5. N. A. Shul'ga, "Prorozv'yazok zadach mekhaniki dlya periodichnikh Struktur," Dopov. Akad. Nauk URSR, Ser. A, No. 11, 1055-1058 (1971).
6. E. G. Henneke, "Reflection—refraction of a stress wave at a plane boundary between anisotropic media," J. Acoust. Soc. Am., 51, No. 1, Part 2, 210-217 (1972).

DIFFERENTIAL EQUATIONS WITH DISPLACED ARGUMENTS IN STATIONARY PROBLEMS IN THE MECHANICS OF A DEFORMABLE BODY

G. G. Onanov and A. L. Skubachevskii

UDC 539.3: 517.946

1. The Elastic Model. Figure 1 shows an elastic system consisting of two parallel plates connected by a regular system of vertical and slanting ribs (at an angle α to the vertical), all oriented in the same direction.

It is natural to contrast this discrete—continuous system with a continuous model, "spreading out" in the space between the plates both the vertical ribs (O-braces) and the slanting ribs (α-braces). To do this, we must introduce kinematically independent continuous fields of elastic displacements of the O-braces and α-braces uniformly distributed in the space between the plates. As a result, we arrive at a three-layer plate with a "two-phase" model of a filler, which combines in itself a medium of O-braces and a medium of α-braces.

We introduce a unified system of Cartesian coordinates x, y, z, making the middle surfaces of the plates coincide with the planes $z = \pm h$ in such a way that the ribs will be directed along the axis Ox. We also introduce the local Cartesian coordinates x_β, O_β, y_β ($\beta = 0, \alpha$) in the planes of the ribs, making the axis $O_\beta x_\beta$ coincide with the line of intersection of the corresponding rib and the plane xOy. We shall assume for the sake of simplicity that both the planes themselves and the ribs, both vertical and slanting, are moment-free (zero rigidity out of the plane); the ribs offer no resistance to tension or compression in the longitudinal direction but are absolutely rigid in the transverse direction (in the plane of the ribs).

Let us examine a single rib. Obviously, by virtue of the simplifications we have made, the cross section $x_\beta = \text{const}$ of the rib is displaced in the plane of the rib like a rigid body. Consequently the elastic displacements u_β, v_β of an arbitrary point of the rib in the direction of the axes x_β, y_β can be represented in the form

$$u_\beta(x_\beta, y_\beta) = \varphi_\beta(x_\beta) - \psi_\beta(x_\beta) y_\beta; \quad v_\beta(x_\beta, y_\beta) = v_\beta(x_\beta), \tag{1.1}$$

where φ_β, ψ_β are the translational displacement in the direction of the axis $O_\beta x_\beta$ and the rotation in the plane $x_\beta O_\beta y_\beta$ of the rib cross section $x_\beta = \text{const}$.

Having established the explicit relation (1.1) between the displacements of an individual rib and the coordinate y_β, we turn to the continuous model. The local coordinates are connected with the unit coordinates

$$x_\beta = x; \quad y_\beta = z/\cos\beta \quad (\beta = 0, \alpha). \tag{1.2}$$

The equation of the plane of the rib in the unified system of coordinates has the form

$$y - z\,\mathrm{tg}\,\beta = \text{const} \quad (\beta = 0, \alpha). \tag{1.3}$$

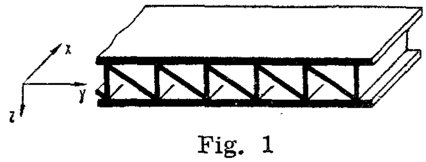

Fig. 1

Moscow Aviation Institute. Translated from Prikladnaya Mekhanika, Vol. 15, No. 5, pp. 39-47, May, 1979. Original article submitted June 30, 1978.

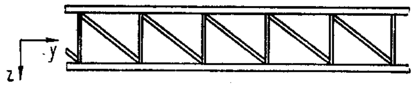

Fig. 2

Spreading out the ribs in the space between the plates, we must introduce the continuous fields $u_\beta(x, y, z)$, $v_\beta(x, y, z)$ of the elastic displacements of the β-braces ($\beta = 0, \alpha$). To do this, we must, after first passing from the local coordinates to the unified system of coordinates, "extend" the expressions (1.1), which relate to an individual rib, in accordance with the condition

$$u_\beta(x, \mathrm{const} + z\,\mathrm{tg}\,\beta, z) = u_\beta(x_\beta, y_\beta); \quad v_\beta(x, \mathrm{const} + z\,\mathrm{tg}\,\beta, z) = v_\beta(x_\beta, y_\beta) \tag{1.4}$$

to the entire space between the plates.

From (1.1)-(1.3) it follows that we have, corresponding to the condition (1.4), the representations

$$u_\beta(x, y, z) = \Phi_\beta(x, y - z\,\mathrm{tg}\,\beta) - \Psi_\beta(x, y - z\,\mathrm{tg}\,\beta)\, z/\cos\beta;$$
$$v_\beta(x, y, z) = V_\beta(x, y - z\,\mathrm{tg}\,\beta), \tag{1.5}$$

where φ_β, Ψ_β, V_β are some functions of two variables each.

The expressions (1.5) for $\beta = 0, \alpha$ represent kinematically independent continuous fields of elastic displacements of the two-phase filler of a three-layer plate. These fields must be subjected to the conditions of the kinematic connection of the filler with the supporting layers

$$u_\beta(x, y, \pm h) = u^\pm(x, y);$$
$$v_\beta(x, y, \pm h) = v^\pm(x, y)\sin\beta + w^\pm(x, y)\cos\beta, \tag{1.6}$$

where $u^\pm$, $v^\pm$, $w^\pm$ are the displacements of the surfaces of the lower plate ($z = h$) and the upper plate ($z = -h$); $\beta = 0, \alpha$.

From (1.5), (1.6) it follows that the field of elastic displacements of a three-layer plate with a two-phase filler is determined by 12 functions of two independent variables u^-, v^-, w^-, u^+, v^+, w^+, Φ_0, Ψ_0, V_0, Φ_α, Ψ_α, V_α, connected by the eight relations

$$u^\pm(x, y) = \Phi_\beta(x, y \mp h\,\mathrm{tg}\,\beta) \mp \Psi_\beta(x, y \mp h\,\mathrm{tg}\,\beta)\, h/\cos\beta;$$
$$v^\pm(x, y)\sin\beta + w^\pm(x, y)\cos\beta = V_\beta(x, y \mp h\,\mathrm{tg}\,\beta) \quad (\beta = 0, \alpha). \tag{1.7}$$

2. Variational and Boundary-Value Problems. The boundary-value problems corresponding to the proposed adequate continuous interpretation of the discrete—continuous systems under consideration can be naturally formulated on the basis of Lagrange's principle.

As our main unknown, we introduce the four-dimensional vector function $\bar{u}$ of two variables:

$$u^1 = \Phi_\alpha;\; u^2 = \Psi_\alpha;\; u^3 = V_\alpha;\; u^4 = V_0. \tag{2.1}$$

Then

$$u^\pm = u^1_{\mp\tau} \mp h\sec\alpha\, u^2_{\mp\tau};\quad v^\pm = \mathrm{cosec}\,\alpha\, u^3_{\mp\tau} - \mathrm{ctg}\,\alpha\, u^4;$$
$$w^\pm = u^4;\; \Phi_0 = 0{,}5\,(u^1_{+\tau} + u^1_{-\tau}) + 0{,}5\, h\sec\alpha\,(u^2_{+\tau} - u^2_{-\tau});$$
$$\Psi_0 = 0{,}5 h^{-1}(u^1_{+\tau} - u^1_{-\tau}) + 0{,}5\sec\alpha\,(u^2_{+\tau} + u^2_{-\tau}), \tag{2.2}$$

where $\pm\tau$ denotes the displacement of the second argument by $\tau = h\tan\alpha$, e. g., $u^i_{\pm\tau} = u^i(x, y \pm \tau)$.

We shall assume that the plates are rectangular $\{0 \le x \le a;\ 0 \le y \le b\}$ and the β-media are included between the plates in a rectangular parallelepiped $V^\beta = \{0 \le x \le a;\ 0 \le y \le b;\ -h \le z \le h\}$.

The functional of the total potential energy of the three-layer plate with a two-phase filler can, by virtue of (1.5), (2.1), (2.2), be represented in the form

$$\text{Э}(\bar{u}) = \int_0^a\int_0^b \Big\{ \sum_{i=1,-1} G\delta\,[(1-\nu)^{-1}((u^1_{i\tau})_x + ih\sec\alpha\,(u^2_{i\tau})_x)^2 +$$
$$+ 0{,}5\,((u^1_{i\tau})_y + ih\sec\alpha\,(u^2_{i\tau})_y)^2 + (1-\nu)^{-1}(\mathrm{cosec}\,\alpha\,(u^3_{i\tau})_y - \mathrm{ctg}\,\alpha\, u^4_y)^2 +$$
$$+ 0{,}5\,(\mathrm{cosec}\,\alpha\,(u^3_{i\tau})_x - \mathrm{ctg}\,\alpha\, u^4_x)^2 + 2\nu(1-\nu)^{-1}((u^1_{i\tau})_x + ih\sec\alpha\,(u^2_{i\tau})_x) \times$$

$$\times (\operatorname{cosec}\alpha\,(u^3_{i\tau})_y - \operatorname{ctg}\alpha u^4_y) + ((u^1_{i\tau})_y + ih \sec\alpha\,(u^2_{i\tau})_y)\ (\operatorname{cosec}\alpha\,(u^3_{i\tau})_x -$$
$$- \operatorname{ctg}\alpha u^4_x)] + Gh\mu_\alpha (u^3_x - u^2)^2 + Gh\mu_0 [0{,}5h^{-1}(u^1_\tau - u^1_{-\tau}) + 0{,}5 \sec\alpha\,(u^2_\tau + u^2_{-\tau}) -$$
$$- u^4_x]^2 - \sum_{=,-1} [X^{-i}(u^1_{i\tau} + ih\cos\alpha u^2_{i\tau}) + Y^{-i}(\operatorname{cosec}\alpha u^3_{i\tau} - \operatorname{ctg}\alpha u^4) + Z^i u^4]\Big\}\, dxdy,$$

where G, ν are the shear modulus and the Poisson coefficient, respectively; δ is the thickness of the plates; μ_β ($\beta = 0, \alpha$) are the volumetric content of β-braces in a unit volume V^β after spreading out; X^i, Y^i, Z^i (i = 1, − 1) are the components of the external loading on the upper and lower plates in the unified system of coordinates.

We calculate the variation of the functional ϑ ($\bar{u}$). We write $Q_1 = \{0 < x < a;\ \tau < y < b - \tau\}$; $Q_2 = \{0 < x < a;\ 0 < y < b\}$; $G_1 = \{0 < x < a;\ -\tau < y < \tau\}$; $G_2 = \{0 < x < a;\ b - \tau < y < b + \tau\}$, using $\bar{v}$ to denote the variation of the function $\bar{u}$; we subdivide the resulting integral into the sum of integrals of the form

$$\iint_{Q_2} k^{ij}_{rm} L_r u^i L_m v^j dxdy,$$

where the operators $L_1 u^i = u^i$, $L_2 u^i = u^i_x$, $L_3 u^i = u^i_y$, $L_4 u^i = u^i_{+\tau}$, $L_5 u^i = (u^i_{+\tau})_x$, $L_6 u^i = (u^i_{+\tau})_y$, $L_7 u^i = u^i_{-\tau}$, $L_8 u^i = (u^i_{-\tau})_x$, $L_9 u^i = (u^i_{-\tau})_y$; k^{ij}_{rm} are constants (i, j = 1, ..., 4; r, m = 1, ..., 9).

In the integrals containing the functions $v^i_{\pm\tau}$, $(v^i_{\pm\tau})_x$, $(v^i_{\pm\tau})_y$ (i = 1, ..., 4) we make the change of variables $y' = y \pm \tau$, thereby passing from integrals over the region Q_2 to integrals over the region $Q_1 \cup G_2$ if $y' = y + \tau$ and over the region $Q_1 \cup G_1$ if $y' = y - \tau$. Reducing similar terms and setting the variation of the functional ϑ ($\bar{u}$) equal to zero, we obtain

$$\sum_{i=1}^{3} \iint_{Q_1} \left[\sum_{j=1}^{4} (D_{ij1} u^j \cdot v^i + D_{ij2} u^j \cdot v^i_x + D_{ij3} u \cdot v^i_y) \right] dxdy + \sum_{k=1}^{2} \sum_{i=1}^{3} \iint_{G_k} \left[\sum_{j=1}^{4} (B_{kij1} u^j \cdot v^i + B_{kij2} u^j \cdot v^i_x + B_{kij3} u^j \cdot v^i_y) \right] dxdy +$$
$$+ \iint_{Q_2} \left[\sum_{j=1}^{4} (D_{4j1} u^j \cdot v^4 + D_{4j2} u^j \cdot v^4_x + D_{4j3} u^j \cdot v^4_y) \right] dxdy = \sum_{i=1}^{3} \iint_{Q_1} f^i v^i dxdy + \iint_{Q_2} f^4 v^4 dxdy, \tag{2.3}$$

where D_{ijn} (i, j = 1, ..., 4; n = 1, 2, 3); B_{kijn} (k = 1, 2; i, n = 1, 2, 3; j = 1, ..., 4) are some differential–difference operators; $f^1 = X^{-1}_{-\tau} + X^1_{+\tau}$; $f^2 = h\sec\alpha\,(X^{-1}_{-\tau} - X^1_{+\tau})$; $f^3 = \operatorname{cosec}\alpha\,(Y^{-1}_{-\tau} + Y^1_{+\tau})$; $f^4 = \operatorname{cosec}\alpha\,[(Z^{-1} + Z^1)\ \sin\alpha - (Y^{-1} + Y^1)\cos\alpha]$.

If on the left side of (2.3) we formally integrate by parts integrals over Q_1 and Q_2, we find that $\bar{u}$ satisfies a system of partial differential equations with displaced argument:

$$\begin{gathered}
-2G\delta[\Delta u^1 + (1+\nu)(1-\nu)^{-1} u_{xx}] - 2G\delta \operatorname{cosec}\alpha\,(1+\nu)(1-\nu)^{-1} u^3_{xy} + \\
+ G\delta \operatorname{ctg}\alpha\,(1+\nu)(1-\nu)^{-1}(R_1 u^4)_{xy} + G\mu_0 (R_{-1} u^4)_x + 0{.}5G\mu_0 h^{-1} R_2^- u^1 - \\
- 0{,}5G\mu_0 \sec\alpha\, R_{-2} u^2 = f^1; \\
-2G\delta h^2 \sec^2\alpha\,[\Delta u^2 + (1+\nu)(1-\nu)^{-1} u^2_{xx}] - G\delta h \operatorname{cosec}\alpha\,(1+\nu)(1-\nu)^{-1}(R_{-1} u^4)_{xy} - \\
- 2Gh\mu_\alpha u^3_x - Gh\mu_0 \sec\alpha\,(R_1 u^4)_x + 0.5G\mu_0 \sec\alpha R_{-2} u^1 + \\
+ 0.5Gh\mu_0 \sec^2\alpha R_2^+ u^2 + 2Gh\mu_\alpha u^2 = f^2; \\
-2G\delta \operatorname{cosec}\alpha\,(1+\nu)(1-\nu)^{-1} u^1_{xy} - 2G\delta \operatorname{cosec}^2\alpha\,[\Delta u^3 + (1+\nu)(1-\nu)^{-1} u^3_{yy}] + \\
+ G\delta \operatorname{ctg}\alpha \operatorname{cosec}\alpha\,[\Delta(R_1 u^4) + (1+\nu)(1-\nu)^{-1}(R_1 u^4)_{yy}] - \\
- 2Gh\mu_\alpha u^3_{xx} + 2Gh\mu_\alpha u^2_x = f^3; \\
G\delta \operatorname{ctg}\alpha\,(1+\nu)(1-\nu)^{-1}(R_1 u^1)_{xy} + G\delta h \operatorname{cosec}\alpha\,(1+\nu)\ (1-\nu)^{-1}(R_{-1} u^2)_{xy} + \\
+ G\delta \operatorname{ctg}\alpha \operatorname{cosec}\alpha\,[\Delta(R_1 u^3) + (1+\nu)\ (1-\nu)^{-1}(R_1 u^3)_{yy}] - \\
- 2G\delta \operatorname{ctg}^2\alpha\,[\Delta u^4 + (1+\nu)(1-\nu)^{-1} u^4_{yy}] - 2Gh\mu_0 u^4_{xx} + \\
+ G\mu_0 (R_{-1} u^1)_x + Gh\mu_0 \sec\alpha\,(R_1 u^2)_x = f^4,
\end{gathered} \tag{2.4}$$

where, as we can see from (2.3), the first three equations are determined for $(x, y) \in Q_1$, and the last equation for $(x, y) \in Q_2$. Here R means the difference operators:

$$R_{-1}u = u_{+\tau} - u_{-\tau};\ R_1 u = u_{+\tau} + u_{-\tau};\ R_{-2}u = u_{+2\tau} - u_{-2\tau};$$
$$R_2^- u = 2u - u_{-2\tau} - u_{+2\tau};\ R_2^+ u = 2u + u_{-2\tau} + u_{+2\tau}.$$

We shall assume that in addition to the internal connections, the system is subjected to absolutely rigid external geometric connections which ensure the existence and uniqueness of the fields of elastic displacements (homogeneous geometric boundary conditions).

We take:

a) for x = 0, a there are no displacements of the β-media ($\beta = 0, \alpha$), i. e., for x = 0, a, $u_\beta(x, y, z) = v_\beta(x, y, z) = 0$, and hence, by virtue of (1.5), (2.1), we obtain

$$u^1(x, y) = u^2(x, y) = u^3(x, y) = 0 \quad ((x, y) \in \{x = 0, a;\ -\tau \leqslant y \leqslant b + \tau\});$$
$$u^4(x, y) = 0 \quad ((x, y) \in \{x = 0, a;\ 0 \leqslant y \leqslant b\}); \tag{2.5}$$

b) for $(x, y, z) \in \{0 \leqslant x \leqslant a;\ -\tau \leqslant y - z\,\mathrm{tg}\,\alpha \leqslant \tau;\ -h \leqslant z \leqslant h\} \cup \{0 \leqslant x \leqslant a;\ b - \tau \leqslant y - z\,\mathrm{tg}\,\alpha \leqslant b + \tau;\ -h \leqslant z \leqslant h\}$ there are no displacements of the α-braces, i. e., $u_\alpha(x, y, z) = v_\alpha(x, y, z) = 0$, and for $(x, y, z) \in \{0 \le x \le a;\ y = 0,\ b;\ -h \le z \le h\}$ there are no displacements of the O-braces, i. e., $u_0(x, y, z) = v_0(x, y, z) = 0$. From this, by (1.5), (2.1) we obtain

$$u^1(x, y) = u^2(x, y) = u^3(x, y) = 0 \quad ((x, y) \in G_1 \cup G_2);$$
$$u^4(x, y) = 0 \quad ((x, y) \in \{0 \leqslant x \leqslant a;\ y = 0, b\}). \tag{2.6}$$

Since the vector function $\bar{v}$ satisfies the boundary conditions (2.6), the integrals over the regions G_1 and G_2 in formula (2.3) vanish. Consequently, if $\bar{u}$ gives us the minimum of the functional $\mathcal{E}(\bar{u})$ with boundary conditions (2.5), (2.6), then it is the solution of the boundary-value problem (2.4)-(2.6).

It should be pointed our that the conditions for fixing the α-braces are not traditional, since they are given not only on the lateral faces of the three-layer plate but also in some volumes. It should be noted that the conditions for fixing only on the lateral faces lead to some additional equations in the regions G_1 and G_2, which are specific for differential equations with displaced arguments. In the present paper we shall not consider such fixing conditions.

3. Generalized Solutions of Two-Dimensional Boundary-Value Problems. As in the simple examples (see [2]), Eq. (2.3) with the boundary conditions (2.5), (2.6) can have solutions whose derivatives are discontinuous inside the region. Therefore the integration by parts which we used to derive the system of equations with displaced argument (2.4) is not valid in the general case. The equivalence of Eq. (2.3) and the system of equations (2.4) can be established by using the concept of generalized functions.

We introduce the real spaces of vector functions

$$L_2^4 = L_2(Q_1) \times L_2(Q_1) \times L_2(Q_1) \times L_2(Q_2);$$
$$\overset{0}{H}{}^{1,4} = \overset{0}{H}{}^1(Q_1) \times \overset{0}{H}{}^1(Q_1) \times \overset{0}{H}{}^1(Q_1) \times \overset{0}{H}{}^1(Q_2).$$

Here $L_2(Q_i)$ (i = 1, 2) is the space of functions which are square integrable over the region Q_i; $\overset{0}{H}{}^1(Q_i)$ (i = 1, 2) is the Sobolev space of functions which are square integral over Q_i together with their first generalized derivatives and are such that the traces of these functions on the boundary of Q_i are equal to zero (see [3], Ch. III).

We shall try to find the extremum of the functional $\mathcal{E}(\bar{u})$ with the boundary conditions (2.5), (2.6) and the corresponding solution of the boundary-value problem (2.4)-(2.6) in the space $\overset{0}{H}{}^{1,4}$, setting $u^i(x, y) = 0$ for $(x, y) \in G_1 \cup G_2$ (i = 1, 2, 3). We assume that $\bar{f} = \{f^1, f^2, f^3, f^4\} \in L_2^4$. We shall assume that the differential and difference operators on the left sides of Eqs. (2.4) act in the space of generalized functions (see [1], Ch. XIV), where we assume $u^i(x, y) = 0$ (i = 1, 2, 3) for $(x, y) \in G_1 \cup G_2$. We shall call $\bar{u} \in \overset{0}{H}{}^{1,4}$ the generalized solution of the system of equations (2.4) with the boundary conditions (2.5), (2.6) if $\bar{u}$ satisfies the system of equations (2.4) in the sense indicated above.

From the rules for operations on generalized functions it follows that the vector function $\bar{u} \in \overset{0}{H}{}^{1,4}$ satisfies Eq. (2.3) with the boundary conditions (2.5), (2.6) if and only if it is a generalized solution of the system of equations (2.4).

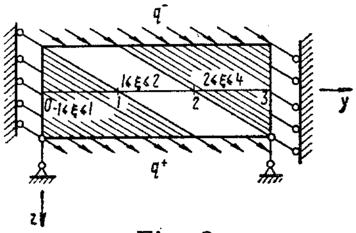

Fig. 3

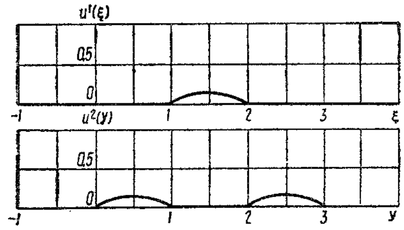

Fig. 4

Earlier we proved that if the vector function $\bar{u} \in \overset{0}{H}{}^{1,4}$ yields a minimum of the functional $\mathcal{Э}(\bar{u})$ with the boundary conditions (2.5), (2.6), then $\bar{u}$ is a general solution of the system of equations (2.4) with boundary conditions (2.5), (2.6). The contrary can also be proved. We state this in the form of a theorem.

THEOREM 1. The vector function $\bar{u} \in \overset{0}{H}{}^{1,4}$ yields a minimum of the functional $\mathcal{Э}(\bar{u})$ with boundary conditions (2.5), (2.6) if and only if it is a generalized solution of the system of equations (2.4) with boundary conditions (2.5), (2.6).

We rewrite the system of equations (2.4) in the form

$$L\bar{u} = \bar{f}, \tag{3.1}$$

where $D(L) = \{\bar{u} \in \overset{0}{H}{}^{1,4}: L\bar{u} \in L_2^4\}$ is the domain of definition of the operator L and the differential and difference operators constituting it act in the spaces of generalized functions; here $u^i(x, y) = 0$ $(i = 1, 2, 3)$ for $(x, y) \in G_1 \quad G_2$.

We denote by $B(\bar{u}, \bar{v})$ the left side of Eq. (2.3). We can prove the following lemma.

LEMMA. The bilinear form $B(\bar{u}, \bar{v})$ is an equivalent scalar product in the space $\overset{0}{H}{}^{1,4}$.

Making use of this lemma, using ordinary methods (see [3], Ch. IV, and [1], Ch. XIV), we can easily establish the following assertions.

THEOREM 2. The solution of the system of equations (2.4) in the class of vector functions $\bar{u} \in D(L)$ exists and is unique for any $\bar{f} \in L_2^4$, where $\|\bar{u}\|_{\overset{0}{H}{}^{1,4}} \le c\,\|\bar{f}\|_{L_2^4}$, with $c > 0$.

THEOREM 3. The spectrum of $\sigma(L)$ is discrete and of finite multiplicity and has no finite limit points, and $\sigma(L) \subset (0, \infty)$.

In order to apply variational methods to the solution of the corresponding boundary-value problem, instead of the total-energy functional $\mathcal{Э}(\bar{u})$, it is more convenient to introduce in $\overset{0}{H}{}^{1,4}$ the functional $E(\bar{v}) = B(\bar{v}, \bar{v}) - 2(\bar{f}, \bar{v})_{L_2^4}$. It can be shown that $d = \inf\limits_{v \in \overset{0}{H}{}^{1,4}} E(\bar{v}) > -\infty$. The sequence $\bar{v}_m \in \overset{0}{H}{}^{1,4}$ $(m = 1, 2, \dots)$ is called the sequence minimizing the functional E on $\overset{0}{H}{}^{1,4}$ if $\lim\limits_{m \to \infty} E(\bar{v}_m) = d$.

In $\overset{0}{H}{}^{1,4}$ we take an arbitrary independent system $\bar{e}_k$ $(k = 1, 2, \dots)$, whose linear envelope is dense in $\overset{0}{H}{}^{1,4}$. We denote by $\bar{v}_k$ the element which realizes the minimum of the functional E on the linear manifold spanned over the vector functions $\bar{e}_1, \dots, \bar{e}_k$. It is not difficult to show that there exists exactly one such element $\bar{v}_k = c_{k1}\bar{e}_1 + \dots + c_{kk}\bar{e}_k$. The sequence $\bar{v}_k$ $(k = 1, 2, \dots)$ is called the Ritz sequence for the functional E with respect to the system $\bar{e}_1, \dots, \bar{e}_k$.

We have the following assertion.

THEOREM 4. The Ritz sequence of the functional E constructed with respect to an arbitrary linearly independent system of functions, whose linear envelope is everywhere dense in $\overset{0}{H}{}^{1,4}$, converges in $\overset{0}{H}{}^{1,4}$ to the solution of the system of equations (2.4).

4. The Boundary-Value Problem in the One-Dimensional Case. The one-dimensional analog of the three-layer plate with a two-phase filler which is under consideration here is the continuous interpretation of a two-belt rod system — a truss with a regular set of absolutely rigid vertical and diagonal braces (Fig. 2). A representation of the field of elastic displacements of this model can be obtained from the corresponding expressions for a three-layer plate, assuming that the displacements take place only in the plane yOz and are

independent of the coordinate x. Making use of (1.5) and (1.7), we have ($\beta = 0,\ \alpha$)

$$v_\beta(y, z) = V_\beta(y - z\,\mathrm{tg}\,\beta); \tag{4.1}$$

$$v^{\pm}(y)\sin\beta + w^{\pm}(y)\cos\beta = V_\beta(y \mp h\,\mathrm{tg}\,\beta), \tag{4.2}$$

from which it follows that the field of displacements of a three-layer beam with a two-phase filler is determined by six functions of one variable (v^-, w^-, v^+, w^+, V_0, V_α), connected by four relations (4.2).

Introducing the two-dimensional vector function $\bar{u}$, we have

$$u^1 = V_\alpha;\ u^2 = V_0; \tag{4.3}$$

$$v^{\pm} = \mathrm{cosec}\,\alpha u^1_{\mp\tau} - \mathrm{ctg}\,\alpha u^2;\ w^{\pm} = u^2. \tag{4.4}$$

On the basis of (4.1)-(4.4), in a manner analogous to the previous case, we arrive at a problem involving the minimum of the functional

$$\textit{Э}(\bar{u}) = \int_0^b \Big\{ \sum_{i=1,-1} [EF(1-\nu)^{-1}[\mathrm{cosec}\,\alpha\,(u^1_{i\tau})' - \mathrm{ctg}\,\alpha\,(u^2)']^2 - Y^i[\mathrm{cosec}\,\alpha u^1_{i\tau} - \mathrm{ctg}\,\alpha u^2] - Z^i u^2] \Big\}\, dy \tag{4.5}$$

with the boundary conditions

$$u^1(y) = 0 \quad (y \in [-\tau, \tau] \cup [b-\tau, b+\tau]);$$
$$u^2(y) = 0 \quad (y = 0, b), \tag{4.6}$$

where E is Young's modulus; F is the cross-sectional area of the belts; Y^i, Z^i are the components of the external load on the belts.

As in the previous case, we can show that the vector function $\bar{u} \in \overset{0}{H}{}^{1,2} = \overset{0}{H}{}^1((\tau,\ b-\tau)) \times \overset{0}{H}{}^1((0,\ b))$ yields a minimum of the functional (4.5) with boundary conditions (4.6) if and only if it is a generalized solution of the system of equations

$$-2(u^1)'' + \cos\alpha\,(R_1 u^2)'' = f^1; \qquad \cos\alpha\,(R_1 u^1)'' - 2\cos^2\alpha\,(u^2)'' = f^2, \tag{4.7}$$

where $f^1 = (1-\nu)\ \sin\alpha\,(2EF)^{-1}(Y^{-1}_{-\tau} + Y^1_{+\tau}) \in L_2((\tau, b-\tau))$; $f^2 = (1-\nu)\ \sin\alpha\,(2EF)^{-1}\,[(Z^{-1} + Z^1)\sin\alpha - (Y^{-1} + Y^1)\cos\alpha] \in L_2((0,\ b))$.

It should be noted that the first equation is considered in the interval $(\tau,\ b-\tau)$ and the second in the interval (0, b) and that we set $u^1(y) = 0$ for $y \in [-\tau, \tau] \cup [b-\tau,\ b+\tau]$.

We rewrite the system of equations (4.7) in the form

$$L\bar{u} = \bar{f},$$

where $D(L) = \{\bar{u} \in \overset{0}{H}{}^{1,2} : L\bar{u} \in L_2^2\}$ is the domain of definition of the operator L, which acts in the space of generalized functions, where we set $u^1(y) = 0$ for $y \in [-\tau,\ \tau] \cup [b-\tau,\ b+\tau]$.

It can be shown that, as in the two-dimensional case: 1) the solution of the boundary-value problem (4.7), (4.6) exists and is unique; 2) the spectrum of $\sigma(L)$ is discrete and of finite multiplicity and $\sigma(L) \subset (0, \infty)$; 3) the Ritz method converges.

We write $n = b/\tau$ (for the sake of simplicity we shall assume that n is an integer). We introduce the $(2n-2)$-dimensional vector function $\bar{w} = \{w^1, \ldots, w^{2n-2}\}$, defined in the interval $[0,\ \tau]$:

$$w^j(y) = \begin{cases} v^1(y + j\tau) & (j = 1, \ldots, n-2); \\ v^2(y + (j - n + 1)\tau) & (j = n-1, \ldots, 2n-2). \end{cases} \tag{4.8}$$

Then, passing to the variables (4.8), we reduce the system of differential equations with displaced argument (4.7) to a system of 2n–2 ordinary differential equations in the functions $w^1, \ldots, w^{2n-2}$. Using this method, we can prove the following assertion.

<u>THEOREM 5.</u> Let $f^1 \in C([\tau,\ b-\tau])$, $f^2 \in C([0,\ b])$. Then $u^1 \in C([\tau,\ b-\tau])$, $u^2 \in C([0, b])$ and $u^1 \in C^2([j\tau,\ (j+1)\tau])$ $(j = 1, \ldots, n-2)$, $u^2 \in C^2([j\tau,\ (j+1)\tau])$ $(j = 0, \ldots, n-1)$.

Theorem 5 indicates a method of finding the solution of the system of equations (4.7) in explicit form if the right-hand sides f^1, f^2 are continuous. We first find the general solution of a system of 2n– 2 equations dependent on $(2n-2) \times 2$ arbitrary constants. The conditions for the continuity of the functions u^1, u^2 and the boundary conditions $u^1(j\tau-0) = u^1(j\tau+0)$ $(j = 2,\ldots,n-2)$, $u^1(\tau) = u^1(b-\tau) = 0$; $u^2(j\tau-0) = u^2(j\tau+0)$ $(j = 1,\ldots, n-1)$, $u^2(0) = u^2(b) = 0$ enable us to eliminate 2n constants. By Theorem 5, the functions $r_1(y) = -2(u_1)' + \cos\alpha(R_1u^2)'$ and $r_2(y) = \cos\alpha(R_1u^1)' - 2\cos^2\alpha(u_2)'$ are piecewise continuously differentiable, but, by the definition of a generalized solution, $r_1'(y)$, $r_2'(y)$ must not contain any terms of the δ-function type. The condition for this is the continuity of the functions $r_1(y)$ on $[\tau, b-\tau]$ and $r_2(y)$ on $[0, b]$ $[r_1(j\tau-0) = r_1(j\tau+0)$ $(j = 2, \ldots, n-2)$; $r_2(j\tau-0) = r_2(j\tau+0)$ $(j = 1,\ldots,n-1)]$, which enables us to eliminate the remaining $2n-4$ constants. By the existence and uniqueness theorem, the functions u^1, u^2 we have obtained are the solution of the system of equations (4.7).

It must be noted that the last $2n-4$ constants are eliminated precisely from the condition of the absence of δ-functions on the right-hand sides of the system of equations (4.7), and not from the condition of continuity of the first derivatives of the functions u^1 and u^2 at the corresponding points, which at first glance seems more natural (see [2]).

Example. We consider the three-dimensional beam $(b = 3,\ \tau = 1,\ \alpha = \pi/3)$ shown in Fig. 3; the belts of this beam are loaded with the uniformly distributed load $q^{\pm}(y)$, applied in the direction of the oblique braces $(f^1(y) \equiv 1,\ f^2(y) \equiv 0)$.

The boundary-value problem (4.7), (4.6) takes the form

$$-4(u^1)'' + (R_1u^2)'' = 2; \qquad (R_1u^1)'' - (u^2)'' = 0; \tag{4.9}$$

$$u^1(y) = 0\ (y \in [-1,1] \cup [2,4]); \qquad u^2(y) = 0\ (y = 0, 3). \tag{4.10}$$

We can convince ourselves without difficulty that the solution of the problem is given by the functions $u^1(y) = -0.5(y^2-3y+2)$;

$$u^2(y) = \begin{cases} -0.5(y^2-y) & (y \in [0,1]); \\ 0 & (y \in [1,2]); \\ -0.5(y^2-5y+6) & (y \in [2,3]). \end{cases}$$

To see this, we note that $u^1(1) = u^1(2) = 0$; $u^2(0) = u^2(3) = 0$. In addition, it is obvious that $u^1 \in \overset{0}{H}{}^1((1, 2))$, $u^2 \in \overset{0}{H}{}^1((0, 3))$. We can convince ourselves without difficulty that in the interval (1, 2) the functions $u^1(y)$, $u^2(y)$ satisfy the first equation of the system (4.9). Taking account of the fact that $u^1(y) = 0$ for $y \in [-1, 1] \cup [2, 4]$, we find that $R_1u^1(y) = u^2(y)$ for $y \in (0, 3)$. Consequently the functions $u^1(y)$, $u^2(y)$ satisfy the second equation of the system (4.9). We note that the function $(R_1u^1)'' = (u^2)''$ is singular, since $(u^2)'' = \psi(y) + 0.5\delta(y-1) + 0.5\delta\cdot(y-2)$, where $\psi(y) = -\theta(1-y) + \theta(y-2)$; $\theta(y)$ is the Heaviside unit function.

The resulting solution $\bar{u}$ (Fig. 4) is quite obvious, even though it is rather unusual. We introduce the variable $\xi = y - z\tan\alpha$, which, as can be seen from Fig. 3 is the y coordinate of the point of intersection of the α-brace passing through the point (y, z) with the axis Oy. Then from (4.1), (4.3), (4.4) it follows that

$$u^1(\xi) = V_\alpha(y - z\,\mathrm{tg}\,\alpha) = v_\alpha(y, z); \qquad u^2(y) = V_0(y) = v_0(y) = w^{\pm}(y). \tag{4.11}$$

From the conditions of fixing of the O-braces and α-braces, for $y = 0, 3$, $-h \le z \le h$, in accordance with Fig. 4, we have $u^1(\xi) = 0$ $(-1 \le \xi \le 1;\ 2 \le \xi \le 4)$; $u^2(0) = u^2(3) = 0$.

The load q^- in the interval $1 \le y \le 3$ and the load q^+ in the interval $0 \le y \le 2$ are directly taken up by the absolutely rigid fixed α-braces $(-1 \le \xi \le 1;\ 2 \le \xi \le 4$. Therefore, as a result of the simultaneousness of the deformations of the braces and belts, there are no displacements of the vertical braces in the interval $1 \le y \le 2$ (Fig. 4). In the interval $1 \le \xi \le 2$ the displacements of the α-braces v_α are caused by the load q^- in the interval $0 \le y \le 1$ and by the load q^+ in the interval $2 \le y \le 3$. Because of the compatibility of the deformations of the braces and belts, simultaneously with these deformations there are deformations of the vertical braces.

LITERATURE CITED

1. N. Dunford and J. T. Schwartz, Linear Operators, Wiley-Interscience.
2. G. A. Kamenskii and A. D. Myshkis, "The formulation of boundary-value problems for differential equations with a displaced argument and several leading terms," Differents. Uravn., 10, No. 3, 409-418 (1974).
3. V. P. Mikhailov, Partial Differential Equations [in Russian], Nauka, Moscow (1976).