ON THE METHOD OF GENERALIZED STOCHASTIC GRADIENTS AND QUASI-FÉJER SEQUENCES

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The concepts of the method of generalized stochastic gradients and a random quasi-Féjer sequence were introduced in [1, 7, 8, 18]. In this paper we note a connection between these concepts and give more exact and more general conditions for the convergence of the method of stochastic gradients. As examples we consider the problems of adaptive minimization, random search, and programmed control of an object subject to random influences.

FUNDAMENTAL DEFINITIONS

Let $\mathfrak{A}$ be a closed set in R^n and $C(\mathfrak{A})$ its convex hull.

The random sequence $\{z^k(\omega)\}$, $k = 0, 1, \ldots$, is said to be random quasi-Féjer with respect to the set $\mathfrak{A}$, if $M \| z^0 \| \leq \text{const} < \infty$ and

$$M(\| y - z^{k+1}(\omega) \|^2 (z^0, z^1, \ldots, z^k) \leqslant \| y - z^k(\omega) \|^2 + g_k \tag{1}$$

for arbitrary $y \in \mathfrak{A}$, , $k = 0, 1, \ldots$. The numbers g_k are such that $\sum_{k=0}^{\infty} g_k < \infty$.

Obviously, if $\{z^k\}$ is a random quasi-Féjer sequence with respect to the set $\mathfrak{A}$, it is a random quasi-Féjer sequence with respect to the set $C(\mathfrak{A})$.

A random quasi-Féjer sequence is said to be a random Féjer sequence if $g_k = 0$. The following lemma reflects the fundamental properties of random quasi-Féjer sequences which, with a small change, are analogous to the properties of ordinary dominate Féjer sequences [12, 13].

LEMMA. If the sequence $\{z^k(\omega)\}$ is a random quasi-Féjer sequence, then:

a) the set of limit points of $\{z^k(\omega)\}$ is not empty for almost all ω;

b) if $z'(\omega)$ and $z''(\omega)$ are any limit points of $\{z^k(\omega)\}$ for some ω not belonging to $C(\mathfrak{A})$, then $C(\mathfrak{A})$ lies in the plane which is the geometrical locus of points equidistant from $z'(\omega)$ and $z''(\omega)$.

This lemma follows directly from the fact that for any $y \in \mathfrak{A}$ the sequence $\rho_k = \| y - z^k \|^2 + \sum_{s=k}^{\infty} g_s$, $k = 0, 1, \ldots$, by (1), is a semimartingale [14] and converges for almost all ω and, hence, the sequence $\{\| y - z^k(\omega) \|^2\}$ converges for almost all ω.

COROLLARY 1. If the set $C(\mathfrak{A})$ is of dimension n, then $\{z^k(\omega)\}$ has a unique limit point for each ω.

COROLLARY 2. If the limit point $z(\omega)$ of the sequence $\{z^k(\omega)\}$ for some ω belongs to $C(\mathfrak{A})$, then $z(\omega)$ is the unique limit point for that ω.

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These properties of random quasi-Féjer sequences make it possible to standardize and simplify the proofs of certain known stochastic methods of optimization and solutions of sets of inequalities.

Let $F(x_1, \ldots, x_n)$ be a convex, but not necessarily differentiable function. The generalized gradient vector at the point $x = (x_1, \ldots, x_n)$ is any vector $\hat{F}_x(x)$ satisfying the inequality

$$F(y) - F(x) \geqslant (\hat{F}_x(x), y - x) \tag{2}$$

for arbitrary $y = (y_1, \ldots, y_n)$.

Thus, the vector $\hat{F}_x(x)$ is directed along the outward normal to one of the support hyperplanes of the set $\{y: F(y) \leq F(x)\}$; if $F(x)$ is differentiable, then $\hat{F}_x(x)$ coincides with the gradient $F_x(x) = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)$.

The convergence of $F(x)$ to the minimum point was investigated in [2-6] by the method of generalized gradient descent, defined by the equation:

$$x^{s+1} = x^s - \varrho_s \gamma_s \hat{F}_x(x^s), \quad s = 0, 1, \ldots, \tag{3}$$

where x^0 is an arbitrary point, ϱ_s is the step length, and γ_s is a normalizing factor. A feature of this method is that in general from one iteration to the next no monotonic diminishing of the values of $F(x)$ is observed and rigid control of the step length ϱ_s has to be stipulated for convergence. In [3, 4] it was proposed to choose the step length so that $\varrho_s \geq 0$, $\varrho_s \to 0$, $\sum_{s=0}^{\infty} \varrho_s = \infty$. In [5] another control method was was proposed which ensures that under very general assumptions (which are usual in such cases), the method (3) converges like a geometrical progression.

The process (3) can be used when it is easy to calculate the value of the vector $\hat{F}_x(x^s)$ at each point x^s. A theoretical formalism has been developed for calculating this vector, which, in its generality, recalls the formalism of ordinary differentiation.

But in practice, in nonlinear problems the exact value of even the ordinary gradient is known only in exceptional cases (for example, if $F(x)$ is specified as a polynomial and if rounding-off errors are neglected). Very frequently the value of the gradient can be calculated using certain difference analogs which are very sensitive to various kinds of random error, for example, from the equation:

$$F_x^{\Delta}(x) = \sum_{j=1}^{n} \frac{F(x + \Delta e^j) - F(x)}{\Delta} e^j, \tag{4}$$

where e^j is the unit vector along the j-th axis. There arises the not altogether obvious problem of the stability to random noise of methods of mathematical programming. It will become clear from what follows that the condition guaranteeing the convergence of the method (3) in the absence of random noise is no longer sufficient when noise is present.

For the systematic investigation of this problem we introduce the following definition. We shall say that the generalized stochastic gradient vector, or briefly, the stochastic quasi-gradient vector of the function $F(x)$ at the point x is any random vector $\xi(x)$ whose relative mathematical expectation (for each component separately) is $F_x(x)$ for fixed x, i.e.,

$$\mathbf{M}(\xi(x)/x) = \hat{F}_x(x). \tag{5}$$

Here it is assumed that the distribution of $\xi(x)$ depends only on the point x, and that the average value of the error is zero. This assumption is strong enough, since the approximate value of the gradient or the generalized gradient can be calculated over some set of points and, as (4) shows, has a nonzero average error.

Hence, we shall regard the vector $\xi(x)$ as of a more general form in which the mean value can be determined by a whole set of points $H(x)$ and the relative mathematical expectation has the form

$$\mathbf{M}(\xi(x)/H(x)) = c\hat{F}_x(x) + m, \tag{6}$$

where c is a number and m is a vector depending in general on $H(x)$.

Consider the problem of minimizing the downward-convex function

$$F(x_1, \ldots, x_n) \tag{7}$$

subject to the condition

$$x = (x_1, \ldots, x_n) \in \mathfrak{D}, \tag{8}$$

where $\mathfrak{D}$ is a convex closed set in $\mathbf{R}^n$. Let $\pi(x)$ denote the projection operator on $\mathfrak{D}$, i.e., for some $\pi(x)$ $\pi(x) \in \mathfrak{D}$, $\| y - \pi(x) \|^2 \leq \| y - x \|^2$, for any $y \in \mathfrak{D}$.

Consider the random sequence of points $\{x^s\}$ defined by

$$x^{s+1} = \pi(x^s - \varrho_s \gamma_s \xi^s), \quad s = 0, 1, \ldots \tag{9}$$

Here x^0 is an arbitrary point for which $\mathbf{M} \| x^0 \|^2 \leq \text{const} < \infty$, ϱ_s is the step length, γ_s is a normalizing factor, $\xi^s = (\xi_1^s, \ldots, \xi_n^s)$ is a random vector whose relative mathematical expectation (for each component) is

$$\mathbf{M}(\xi^s / x^0, x^1, \ldots, x^s) = c_s \hat{F}_x(x^s) + m^s, \quad s = 0, 1, \ldots, \tag{10}$$

where c_s is a nonnegative number, $m^s = (m_1^s, \ldots, m_n^s)$ is a vector, $\hat{F}_x(x^s)$ is the generalized gradient vector, i.e., the vector ξ^s satisfies a relation of the form (6). When $\mathfrak{D} = R^n$ and $\pi(x) = x$, the method (9)-(10) is called the method of generalized stochastic gradients, or, simply, the method of stochastic quasi-gradients.

The values of c_s and m^s in (10) may depend on $x^0, \ldots, x^s$, but we assume that we know constants l_s and r_s, depending only on s, for which $c_s(x^0, \ldots, x^s) \geq l_s$, $\| m^s(x^0, \ldots, x^s) \| \leq r_s$.

Let $\mathfrak{D}^*$ denote the set of solutions of the problem (7)-(8).

<u>THEOREM 1.</u> Suppose that we know the value of $h_s(x^0, \ldots, x^s)$, such that

$$\mathbf{M}(\| \xi^s \|^2 / x^0, x^1, \ldots, x^s) \leqslant h_s^2 \leqslant M_B < \infty \text{ for } \| x^k \| \leqslant B < \infty, \quad k = 0, 1, \ldots, s; \tag{11}$$

let the normalizing factor γ_s satisfy the condition

$$0 < \underline{\gamma} \leqslant \gamma_s(\tau_s \| x^s \| + h_s) \leqslant \bar{\gamma} < \infty, \tag{12}$$

where $\tau_s = 1$, if $\| m^s \| > 0$, and $\tau_s = 0$, if $\| m^s \| = 0$, and let the quantities ϱ_s, c_s, r_s be such that

$$\varrho_s \geqslant 0, \; c_s \geqslant 0, \; \sum_{s=0}^{\infty} \varrho_s r_s < \infty, \quad \sum_{s=0}^{\infty} \varrho_s^2 < \infty. \tag{13}$$

Then the sequence of points $\{x^s(\omega)\}$, defined by (9) and (10), is a random quasi-Féjer sequence with respect to the set $\mathfrak{D}^*$. But if, in addition,

$$\sum_{s=0}^{\infty} \varrho_s l_s = \infty, \tag{14}$$

then for almost all ω the sequence $\{x^s(\omega)\}$ converges to the solution of the problem (7)-(8).

We can easily verify the conditions of the theorem when solving actual problems, as will be shown below. Here we note only that

$$\mathbf{M}(\| \xi^s \|^2 / x^0, x^1, \ldots, x^s) = \sum_{i=1}^{n} \mathbf{D}(\xi_i^s / x^0, x^1, \ldots, x^s) + c_s^2 \| \hat{F}_x(x^s) \|^2 + 2c_s(\hat{F}_x(x^s), m^s) + \| m^s \|^2. \tag{15}$$

It follows from this, for example, that if the sum of the variances $\sum_{i=1}^{n} \mathbf{D}(\xi_i^s / x^0, x^1, \ldots, x^s)$ of the components of the vector $(\xi^s = \xi_1^s, \ldots, \xi_n^s)$ are bounded in $\mathfrak{D}$, and $\| F_x(x^s) \|$ is also bounded, then $h_s = \text{const}$, i.e., condition (11) holds. Obviously in actual problems the validity of (11) is a corollary of the boundedness of the domain $\mathfrak{D}$. It is also essential that F(x) be differentiability (see Note 3).

We begin by proving the first part of the theorem. Let x^* denote an arbitrary solution of the problem (7)-(8). Then

$$\|x^* - x^{s+1}\|^2 \leqslant \|x^* - x^s + \varrho_s \gamma_s \xi^s\|^2 = \|x^* - x^s\|^2 + 2\varrho_s \gamma_s (\xi^s, x^* - x^s) + \varrho_s^2 \gamma_s^2 \|\xi^s\|^2.$$

We take the relative mathematical expectation of both sides of this inequality:

$$\mathbf{M}(\|x^* - x^{s+1}\|^2/x^0, x^1, \ldots, x^s) \leqslant \|x^* - x^s\|^2 + 2\varrho_s \gamma_s c_s (\hat{F}_x(x^s), x^* - x^s) + 2\varrho_s \gamma_s (m^s, x^* - x^s) + \varrho_s^2 \gamma_s^2 \mathbf{M}(\|\xi^s\|^2/x^0, x^1, \ldots, x^s).$$

From this, taking note of (2), the Cauchy–Bunyakovskii inequality, and the fact that we can always assume that $\gamma_s \leq \gamma^* < \infty$, we obtain

$$\mathbf{M}(\|x^* - x^{s+1}\|^2/x^0, x^1, \ldots, x^s) \leqslant \|x^* - x^s\|^2 + 2\varrho_s r_s (\gamma^* \|x^*\| + \bar{\gamma}) + \bar{\gamma}^2 \varrho_s^2. \tag{17}$$

This inequality and (13) prove the first part of the theorem. We now show that if (14) holds, one of the limit points of the sequence $\{x^s(\omega)\}$ for almost all ω belongs to the set of solutions of the problem (7)-(8). From this, by Corollary 2, follows the proof of the second part of the theorem. From (16) we have

$$\mathbf{M}\|x^* - x^{s+1}\|^2 \leqslant \mathbf{M}\|x^* - x^0\|^2 + 2\sum_{k=0}^{s} \varrho_k l_k \mathbf{M}\gamma_k (\hat{F}_x(x^k), x^* - x^k) + 2(\|x^*\|\gamma^* + \bar{\gamma})\sum_{k=0}^{s} \varrho_k r_k + \bar{\gamma}^2 \sum_{k=0}^{s} \varrho_k^2 .$$

It follows from (17) that $\mathbf{M}\|x^* - x^{s+1}\|^2$ is uniformly bounded, and thus

$$\sum_{k=0}^{\infty} \varrho_k l_k \mathbf{M}\gamma_k (\hat{F}_x(x^k), x^* - x^k) > -\infty.$$

Since $\sum_{k=0}^{\infty} \varrho_k l_k = \infty$, we have $\mathbf{M}\gamma_k(\hat{F}_x(x^k), x^* - x^k) \to 0$ as $k \to \infty$. Hence, there is a subsequence $\{s_t\}$, t = 0, 1, ..., for which $\gamma_{s_t}(\omega)(\hat{F}_x(x^{s_t}(\omega)), x^* - x^{s_t}(\omega)) \to 0$ with probability 1 as $t \to \infty$. For almost all ω the sequence $\{\|x^s(\omega)\|\}$ is bounded and thus, noting (12), for almost all ω the sequence $\gamma_s(\omega)$ is bounded below, i.e., for almost all ω $(\hat{F}_x(x^{s_t}(\omega)), x - x^{s_t}(\omega)) \to 0$. Hence as $t \to \infty$, the sequence $\{x^{s_t}(\omega)\}$ converges to a solution of the problem (7)-(8), which is what we wished to prove.

Note 1. Obviously if the domain $\mathfrak{D}$ is bounded, Theorem 1 remains valid when $\gamma_s \equiv \text{const} > 0$.

Assume that $r_s/l_s \to 0$ as $s \to \infty$, and that there is a number Q such that for $\|x^* - x\| \geq Q$,

$$(\hat{F}_x(x), x^* - x) \leqslant -\alpha\|x^* - x\|, \tag{18}$$

where $\alpha > 0$. Then Theorem 1 remains valid if we replace (12) by the condition $0 < \gamma \leq \gamma_s h_s \leq \bar{\gamma} < \infty$. Indeed, it follows from (16) that

$$\mathbf{M}(\|x^* - x^{s+1}\|^2/x^0, x^1, \ldots, x^s) \leqslant \|x^* - x^s\|^2 + 2\varrho_s \gamma_s l_s \lambda_s \left[(\hat{F}_x(x^s), x^* - x^s) + \frac{r_s}{l_s}\|x^* - x^s\|\right] + 2\varrho_s \gamma_s (1 - \lambda_s)[l_s(\hat{F}_x(x^s), x^* - x^s) + r_s\|x^* - x^s\|] + \bar{\gamma}^2 \varrho_s^2 \tag{19}$$

for any λ_s. But if we take

$$\lambda_s = \begin{cases} 1, & \|x^* - x^s\| \geqslant Q, \\ 0, & \|x^* - x^s\| < Q, \end{cases}$$

we obtain

$$\mathbf{M}(\|x^* - x^{s+1}\|^2/x^0, x^1, \ldots, x^s) \leqslant \|x^* - x^s\|^2 + 2\varrho_s \gamma_s l_s \lambda_s \left[-\alpha\|x^* - x^s\| + \frac{r_s}{l_s}\|x^* - x^s\|\right] + 2\varrho_s \gamma_s (1 - \lambda_s)[l_s(\hat{F}_x(x^s), x^* - x^s) + r_s\|x^* - x^s\|] + \bar{\gamma}^2 \varrho_s^2.$$

Since, beginning with some constant time $s \geq S_0$,

$$\lambda_s\left[-\alpha\|x^* - x^s\| + \frac{r_s}{l_s}\|x^* - x^s\|\right] \leqslant 0,$$

for $s \geq S_0$ we have

$$M(\|x^* - x^{s+1}\|^2/x^0, x^1, \ldots, x^s) \leq \|x^* - x^s\|^2 + 2\varrho_s \gamma_s (1-\lambda_s)[l_s(\hat{F}_x(x^s), x^* - x^s)$$

$$+ r_s\|x^* - x^s\|] + \bar{\gamma}^2\varrho_s^2 \leq \|x^* - x^s\|^2 + 2Q\varrho_s r_s \gamma_s + \bar{\gamma}^2\varrho_s^2.$$

Hence, beginning at $s \geq S_0$ the sequence $\{x^s(\omega)\}$ is a quasi-Féjer sequence. We shall now prove the second part of the theorem. From (16) we have

$$M\|x^* - x^{s+1}\|^2 \leq M\|x^* - x^0\|^2 + 2\sum_{k=0}^{s} \varrho_k l_k M\gamma_k (\hat{F}_x(x^k), x^* - x^k) + 2\gamma^* \sum_{k=0}^{s} \varrho_k r_k M\|x^* - x^k\| + \bar{\gamma}^2 \sum_{k=0}^{s} \varrho_k^2.$$

By (17) the quantities $M\|x^* - x^k\|^2$ are uniformly bounded and thus

$$\sum_{k=0}^{\infty} \varrho_k l_k M\gamma_k (\hat{F}_x(x^k), x^* - x^k) > -\infty.$$

From this, as in the proof of the theorem, it follows that there is a subsequence $\{x^{s_t}(\omega)\}$ which converges to $x^*(\omega) \in \mathfrak{D}^*$ with probability 1.

Note 2. The permissible domain $\mathfrak{D}$ is usually specified by a set of inequalities, i.e., it can be represented as the intersection domains $\mathfrak{D} = \bigcap_{i=1}^{m} \mathfrak{D}_i$ of the number of domains. In this case we can use the operator for sequential projection, first onto $\mathfrak{D}_1$ and then onto $\mathfrak{D}_1 \cap \mathfrak{D}_2$, etc., and thus eventually onto $\bigcap_{i=1}^{m} \mathfrak{D}_i$. It is usually easier to make such a projection than to make a direct projection at once onto the whole domain $\mathfrak{D}$.

Let us now consider briefly the problem of the local convergence of the method (9)-(10), i.e., we shall not assume that $F(x)$ is convex downwards. First of all, analysis of the proof of Theorem 1 shows that the theorem remains valid for any $F(x)$ for which there is a set of minimum points $\mathfrak{D}^{**}$, such that $(\hat{F}_x(x), x^* - x) \leq 0$ for $x \bar{\in} \mathfrak{D}^{**}$ and $(\hat{F}_x(x), x^* - x) = 0$ for $x \in \mathfrak{D}^{**}$. Now assume that $F(x)$ is continuously differentiable that, $\{x : \|F_x(x)\| = 0\}$ is bounded, and that $\|F_x(x) - F_x(y)\| \leq \beta \|x - y\|$.

We consider only the simplest case, when $\mathfrak{D} = R^n$, so that the method (9)-(10) degenerates into the following:

$$x^{s+1} = x^s - \varrho_s \gamma_s \xi^s, \quad s = 0, 1, \ldots, \tag{20}$$

$$M(\xi^s/x^0, x^1, \ldots, x^s) = c_s F_x(x^s) + m^s. \tag{21}$$

Assume that $c_s(x^0, \ldots, x^s) \geq l_s$, $\|m^s(x^0, \ldots, x^s)\| \leq r_s$.

THEOREM 2. Let $M(\|\xi^s\|^2/x^0, x^1, \ldots, x^s) \leq h_s^2 \leq M_B < \infty$ for $\|x^k\| \leq B < \infty$, $k = 0, 1, \ldots, s$, and let the normalizing factor γ_s satisfy the condition $0 \leq \underline{\gamma} \leq \gamma_s h_s \leq \bar{\gamma} < \infty$. Moreover, let

$$\varrho_s \geq 0, \; l_s \geq 0, \; \frac{r_s}{l_s} \to 0, \; \sum_{s=0}^{\infty} \varrho_s l_s = \infty, \; \sum_{s=0}^{\infty} \varrho_s r_s < \infty, \; \sum_{s=0}^{\infty} \varrho_s^2 < \infty.$$

Then the sequence of points $\{x^s(\omega)\}$ defined by (20) and (21) is such that for almost all ω the sequence $\{F(x^s)\}$ converges; a subsequence $\{s_t\}$ exists for which $\|F_x(x^{s_t})\| \to 0$ for almost all ω. From this we have, in particular, that if $F(x)$ is convex downwards, $F(x^s)$ tends to the minimum of $F(x)$.

Indeed,

$$F(x^{s+1}) - F(x^s) = \int_0^1 F_\alpha(x^s - \alpha\varrho_s\gamma_s\xi^s)\, d\alpha$$

$$= \varrho_s\gamma_s \int_0^1 (F_x(x^s) - F_x(x^s - \alpha\varrho_s\gamma_s\xi^s), \xi^s)\, d\alpha - \varrho_s\gamma_s \int_0^1 (F_x(x^s), \xi^s)\, d\alpha \leq -\varrho_s\gamma_s(F_x(x^s), \xi^s) + \beta\varrho_s^2\gamma_s^2\|\xi^s\|^2.$$

After taking the relative mathematical expectation of both sides of this inequality, we obtain

$$M(F(x^{s+1})/x^0, x^1, \dots, x^s) \leq F(x^s) + \varrho_s \gamma_s l_s \times \left[\|F_x(x^s)\|^2 + \frac{r_s}{l_s} \|F_x(x^s)\| \right] + \bar{\gamma}\beta\varrho_s^2.$$

We introduce the factor λ_s such that

$$\lambda_s = \begin{cases} 1, & \|F_x(x^s)\| \geq Q, \\ 0, & \|F_x(x^s)\| < Q, \end{cases}$$

where $Q > 1$. Then from the preceding inequality, for some S_0 and $s \geq S_0$ (for simplicity assume that $S_0 = 0$), we obtain

$$M(F(x^{s+1})/x^0, x^1, \dots, x^s) \leq F(x^s) + (1-\lambda_s)\varrho_s\gamma_s[-l_s\|F_x(x^s)\|^2 + r_s\|F_x(x^s)\|] + \varrho_s^2\bar{\gamma}^2\beta \leq F(x^s) + Q\varrho_s r_s \gamma_s + \bar{\gamma}^2\beta\varrho_s^2.$$

This is equivalent to

$$M(z_{s+1}/z_0, z_1, \dots, z_s) \leq z_s, \quad s = 0, 1, \dots,$$

where $z_s = F(x^s) + \sum_{k=s}^{\infty} (Q\varrho_s r_s \gamma_s + \bar{\gamma}^2\beta\varrho_s^2)$. We can always assume that $F(x) \geq 0$, $\gamma_s \leq \gamma^*$ and thus the sequence $\{z_s\}$ forms a submartingale. Therefore, the sequence $\{F(x^s)\}$ converges with probability 1. As with Note 1, it is easy to show also that there is a sequence $\{s_t\}$, $t = 0, 1, \dots$, for which $\|F_x(x^{s_t})\| \to 0$ with probability 1.

Note 3. If the sum of the variances $\sum_{i=1}^{n} \mathfrak{D} \times (\xi_i^s/x^0, x^1, \dots, x^s)$ is bounded, then the method (20)-(21) converges for $\gamma_s = 1$ and the first condition in Theorem 2 can be dropped.

MINIMIZATION IN A LARGE NUMBER OF DIMENSIONS

In solving extremal problems with a large number of unknowns, conventional methods based on calculation of the gradient may become convenient because of time-consuming computations. In this case the method of random search can frequently be applied. It appears that a general class of such methods can be considered as a special case of the process (9)-(10).

Suppose that it is required to solve a problem of the form (7)-(8) in which $F(x)$ has bounded second derivatives. Then the vector ξ^s can be defined as follows. We consider the vector $\theta = (\theta_1, \dots, \theta_n)$ with independent components uniformly distributed in $[-1, 1]$.

Put

$$\xi^s = \sum_{k=1}^{p_s} \frac{F(x^s + \Delta_s\theta^{sk}) - F(x^s)}{\Delta_s} \theta^{sk}, \tag{22}$$

where θ^{sk}, $k = 0, 1, \dots, p_s$ is a series of independent observations of the vector θ at the s-th iteration and $p_s \geq 1$; $\Delta_s \geq 0$. It is easy to show that

$$M(\xi^s/x^s) = \frac{p_s}{3} F_x(x^s) + W^s\Delta_s, \tag{23}$$

where the vector W^s has bounded components, i.e., $\|W^s\| \leq$ const. Equation (22) is similar to (5). But whereas the calculation of the vector ξ^s from (22) requires $p_s + 1$ computations of the function $F(x)$, where $p_s \geq 1$, the calculation of F_x^Δ from (5) always requires $n + 1$ computations of $F(x)$ (and therefore the process (9), with the vector ξ^s in the form (22), can be more suitable than the corresponding determinate gradient method).

In analyzing and synthesizing a complex system there very frequently is no single analytic model describing its behavior, but rather one or more scenarios of the way in which the activity of the system develops. Each scenario may consist of a number of analytic models linked by definite logical and probabilistic transitions in which the elements are computing machines, games, or even actual objects.

In such a situation it may be possible to observe only the separate random results of the scenarios which are enacted and from this information we have to construct an adaptive search process for the unknown optimal values of the system parameters.

Since the information on the basis of which we must organize a purposeful search for the unknown parameters is random, the adaptive process itself is random. It is very natural to use the method (9)-(10) for this purpose since in it we take as the direction of motion from an arbitrary intermediate point x^s any vector ξ^s which is a statistical estimate of the gradient (and even the generalized gradient) of the function to be minimized.

Methods of adaptive minimization, or more accurately of adaptive maximization, of the form (9)-(10) were first considered in [10].

1. The Method of Stochastic Approximation. The following problem was considered in [10]. We have a random quantity $Y(x, \omega)$ whose distribution depends on the unknown vector $x = (x_1, \ldots, x_n)$. It is required to minimize the function*

$$F(x) = \mathbf{M} Y(x, \omega) \tag{24}$$

under the assumption that it is possible only to observe individual realizations of $Y(x, \omega)$ for any ω.

To solve this problem the method of stochastic approximation was proposed in [10] (see also [16]):

$$x^{s+1} = x^s - \varrho_s \sum_{j=1}^{n} \frac{Y(x^s + \Delta_s e^j, \omega^{sj}) - Y(x^s, \omega^{s0})}{\Delta_s} \times e^j, \quad s = 0, 1, \ldots, \tag{25}$$

where e^j is the unit vector along the j-th axis; $\omega^{s\nu}$, $\nu = 0, 1, \ldots$, are independent trials in the s-th iteration. It is assumed that the second derivatives of $F(x)$ are bounded. It is easy to see then that

$$\mathbf{M}\left(\sum_{j=1}^{n} \frac{Y(x^s + \Delta_s e^j, \omega^{sj}) - Y(x^s, \omega^{s0})}{\Delta_s} \times e^j / x^s\right) = F_x(x^s) + W^s \Delta_s, \tag{26}$$

where the norm of the vector $\| W^s \| \le \text{const}$, i.e., the method (25) is a process of the form (9)-(10) for $\pi(x) \equiv x$, $\mathfrak{D} = R^n$, and

$$\xi^s = \sum_{j=1}^{n} \frac{Y(x^s + \Delta_s e^j, \omega^{sj}) - Y(x^s, \omega^{s0})}{\Delta_s} e^j. \tag{27}$$

In (25) $\gamma_s \equiv 1$, since it was assumed in [10] (see Note 3) that the sum of the variances $\sum_{j=1}^{n} \mathfrak{D}(\xi_j^s / x^0, \ldots, x^s)$ is bounded. The problem of minimizing the functions (22) can be interpreted as follows.

We have a situation, each enactment of which yields the quantity $Y(x, \omega)$ defining the effectiveness of the system for fixed parameters $x = (x_1, \ldots, x_n)$. It is required to find the x for which the mathematical expectation (24) is minimal.

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* In order to avoid complicating the description of the set-theoretic assumptions about measurability and in integrability, we do not intent to dwell on them in this paper and we shall not adhere to the appropriate terminology.

The unique adaptive process proceeds in accordance with (25) to search for the required minimum of (21) omitting the complex and, for all practical purposes, unrealizable process of searching for the unknown distributions. But for large n this method may be absolutely ineffective. To calculate ξ^s from (27), which is the stochastic analog of (5), requires, as indicated in the previous section, n + 1 observations of $Y(x, \omega)$. In actual problems, one observation (game) takes a considerable time. Even if it lasts 0.5 min, with n = 60, one iteration of the method (25) takes 0.5 h.

In this case we can use instead of (25) a stochastic variant of the method considered in the previous section, i.e., we can calculate ξ^s from

$$\xi^s = \sum_{k=1}^{p_s} \frac{Y(x^s + \Delta_s \theta^{sk}, \omega^{sk}) - Y(x^s, \omega^{s0})}{\Delta_s} \theta^{sk}. \tag{28}$$

Moreover, it is necessary that $\mathfrak{D} = R^n$. If $F(x)$ has bounded second derivatives in $\mathfrak{D}$, a relation of the form (23) is also valid for (28).

2. <u>On a Stochastic Game Problem.</u> Let us suppose that we have a series of situations $i = 1, 2, \ldots, m$, in each of which $Y_i(x, \omega)$ defines the effectiveness of the plan $x = (x_1, \ldots, x_n)$. By playing through the scenario it is required to find the plan x which minimizes $F(x) = M \max_i Y_i(x, \omega)$ for $x \in \mathfrak{D}$.

Let $Y_{i(x, \omega)}(x, \omega) = \max_i Y_i(x, \omega)$; $Y_i(x, \omega)$ is twice differentiable; for any x, y in $\mathfrak{D}$, we have

$$F(y) - F(x) \geqslant (M \operatorname{grad} Y_{i(x,\omega)}(x, \omega), y - x). \tag{29}$$

where $\operatorname{grad} Y_{i(x, \omega)}(x, \omega)$ is calculated for $x = (x_1, \ldots, x_n)$ in $Y_i(x, \omega)$, for $i = i(x, \omega)$. In this case the vector ξ^s of the process (9)-(10) is defined by

$$\xi^s = \sum_{j=1}^{n} \frac{Y_{i_s}(x^s + \Delta_s e^j, \omega^s) - Y_{i_s}(x^s, \omega^s)}{\Delta_s} e^j, \tag{30}$$

or

$$\xi^s = \sum_{k=1}^{p_s} \frac{Y_{i_s}(x^s + \Delta_s \theta^{sk}, \omega^s) - Y_{i_s}(x^s, \omega^s)}{\Delta_s} \theta^{sk}, \tag{31}$$

where the vectors e^j and θ^{sk} are as defined in Section 3, and $i_s = i(x^s, \omega^s)$. It is easy to see that, by (29), the vectors (30) and (31) satisfy (10). We note that (29) holds, for example, if $Y_i(x, \omega)$ is convex downwards for any ω.

Consider briefly one iteration of the above adaptive process with the vector (31). Suppose that we have already found the point x^s. We observe a realization $Y_i(x^s, \omega^s)$ and find i_s from $Y_{i_s}(x^s, \omega^s) = \max_i Y_i(x^s, \omega^s)$. We make $p_s \geq 1$ independent observations of the vector θ, and find ϱ_s, Δ_s from (13) and and (14); we compute $Y_{i_s}(x + \Delta_s \theta^{sk}, \omega^s)$, for which we must fix ω^s; we compute ξ^s from (31), x^{s+1} from (9), etc.

3. <u>A Problem in Two-Stage Stochastic Programming.</u> These very important problems in decision-making for an indeterminate future were first considered in [11]. The term "two-stage" should not be taken to mean that there are only two variable planning stages. There are only two stages in the determination of the solution: construction of the plan while the future is unknown, and correction of that plan as the future becomes known. The plan is chosen so that the cost of realization and correction are minimal "on the average."

The formal formulation of the linear problem in two-stage stochastic programming is as follows. Assume that the plan $x = (x_1, \ldots, x_n)$, adopted for some interval of time in the future, must satisfy the condition

$$A(\omega) x + B(\omega) y \leqslant b(\omega). \tag{32}$$

The plan x is adopted until the values $A(\omega)$, $B(\omega)$, $b(\omega)$ become known. Then when they become known, (32) is corrected by the vector y. If $(d(\omega), y)$ is the cost of correction, we can find a $y(x, \omega)$ which minimizes

$$(d(\omega), y) \tag{33}$$

subject to (32), where the vector x, and also $A(\omega)$, $B(\omega)$, and $b(\omega)$ are fixed. The problem consists in finding the vector x which minimizes

$$F(x) = (c, x) + \mathbf{M}(d(\omega), \ y(x, \omega)) \tag{34}$$

subjet to $x \in \mathfrak{D}$, where (c, x) is the cost of the realization of the plan x. The case in which the matrices $A(\omega)$ and $B(\omega)$ and the vector $d(\omega)$ are determinate was considered in [11]. Only the vector $b(\omega)$ is random, and it takes a finite number of values with given probabilities, and $\mathfrak{D} \equiv R^n$. The method (9)-(10) makes it possible to solve the problem of two-stage stochastic programming in more general form.

Suppose that in addition to $y(x, \omega)$ we can obtain $u(x, \omega) = (u_1(x, \omega), \ldots, u_m(x, \omega))$, which are dual variables corresponding to $y(x, \omega)$. Let x^s denote the approximation obtained at the s-th iteration. We observe $A(\omega^s)$, $B(\omega^s)$, $b(\omega^s)$, $d(\omega^s)$ and solve the minimization problem (33) for $x = x^s$, $\omega = \omega^s$ under the conditions (32). We determine $y(x^s, \omega^s)$, $u(x^s, \omega^s)$, and calculate x^{s+1} from (9) for

$$\xi^s = c - A^T(\omega^s)\, u(x^s, \omega^s).$$

It is easy to show [8] that if F(x) is defined by (34), and ξ^s by the above equation, then

$$\mathbf{M}(\xi^s / x^s) = \hat{F}_x(x^s).$$

In [8] the adaptive process described here is extended to general nonlinear problems in two-stage stochastic programming. We note that the problem of minimizing the nondifferentiable function (34) can be considered as a stochastic problem in parametric programming and the above process as a unique method for separable programming (if the matrix B is a block matrix).

4. Programmed Control of a Random Process. Suppose that the behavior of an object on which random perturbations act is described by a set of difference equations:

$$x_i(k+1) = x_i(k) + f_i(x, y, k, \omega), \quad x_i(0) = x_i^0, \ i = 1, 2, \ldots, n; \quad k = 0, 1, \ldots, N-1, \tag{35}$$

where the control vector $y(k) = (y_1(k), \ldots, y_m(k))$ at each moment of time $k = 0, 1, \ldots, N-1$, belongs to the domain $\mathfrak{D}$, i.e.,

$$y(k) \in \mathfrak{D}, \quad k = 0, 1, \ldots, N-1. \tag{36}$$

It is required to find a programmed control, i.e., a vector function $y(k)$, $k = 0, 1, \ldots, N-1$ satisfying (36), for which the function

$$F(y(0), \ldots, y(N-1)) = \mathbf{M}Y(x(N), \omega) \tag{37}$$

has its least value for fixed N.

In this case the method (9)-(10) corresponds to the following adaptive process. Suppose that after the s-th iteration we have obtained the approximation $y^s(k)$, $k = 0, 1, \ldots, N-1$.

We observe ω^s and from (35) find the trajectory $x^s(k)$, $k = 0, 1, \ldots, N$ and also the solution $\lambda^s(k)$, $k = N, \ldots, 0$ of the following system of paired equations for $x(k) = x^s(k)$, $y(k) = y^s(k)$, $\omega = \omega^s$.

$$\lambda_j(k) = \lambda_j(k+1) + \sum_{i=1}^{n} \lambda_i(k+1) f_{ix_j}(x, y, k, \omega),$$

$$\lambda_j(N) = -Y_{x_j}(x(N), \omega), \quad k = N-1, \ldots, 0; \quad j = 1, 2, \ldots, n.$$

We put

$$\xi^s(k) = \sum_{i=1}^{n} \lambda_i^s(k+1) f_{iy}(x^s, y^s, k, \omega^s)$$

and consider the set of vectors $\xi^s = (\xi^s(0), \ldots, \xi^s(N-1))$.

The set ξ^s is the stochastic gradient of (37) as a function of the variables $y(0), \ldots, y(N-1)$, i.e.,

$$\mathbf{M}(\xi^s / y^s(0), \ldots, y^s(N-1)) = \operatorname{grad} F(y^s(0), \ldots, y^s(N-1)),$$

and thus a new approximation y^{s+1} can be obtained using (9). This method is valid when the function (37) is convex downwards, and $\mathfrak{D}$ is a convex set. If this is not the case, we can discuss its convergence to a local minimum.

5. Application of the Stochastic Principle of the Maximum. Assume that the behavior of the object is described by the following system of difference equations:

$$x(k+1) = x(k) + A(k, \omega) x(k) + g(y(k), k, \omega), \quad x(0) = x^0, \quad k = 0, 1, \ldots, N-1, \tag{38}$$

where $x(k) = (x_1(k), \ldots, x_n(k))$, $g(y, k, \omega) = (g_1(y, k, \omega), \ldots, g_n(u, k, \omega))$.

It is required to find a programmed control $y(k)$ minimizing the mathematical expectation

$$\mathbf{M}(c(\omega), x(N)) \tag{39}$$

subject to the conditions

$$y(k) \in \mathfrak{D}, \; k = 0, 1, \ldots, N-1. \tag{40}$$

Using the theory of duality in mathematical programming as employed in [15], we can show that the following decomposition principle (the stochastic principle of the maximum) is valid for the problem (38)-(40): the required control $y(k)$ at times $k = 0, 1, \ldots, N-1$, must be chosen so that

$$\mathbf{M}(\lambda(k+1), g(y(k), k, \omega)) = \max_{v \in \mathfrak{D}} \mathbf{M}(\lambda(k+1), g(v, k, \omega)), \tag{41}$$

$$\lambda(k) = \lambda(k+1) + A'(k, \omega) \lambda(k+1), \quad \lambda(N) = -c(\omega), \quad k = N-1, \ldots, 0. \tag{42}$$

To solve the N problems (41) we can use the methods described at the beginning of this section.

6. Choice of Initial State for the Controlled Object. Usually the behavior of the controlled object can be determined by the choice of the control and the initial state. In certain practical problems the initial state is unknown and it is required to choose it optimally. We consider one of these problems. Let the behavior of the object be described by the system of difference equations

$$x(k+1) = x(k) + A(k, \omega) x(k) + g(y(k), k, \omega), \; x(0) = a, \tag{43}$$

$$y(k) \in \mathfrak{D}, k = 0, 1, \ldots, N-1. \tag{44}$$

The state a is not determined but it is known that $a \in \mathfrak{B}$, where $\mathfrak{B}$ is a convex, closed set. Fixing the state $a \in \mathfrak{B}$, the random event ω and the control $y(k)$, $k = 0, 1, \ldots, N-1$, we obtain a certain value for the goal function $y(a, x(N), \omega)$.

Let $\Phi(a, \omega)$ denote the least value of the function $Y(a, x(N), \omega)$ for fixed a and ω, and let $y(a, k, \omega)$ denote the optimal control and $x(a, k, \omega)$ the optimal trajectory.

It is required to choose an $a \in \mathfrak{B}$ for which the mathematical expectation $G(a) = \mathbf{M}\Phi(a, \omega)$ takes on its least value.

Suppose that for any $x(N)$ and ω the function $Y(a, x(N), \omega)$ is convex downwards in the variables $a = (a_1, \ldots, a_n)$, where $\hat{Y}_a$ is the generalized gradient vector. For any a and ω this function is linear in $x(N)$. Together with the system (43) we consider its associated system

$$\lambda(k) = \lambda(k+1) + A'(k, \omega) \lambda(k+1), \quad \lambda(N) = -Y_x(a, x(N), \omega), \; k = N-1, \ldots, 0. \tag{45}$$

Then the following adaptive process can be proposed for the solution of the original problem. We fix some $a^0 \in \mathfrak{B}$ and ω^0 and find $x(a^0, N, \omega^0)$. Substituting this in (45), we find $\lambda(a^0, 0, \omega^0)$. Suppose we know a^s at the s-th iteration. We observe ω^s, find $x(a^s, N, \omega^s)$, and from (45), find $\lambda(a^s, 0, \omega^s)$. Then we determine a^{s+1} from

$$a^{s+1} = \pi(a^s - \rho_s \gamma_s \xi^s), \quad s = 0, 1, \ldots,$$

where

$$\xi^s = \hat{Y}_a(a^s, x(a^s, N, \omega^s), \omega^s) - \lambda(a^s, 0, \omega^s).$$

It can be shown that here

$$\mathbf{M}(\xi^s/a^s) = \hat{G}_a(a^s),$$

i.e., this method is a particular case of the method (9)-(10) and hence, for appropriate ϱ_s, γ_s, it converges with probability 1 to the minimum of the function $G(a)$.

SOLUTION OF SYSTEMS OF INEQUALITIES

The solution of systems of inequalities can easily be reduced to the solution of an extremal problem of the form (7)-(8). Suppose it is required to find the solution of the system of inequalities

$$F^i(x_1, \ldots, x_n) \leqslant 0, \quad i = 1, 2, \ldots, m, \tag{46}$$

subject to the conditions

$$(x_1, \ldots, x_n) \in \mathfrak{D}. \tag{47}$$

Consider the function $F(x) = \max_i F^i(x)$. Obviously, for the function $F(x)$ chosen in this way the solution of the problem (7)-(8) satisfies (46)-(47) if a solution of (46)-(47) exists. Hence we can apply the method (9)-(10) to solve the inequalities (46)-(47). But a feature of solution of systems of inequalities by the method of minimizing the function $F(x) = \max_i F^i(x)$ is that it is sufficient to continue minimizing until (46) is satisfied. Hence, whenever there is an effective method of verifying the conditions (46) there is a unique method of solving (46)-(47). A very wide class of methods of solving the nonlinear problem (46)-(47) is based on [13] (this paper is based on an idea closely related to ideas in [12]).

We consider briefly a generalization of the methods of [13] to the case in which the gradient of the nonlinear functions (46) can be calculated exactly [1].

Let $\mathfrak{F} = \{x : f(x) \leq 0\}$ be a convex closed set in $\mathbf{R}^n$, and $\mathfrak{D}$ also a convex closed set such that $\mathfrak{F} \cap \mathfrak{D}$. Consider the random sequence of points $\{x^k, \bar{x}^k\}$, defined (for arbitrary $x^0 \in \mathbf{R}^n$) by

$$x^{s+1} = \pi(\bar{x}^s), \tag{48}$$

$$\bar{x}^s = \begin{cases} x^s - \varrho_s\gamma_s f(x^s)\xi^s, & f(x^s) > 0, \\ x^s, & f(x^s) \leqslant 0. \end{cases} \tag{49}$$

Here ϱ_s is the step length, γ_s is a normalizing factor, and ξ^s is a random vector whose relative mathematical expectation is

$$\mathbf{M}(\xi^s/x^0, \bar{x}^0, \ldots, x^s) = c_s q^s + m^s, \tag{50}$$

where $c_s \geq 0$, m^s is a vactor, and q^s is the vector for which the half-space corresponding to

$$(q^s, y - x^s) + f(x^s) \leqslant 0, \tag{51}$$

contains the set $\mathfrak{F}$, if $x^s \in \mathfrak{F}$. Suppose that $c_x(x^0, \bar{x}^0, \ldots, x^s) \geq l_s$, $\| m^s, (x^0, \bar{x}^0, \ldots, x^s) \| \leq r_s$. The following assertion holds [1].

THEOREM 3. If we know an h_s such that

$$\mathbf{M}(\|\xi^s\|^2/x^0, \bar{x}^0, \ldots, x^s) \leqslant h_s^2 \leqslant M_B < \infty \text{ for } \|x^k\| + \|\bar{x}^k\| \leqslant B < \infty,\ k=0, \ldots, s-1;\ \|x^s\| \leqslant B,$$

where the normalizing factor γ_s satisfies the condition

$$0 < \gamma_s(\tau_s\|x^s\| f^2(x^s) + h_s) < 1, \quad \tau_s = \begin{cases} 1, & \|m^s\| > 0, \\ 0, & \|m^s\| = 0, \end{cases}$$

$$0 \leqslant \varrho_s \leqslant 2l_s - \varepsilon_s, \quad \sum_{s=0}^{\infty} \varrho_s r_s < \infty,$$

then the random sequence $\{x^s, \bar{x}^s\}$, defined (for arbitrary $x^0 \in R^n$ by (48)-(49), is a quasi-Féjer sequence with respect to the set $\mathfrak{F} \cap \mathfrak{D}$. But if in addition

$$\sum_{s=0}^{\infty} \varrho_s \varepsilon_s = \infty,$$

then it converges to some element of the set $\mathfrak{F} \cap \mathfrak{D}$ with probability 1.

Consider an example. Suppose it is required to find the solution of the system of inequalities

$$f^i(x_1, \ldots, x_n) \leqslant 0, \qquad i = 1, 2, \ldots, m, \tag{52}$$

$$x = (x_1, \ldots, x_n) \in \mathfrak{D}. \tag{53}$$

We put $f(x) = \max_i f^i(x)$, $\mathfrak{F} = \{x : f(x) \leq 0\}$,

$$\xi^s = \sum_{k=1}^{p_s} \frac{f^{i_s}(x^s + \Delta_s \theta^{sk}) - f^{i_s}(x^s)}{\Delta_s} \theta^{sk}, \tag{54}$$

where the $p_s \geq 1$, θ^{sk}, $k = 1, \ldots, p_s$ are independent realizations of the random vector $\theta = (\theta_1, \ldots, \theta_n)$ with independent components uniformly distributed in [1, 1]; i_s is such that $f^{i_s}(x^s) = \max_i f^i(x^s)$.

The relative mathematical expectation of the vector (54) can be put in the form

$$\mathbf{M}(\xi^s/x^s) = \frac{p_s}{3} f_x^{i_s}(x^s) + W^s \Delta_s,$$

where the vector W^s has bounded components, i.e., $\| W^s \| \leq$ const if the functions $f^i(x)$ have bounded second derivatives in $\mathfrak{D}$. But if these functions are also convex downwards, the vector $q^s = f_x^{i_s}(x^s)$ satisfies (51).

Thus, in this case the process (48)-(49) corresponds to the random search method for solution of the system (52)-(53).

CONCLUSION

The results described in this paper can be extended to the case of Hilbert spaces, with small changes due mostly to the replacement of the word "vector" by the word "element." Of greater interest from the practical point of view is the generalization of the method (9)-(10) to extremal problems with conditions of the form

$$F^i(x_1, \ldots, x_n) \leqslant 0, \quad i = 1, \ldots, m,$$

for which projection for some reason cannot be made (for example, if the functions $F^i(x)$ are not defined analytically and only statistical estimates of their values can be obtained). A stochastic variant of the Arrow–Hurwitz method was proposed in [7] for the solution of such problems.

It is of particular importance to find new means, different from (13)-(14), of controlling the step length ϱ_s. Apart from the fact that the control (13)-(14) has a rigid, programmed nature, it is determinate, i.e., it does not depend on the actual trajectory of the "descent." The process (9)-(10) is random and defines a whole family of trajectories from the initial point to the minimum point. The control (13)-(14) is calculated once for the whole family, so sufficiently rapid convergence cannot be expected from it. For a stochastic method of the form (9)-(10), stochastic means of control (see [17]) depending on the previous history of the "descent" are more natural.

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