which communicates its control in advance for the entire planning period. Hence if the dynamic equations and the functionals of each subsystem do not depend on the controls of the other subsystems, then we have for them the ordinary problems of optimal control. If the simplifying assumption of independence is dropped, then the behavior of the subsystems cannot be described as simply tending to maximize their functional; we must introduce other principles of behavior, for example, to try and reach an equilibrium situation. For convenience we shall confine ourselves below to the case of one subsystem ($n = 1$). Let the system dynamics be described by the equation

$$x_{t+1} = f_t(x_t, u_t, v_t), \ x_0 \quad \text{is fixed,} \quad = \overline{0, T-1}, \tag{2.1}$$

whereas the performance criteria (the functionals) of the center and of the subsystem are

$$I(\bar{x}, \bar{u}, \bar{v}) = \sum_{t=0}^{T-1} g_t(x_t, u_t, v_t) + g_T(x_T), \tag{2.2}$$

$$J(\bar{x}, \bar{u}, \bar{v}) = \sum_{t=0}^{T-1} h_t(x_t, u_t, v_t) + h_T(x_T); \tag{2.3}$$

the center must ensure the fulfillment of the phase constraints (1.3), and the control spaces are $U_t(x_t)$, $V_t(x_t, u_t)$, $t = \overline{0, T-1}$. The subsystem dynamics can be assigned also by a separate equation, but this makes no difference at all, since we can always go over to a single equation by appropriately increasing the dimension of the phase space.

For reducing the problem of determination of optimal control by the center to several parametric problems of smaller dimension, we shall use a modified maximum principle for a subsystem. The modified

Hamiltonian function for a subsystem for fixed u_t has the form

$$\tilde{H}_t(\tilde{x}_t, \tilde{x}_{t+1}, u_t, v_t, \tilde{\psi}_{t+1}, C) = [\tilde{\psi}_{t+1}, f_t(\tilde{x}_t, u_t, v_t)] + h_t(\tilde{x}_t, u_t, v_t) - C\,\|\tilde{x}_{t+1} - f_t(\tilde{x}_t, u_t, v_t)\|^2, \quad t = \overline{0, T-1}, \tag{2.4}$$

where $\tilde{x}_t$ and $\tilde{\psi}_t$ satisfy the original and the conjugate equations. We shall assume that requirements are met such that the necessary and sufficient conditions of optimality of control consist in a strict global maximum of the modified Hamiltonian function being reached on it [5]. These requirements are weaker than the corresponding requirements for the ordinary Hamiltonian function, but if the latter are satisfied, then it is necessary to use the ordinary Hamiltonian function; i.e., set the constant C = 0 in (2.4).

Although we are now considering the case in which the center selects its control in advance for the entire period T, it is more convenient to determine the optimal control of the center by a recursive procedure.

Suppose that the system has reached the state x_{T-1}; then the optimal center control $\tilde{u}_{T-1}(x_{T-1})$ will be a solution of the problem

$$\Phi_T(x_{T-1}) = \max_{u_{T-1} \in U_{T-1}(x_{T-1})} [g_T(\tilde{x}_T) + g_{T-1}(x_{T-1}, u_{T-1}, \tilde{v}_{T-1})]$$

under the constraints

$$\tilde{x}_T = f_{T-1}(x_{T-1}, u_{T-1}, \tilde{v}_{T-1}) \in X_T,$$

$$\tilde{H}_{T-1}(x_{T-1}, \tilde{x}_T, u_{T-1}, \tilde{v}_{T-1}, \tilde{\psi}_T, C) = \max_{v_{T-1} \in V_{T-1}(x_{T-1}, u_{T-1})} \tilde{H}(x_{T-1}, \tilde{x}_T, u_{T-1}, v_{T-1}, \tilde{\psi}_T, C),$$

$$\tilde{\psi}_T = \frac{\partial h_T(x_T)}{\partial x_T}.$$

This is a static problem of hierarchical control (true, though, a parametric problem, but with a unimodal subsystem criterion $\tilde{H}_{T-1}$); it has been considered, for example, in [3], and therefore we shall assume that we can find its solution:

$$\Phi_T(x_{T-1}), \tilde{u}_{T-1}(x_{T-1}), \tilde{v}_{T-1}(x_{T-1}), \tilde{\psi}_T(x_{T-1}).$$

Then we have to consider an inverse sequence of problems ($t = \overline{T-1, 1}$)

$$\Phi_t(x_{t-1}) = \max_{u_{t-1} \in U_{t-1}(x_{t-1})} [g_{t-1}(x_{t-1}, u_{t-1}, \tilde{v}_{t-1}) + \Phi_{t+1}(\tilde{x}_t)]$$

under the constraints

$$\tilde{x}_t = f_{t-1}(x_{t-1}, u_{t-1}, \tilde{v}_{t-1}) \in X_t,$$

$$\tilde{H}_{t-1}(x_{t-1}, \tilde{x}_t, u_{t-1}, \tilde{v}_{t-1}, \tilde{\psi}_t, C) = \max_{v_{t-1} \in V_{t-1}(x_{t-1}, u_{t-1})} \tilde{H}_{t-1}(x_{t-1}, \tilde{x}_t, u_{t-1}, v_{t-1}, \tilde{\psi}_t, C),$$

$$\tilde{\psi}_t = \tilde{\psi}_{t+1}(\tilde{x}_t)\,\frac{\partial f_t(\tilde{x}_t, \tilde{u}_t(\tilde{x}_t), \tilde{v}_t(\tilde{x}_t))}{\partial x_t} + \frac{\partial h_t(\tilde{x}_t, \tilde{u}_t(\tilde{x}_t), \tilde{v}_t(\tilde{x}_t))}{\partial x_t}.$$

The maximum value of the functional of the center is equal to $\Phi_1(x_0)$, the optimal control at the initial instant is $\tilde{u}_0(x_0)$, and for finding the other optimal controls of the center it is necessary to traverse the chain in the forward direction ($t = \overline{1, T-1}$).

3. Hierarchical Systems with Successive Control Actions of the Center

Now let us consider the case in which the center communicates its control to a subsystem (n = 1) only at a given step. In this case the subsystem does not know the future controls of the center, and for it this problem cannot be reduced to an ordinary optimal control problem. Once again let us move from the end, by assuming that the subsystem also knows the criteria and the principles of behavior of the center (just as the center has knowledge about the subsystem). If the system reaches the state x_{T-1}, then the criteria of the center and of the subsystem will be

$$F_T(x_{T-1}, u_{T-1}, v_{T-1}) = g_{T-1}(x_{T-1}, u_{T-1}, v_{T-1}) + g_T(f_{T-1}(x_{T-1}, u_{T-1}, v_{T-1})),$$
$$G_T(x_{T-1}, u_{T-1}, v_{T-1}) = h_{T-1}(x_{T-1}, u_{T-1}, v_{T-1}) + h_T(f_{T-1}(x_{T-1}, u_{T-1}, v_{T-1})),$$

and the maximal guaranteed results of the center and of the subsystem under our assumptions will be (respectively)

$$\tilde{\Phi}_T(x_{T-1}) = \max_{u_{T-1}\in U^0_{T-1}(x_{T-1})} \min_{v_{T-1}\in\Omega_{T-1}(x_{T-1},u_{T-1})} F_T(x_{T-1}, u_{T-1}, v_{T-1}), \tag{3.1}$$

where

$$\Omega_{T-1}(x_{T-1}, u_{T-1}) = \{v_{T-1} \mid v_{T-1}\in V_{T-1}(x_{T-1}, u_{T-1}),$$

$$G_T(x_{T-1}, u_{T-1}, v_{T-1}) \geqslant G_T(x_{T-1}, u_{T-1}, \omega) \ \ \forall \omega \in V_{T-1}(x_{T-1}, u_{T-1})\}; \tag{3.2}$$

$$U^0_{T-1}(x_{T-1}) = \{u_{T-1} \mid u_{T-1}\in U_{T-1}(x_{T-1}),\ f_{T-1}(x_{T-1}, u_{T-1}, v_{T-1})\in X_T \ \ \forall v_{T-1} \in \Omega_{T-1}(x_{T-1}, u_{T-1})\};$$

$$\tilde{\Psi}_T(x_{T-1}) = \min_{u_{T-1}\in E_{T-1}(x_{T-1})} \max_{v_{T-1}\in V_{T-1}(x_{T-1},u_{T-1})} G_T(x_{T-1}, u_{T-1}, v_{T-1}), \tag{3.3}$$

where

$$E_{T-1}(x_{T-1}) = \{u_{T-1} \mid u_{T-1}\in U^0_{T-1}(x_{T-1}),$$

$$\min_{v_{T-1}\in\Omega_{T-1}(x_{T-1},u_{T-1})} F_T(x_{T-1}, u_{T-1}, v_{T-1}) \geqslant \min_{v_{T-1}\in\Omega_{T-1}(x_{T-1},u_{T-1})} F_T(x_{T-1}, y, v_{T-1}) \ \forall y \in U^0_{T-1}(x_{T-1})\}. \tag{3.4}$$

After that we find the criteria of the center and of the subsystem

$$F_{T-1}(x_{T-2}, u_{T-2}, v_{T-2}) = g_{T-2}(x_{T-2}, u_{T-2}, v_{T-2}) + \tilde{\Phi}_T(f_{T-2}(x_{T-2}, u_{T-2}, v_{T-2})),$$

$$G_{T-1}(x_{T-2}, u_{T-2}, v_{T-2}) = h_{T-2}(x_{T-2}, u_{T-2}, v_{T-2}) + \tilde{\Psi}_T(f_{T-2}(x_{T-2}, u_{T-2}, v_{T-2})),$$

and solve for them problems of the form (3.1)-(3.4), and obtain $\tilde{\Phi}_{T-1}(x_{T-2})$, $\tilde{\Psi}_{T-1}(x_{T-2})$, etc. In final form, the maximal guaranteed results of the center and of the subsystem will be $\tilde{\Phi}_1(x_0)$ and $\tilde{\Psi}_1(x_0)$, and for finding the optimal guaranteed strategies it is necessary to traverse the chain in the forward direction. If the center and the subsystem do not select controls from the sets Ω and E that are the worst for the other side (here, for example, the "benevolence" principle would be appropriate [3]), then their results could be even larger (see Remark 2 in Sec. 1).

Remark. In the general case it is impossible to say which type of control performed by the center (i.e., single or successive) is better [it could be that $\Phi_1(x_0) > \tilde{\Phi}_1(x_0)$ and $\Phi_1(x_0) > \tilde{\Phi}_1(x_0)$]. But if the optimal controls of the center and of the subsystem are unique, then these types will be equivalent [$\Phi_1(x_0) = \tilde{\Phi}_1(x_0)$].

The solving of all these problems is based on a combination of the standard procedure of dynamic programming and of special methods considered in [2-4].

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