SOLITARY WAVES IN A LAYER OF VISCOUS LIQUID

V. Ya. Shkadov

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Solitary waves in a thin layer of viscous liquid which is running down a vertical surface under the action of gravity are investigated. The existence of such waves was demonstrated in the experiments of [1, 2]. The difficulties that must be faced in a theoretical computation were also noted in these studies. Below a solution of the problem of stationary waves is obtained by the method of expansion in the small parameter in two regions with subsequent matching and also by a numerical integration method. It is shown that in each case a solution of solitary wave type exists along with the single-parameter family of periodic solutions (parameter — the wave number α). On decreasing the wave number, the periodic waves go over into a succession of solitary waves.

As the basis of the investigation we take the equation for the thickness of the layer $h(\xi_1)$, which is obtained by integrating the basic equations of motion of a viscous liquid transverse to the layer. In the integration it is assumed that the boundary-layer approximation can be used and a parabolic profile of the longitudinal velocity is taken. In the coordinate system attached to the wave this equation has the following form:

$$Ghh''' + \frac{1}{5}\left[6\left(\frac{1-z}{h}\right)^2 - z^2\right]h' + Hh - E\frac{1+zh-z}{h^2} = 0 \tag{1}$$

$$z = \frac{c}{U_0}, \quad G = \frac{\sigma}{\rho U_0^2 a_0}, \quad H = \frac{g a_0}{U_0^2}, \quad E = \frac{3\nu}{U_0 a_0}, \quad \xi_1 = (x - ct)a_0^{-1}$$

Here c is the wave velocity; U_0 and a_0 are the characteristic values of the velocity and the thickness of the layer. In the case of a solitary wave U_0 and a_0 denote the mean values of the velocity and thickness of the unperturbed layer.

The nonlinear periodic solutions of Eq. (1) were investigated in [3]. The method of Fourier series expansion was used and explicit expressions were obtained for the waveform, the phase velocity, and the layer thickness. For a fixed number of terms considered in this solution, the accuracy decreases with the decreasing wave number α due to the fact that for small values of α the wave profile is very different from a harmonic wave. As an example, some results of direct numerical integration of Eq. (1) in the case of periodic waves in a layer of water are shown in Fig. 1 for $Re = 3a_0U_0\nu^{-1} = 24.41$ and $\alpha = 0.107$ (curve 1) and $\alpha = 0.051$ (curve 3). The integration was done over a wavelength $\xi_{10} \le \xi_1 \le \xi_{10} + 2\pi\alpha^{-1}$. The initial point ξ_{10} was chosen at the crest of the wave $h'(\xi_{10}) = 0$; for given values of Re and α, the values of c, a_0, and the initial data $h(\xi_{10})$, $h''(\xi_{10})$ were chosen in such a way that for $\xi_1 = \xi_{10} + 2\pi\alpha^{-1}$ the periodicity condition is satisfied. For the values of α lying close to the neutral stability curve in the Re, α plane the wave profiles are almost sinusoidal. The effect of the nonlinear terms in Eq. (1) increases with the decrease of α, and the profiles become noticeably deformed, acquiring the form of solitary waves.

The computations were carried out with a small step along parameter α. For obtaining the wave for $\alpha_1 = \alpha + \Delta\alpha$ the characteristics of the wave solution corresponding to α were

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used as the initial conditions. For a given value of Re such computations can be carried out only up to a certain finite value $\alpha_k(\text{Re})$. For $\alpha < \alpha_k$ the iteration process of selecting the initial data at the point ξ_{10} begins to diverge. Thus, although in these computations the tendency for the transition of the periodic waves into solitary waves is detected with decrease of α, a special method of solution is needed for determining the solitary waves.

We assume that there is a solitary wave such that for $\xi_1 - \xi_{10} \to \pm\infty$, $h \to 1$, where ξ_{10} is some characteristic point on the wave. Let us introduce the change of variable $\xi = \text{Re}^{2/9}\gamma^{-1/3}(\xi_1 - \xi_{10})$; then Eq. (1) becomes

$$h^3h''' + \delta[\omega - z^2(h^2-1)]h' + h^3 - 1 - z(h-1) = 0$$
$$\delta = 45^{-1}\text{Re}^{11/9}\gamma^{-1/3},\ \omega = 5z^2 - 12z + 6,\ \gamma = \sigma\rho^{-1}\nu^{-4/3}g^{-1/3} \tag{2}$$

We investigate the asymptotic behavior of the small deviation from $h = 1$ for $\xi \to \pm\infty$. We put $h - 1 = \varepsilon \exp \sigma\xi$. Linearizing Eq. (2) with respect to ε, for σ we obtain the following equation:

$$\sigma^3 + \delta\omega\sigma + 3 - z = 0 \tag{3}$$

For $z < 3$, Eq. (3) has one real root $\mu < 0$ and two complex-conjugate roots with positive real part $m \pm il$, $m = -{}^1/_2\mu$. Accordingly, we can construct two particular solutions of the linearized equation (2):

$$\varphi = \varepsilon_1 \exp \mu\xi, \qquad \psi = \varepsilon_2 \exp m\xi \cos(b + l\xi) \tag{4}$$

with arbitrary constants ε_1, ε_2, and b. The first of these solutions — $\varphi(\xi)$ — decreases to zero as $\xi \to \infty$; the second — $\psi(\xi)$ — decreases to zero as $\xi \to -\infty$. In the solitary wave there must be a continuous transition from $\varphi(\xi)$ to $\psi(\xi)$. This transition, if at all possible, must occur on account of the nonlinear terms of Eq. (2).

For continuing the solutions of (4) into the nonlinear region we use the method of expansion in the small parameter. For φ we write

$$\varphi = \varepsilon_1\varphi_1 + \varepsilon_1^2\varphi_2 + \varepsilon_1^3\varphi_3 + \ldots \tag{5}$$

Substituting (5) into Eq. (2) and equating the coefficients, we obtain

$$\varphi_k''' + \delta\omega\varphi_k' + (3-z)\varphi_k = F_k \tag{6}$$

where the right-hand side F_k is expressed in terms of functions φ_m with smaller numbers and their derivatives. Putting $p = \delta z^2$ and $\varphi_k = \Phi_k \exp k\mu\xi$, we obtain

$$\Phi_k = (k^3\mu^3 + k\delta\omega\mu + 3 - z)^{-1}F_k \qquad (k = 2, 3, \ldots)$$
$$F_2 = 2p\mu - 3 - 3\mu^3$$
$$F_3 = (6p\mu - 6 - 27\mu^3)\Phi_2 + p\mu - 1 - 3\mu^3 \tag{7}$$
$$F_4 = (12p\mu - 6 - 84\mu^3)\Phi_3 - (4p\mu - 3 - 30\mu^3)\Phi_2 - (3 + 24\mu^3)\Phi_2^2 - \mu^4$$

We can take $\Phi_1 = 1$ without any loss of generality, since this can always be achieved by a choice of ε_1. The decrease of ε_1, in turn, is equivalent to displacing the origin for ξ in $\exp \mu\xi$ to the right.

For ψ we write the expansion

$$\psi = \varepsilon_2\psi_1 + \varepsilon_2^2\psi_2 + \varepsilon_2^3\psi_3 + \ldots \tag{8}$$

We introduce the notation $\zeta = b + l\xi$; then the coefficients ψ_k in the expansion of ψ in powers of ε_2 must be of the form

$$\psi_1 = \exp m\xi \cos\zeta, \qquad \psi_2 = \exp 2m\xi(\psi_{20} + \psi_{21}\cos 2\zeta + \psi_{22}\sin 2\zeta)$$
$$\psi_3 = \exp 3m\xi(\psi_{31}\cos\zeta + \psi_{32}\sin\zeta + \psi_{41}\cos 3\zeta + \psi_{42}\sin 3\zeta) \tag{9}$$

We shall restrict the computation to three terms of the expansion (8). Substituting (8) into (2) with (9) taken into consideration, collecting together the terms with equal powers of ε and then the terms with equal harmonics, and equating them to zero, we obtain the equations for ψ_{ki}. In particular, for ψ_{20} we have

$$(8m^3 + 2\delta\omega m + 3 - z)\psi_{20} = -{}^1/_2[3 - 2pm + 3m(m^2 - 3l^2)] \tag{10}$$

For other coefficients we obtain the pairwise system of equations

$$a_{k1}\psi_{k1}+a_{k2}\psi_{k2}=F_{k1},\qquad b_{k1}\psi_{k1}+b_{k2}\psi_{k2}=F_{k2}\qquad (k=2,3,\ldots)\tag{11}$$

$$\begin{aligned}&a_{21}=8m(m^2-3l^2)+2\delta\omega m+3-z, && a_{22}=8l(3m^2-l^2)+2\delta\omega l\\ &a_{31}=9m(3m^2-l^2)+3\delta\omega m+3-z, && a_{32}=(27m^2-l^2)l+\delta\omega l\\ &a_{41}=27m(m^2-3l^2)+3\delta\omega m+3-z, && a_{42}=27l(3m^2-l^2)+3\delta\omega l\end{aligned}\tag{12}$$

The relations $b_{k1} = -a_{k2}$ and $b_{k2} = a_{k1}$ are satisfied for all k and i. For the right-hand side F_{ki} we get

$$\begin{aligned}&F_{21}={}^1/_2[3-2pm+3m(m^2-3l^2)],\qquad F_{22}={}^1/_2[2pl+3l(l^2-3m^2)]\\ &F_{31}={}^3/_4+{}^9/_4(m^2-3l^2-{}^3/_4p)m+[9m(3m^2-l^2)-6pm+6]\psi_{20}\\ &F_{32}={}^1/_4+{}^3/_4m(m^2-3l^2)-{}^1/_4pm+[{}^{27}/_2(m^2-3l^2)m-3pm+3]\psi_{21}-[{}^{27}/_2(l^2-3m^2)+3p]l\psi_{22}\\ &F_{41}={}^3/_4l(l^2-3m^2)+{}^1/_4pl+[3(l^2-3m^2)+2p]l\psi_{20}+[{}^{21}/_2(l^2-3m^2)+p]l\psi_{21}+[{}^{27}/_2m(m^2-3l^2)-3pm+3]\psi_{22}\\ &F_{42}={}^3/_4l(l^2-3m^2)+{}^1/_4pl+[{}^{27}/_2(l^2-3m^2)+3p]l\psi_{21}+[{}^{27}/_2m(m^2-3l^2)-3pm+3]\psi_{22}\end{aligned}\tag{13}$$

We now attempt to join solutions (5) and (8) at the point $\xi = 0$. At this point the following conditions must be satisfied:

$$\varphi=\psi,\qquad \varphi'=\psi',\qquad \varphi''=\psi''\qquad (\xi=0)\tag{14}$$

In order to satisfy these conditions, the three arbitrary constants z, b, and ε_2 must be suitably chosen. As regards ε_1, it can be chosen with a certain degree of arbitrariness, since the choice of the point of joining is to some extent arbitrary. Changing ε_1 we will thereby change the value of $\varphi(0)$ and, hence, the position of the point of joining in relation to the wave front. In view of the fact that only a limited number of terms are considered in expansions (5) and (8), this arbitrariness is actually small and the choice of a suitable joining point becomes important. Practical computations show that it is possible to accomplish such joining and the existence of the solitary wave is thus confirmed.

Let us consider a specific example. There is a layer of water for which Re = 24.41, γ = 2850, and δ = 0.08. We take $\varepsilon_1 = -0.175$; then we find that for z = 2.45, b = 4.71, ε_2 = 0.6, at the point $\xi = 0$ we obtain

$$\begin{aligned}&\varphi(0)=-0.132, && \varphi'(0)=0.056, && \varphi''(0)=-0.010\\ &\psi(0)=-0.134, && \psi'(0)=0.054, && \psi''(0)=-0.021\end{aligned}$$

The joining conditions (14) are satisfied quite accurately. The wave profile computed from formulas (5)-(13) is quite close to that obtained from the numerical solution.

The direct numerical solution of Eq. (2) gives another method of finding the solitary wave, which permits one to bypass the unwieldy procedure of matching the parameters during the joining of the solutions. For a given flow rate, solution (5) depends only on parameter z. Specifying z, $\varphi(0)$, $\varphi'(0)$, and $\varphi''(0)$ can be found from (5), and these values can be used as the initial data for numerical integration of Eq. (2) for negative values of ξ. The value of z should be chosen in such a way that as $\xi \to -\infty$ the solution $h(\xi)$ tends to unity. The computation reduces to repeated numerical solution of the Cauchy problem from the point $\xi = 0$ with the initial data corresponding to different z.

Curve 2 in Fig. 1 shows the solitary wave obtained from the numerical solution corresponding to the value z = 2.335, and it is not very different from the approximate value z = 2.45 obtained by the method of joining the expansions. It is also evident that the periodic wave, which is almost harmonic for α = 0.107, goes over into the solitary wave even for α = 0.051.

The solitary waves for a layer of water with Re = 15, 35, and 55 (curves 1, 2, and 3) are shown in Fig. 2. The corresponding values of z are 2.6075, 2.1386, and 1.8926.

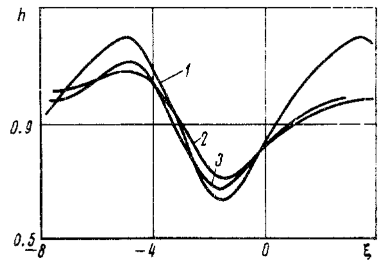

Fig. 1

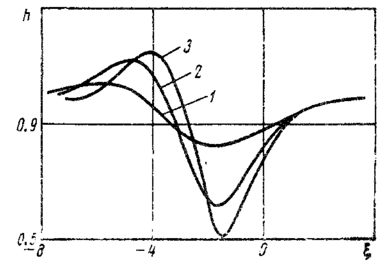

Fig. 2

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LITERATURE CITED

1. P. L. Kapitsa, "Wave flow of thin layers of a viscous liquid. 1, 2," Zh. Éksp. Teor. Fiz., 18, No. 1 (1948).
2. P. L. Kapitsa and S. P. Kapitsa, "Wave flow of thin layers of a viscous liquid. 3," Zh. Éksp. Teor. Fiz., 19, No. 2 (1949).
3. V. Ya. Shkadov, "Wave flow regimes of a thin layer of a viscous liquid under the action of gravity," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 1 (1967).

FORM OF A FREE SURFACE DURING STEADY FLOW OF A CAPILLARY FLUID IN A RECTANGULAR CHANNEL

V. S. Temkina

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The two-dimensional problem of the form of a free surface of an ideal incompressible fluid during steady flow from a rectangular channel through a thin slot with simultaneous uniform delivery of fluid through the side walls is examined. Forces of gravity and surface tension are taken into account. The nonlinear problem of the simultaneous determination of the free surface and velocity field of the fluid is solved by the iteration method. Convergence of the iterations to the solution of the problem for small values of the parameters is investigated. The solution of the linearized problem is obtained in a closed form for a small depth of the discharge and small width of the channel, which is compared with the solution of the problem in a complete formulation. Graphs of the free surface of the fluid for different values of the parameters, obtained as a result of numerical solution of the nonlinear problem, are presented.

1. Statement of the Problem. Iteration Method

A discharge of thickness α is located at point (0, 0) of the plane region of flow Ω (Fig. 1). The free surface S of the fluid, described by the function $y = 1 + f(x)$, and the velocity potential φ satisfy the following system of equations and boundary conditions [1, 2] written in a dimensionless form:

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