DIFFERENTIAL GAMES OF PRESCRIBED DURATION

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In this article differential games with prescribed times of ending are considered. These games constitute a special class which can be investigated in greater detail.

A differential game of prescribed duration can be formulated as follows. Let a certain controllable object be described by the system of differential equations

$$\dot{z} = f(z, u, v), \tag{1}$$

where $z = (z^1, \ldots, z^n)$ is a point of an n-dimensional space E^n, $u = (u^1, \ldots, u^r)$ is the control of player P, and $v = (v^1, \ldots, v^s)$ is the control of player E. At each instant t, the players choose their controls u and v from certain sets U and V, respectively, proceeding only from the knowledge of the phase position of the object z(t) at the given instant.

Thus, we can take that the controls of the players are functions of the phase coordinates $u = u(z)$, $v = v(z)$, with $u(z) \in U$, $v(z) \in v$.

If the controls u(z) and v(z) are chosen, the system (1) is seen to be closed, i.e., for each initial point z_0 the trajectory $z(t) = z(z_0, u, v, t)$ corresponding to the controls u(z) and v(z) is uniquely determined together with the pay-off of the game. The pay-off is given by the functional

$$I(z_0, u, v) = \int_0^{t_1} f_0(z, u, v)\, dt + F(z(t_1)), \tag{2}$$

where t_1 is the instant at which the game ends, $f_0(z, u, v)$ is a function defined on $E^n \times U \times V$, $F(z(t_1))$ is a function of the final state.

The game consists of the opponent P endeavoring at each instant of time to choose his control such that the value of the pay-off would be as small as possible; the aim of player E is the opposite. It is obvious that the question about the existence of admissible strategies does not arise for such games.

Differential games of prescribed durations were considered by Fleming [1-3]. He investigated the problem of existence of optimal strategies and the value of a differential game as a limiting case of a multistep game, when the number of steps increases without bounds. Fleming showed that, for fairly general assumptions, there exists a limit for the value of a discrete game which approximates the original game. However, in a general case it is not clear how this limit is connected with the differential game. Also, it is not clear whether or not the discrete strategies in the limit can generate a certain strategy for the original differential game.

We shall consider a differential game ending at the instant $t = t_1$ whose pay-off is given by the functional

$$I(z_0, u, v) = \begin{cases} +\infty, & \text{if } z(t_1) \bar{\in} M, \\ p(z(t_1)), & \text{if } z(t_1) \in M, \end{cases} \tag{3}$$

where M is a certain given set, $M \subset E^n$, and p(z) is a function defined on all M, $z(t_1) = z(z_0, u, v, t_1)$. The question here is about the existence of an admissible control u which is such that $z(t_1) \in M$ exists for any admissible control of player E.

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Certain sufficient conditions of ending such a game with a value c for a linear object were obtained in [4]. The solution of the problem is reduced to the question: when can a given set be reached exactly at given instant of time?

In the present work, we investigate further properties of games of prescribed duration and a pay-off given by the function (3). For this strategies of a special form are used. These are strategies introduced by B. N. Phsenichnyi [5, 6].

In the first section we present a full description and methods of constructing a region from which the game can be finished with a value c.

For certain linear games, in view of the specific features of the problem, this approach allowed the structure of the game to be investigated fairly fully.

1. The Structure of Differential Games of Fixed Duration

Let a game be described by the system of differential equations

$$\dot{z} = f(z, u, v), \tag{1.1}$$

$z \in E^n$, $u \in U$, $v \in V$; the set U and V are compact.

We assume about the right sides of the system (1.1) that the following conditions are fulfilled:

a) $f(z, u, v)$ is continuous with respect to z, u and v for $z \in E^n$, $u \in U$, $v \in V$, and it is continuously differentiable with respect to z;

b) $|(z, f(z, u, v))| \leqslant C(1 + \|z\|^2)$ for all $z \in E^n, u \in U, v \in V$;

c) the set $f(z, U, v)$, covered by the vector $f(z, u, v)$ when u covers U, is convex for all $z \in E^n$, $v \in V$.

The pay-off of the game is specified in the form

$$I(z_0, u, v) = \begin{cases} +\infty, & \text{if } z(t_1) \bar{\in} M, \\ p(z(t_1)), & \text{if } z(t_1) \in M, \end{cases} \tag{1.2}$$

where M is a closed set, $M \subset E^n$, p(z) is continuous function defined on the entire M, and $z(t_1) = z(z_0, u, t_1)$ is the point where the trajectory $z(t) = z(z_0, u, v, t)$ of the system (1.1), starting from the point $z(0) = z_0$ and corresponding to the controls u and v chosen, is located at the instant $t = t_1$ of the finish of the game.

In the usual formulation of the problem the players P and E must at each instant t choose their controls, proceeding only from knowledge of the phase coordinates, i.e., as functions of the state $u = u(z)$, $v = v(z)$, $u(z) \in U$, $v(z) \in V$. Of the functions u(z) and v(z), we can say that they must be fairly smooth, so that a solution to system (1.1) would exist. Their class must be fairly broad, so that optimal controls would exist.

We now introduce a new form of strategies. These are ε strategies which presuppose a certain discrimination of player E. It is clear that this does not enable us to pose the question about the existence of a saddle point, but in return it allows us to avoid the difficulties connected with a correct choice of a class of admissible controls. We shall investigate the question: what can player P be sure of in this case?

Let the game be started at the point $z(0) = z_0$ and at the instant $t = 0$ player E chooses a certain number $\varepsilon_1 > 0$ and his control $v(\tau)$, $0 \le \tau \le \varepsilon_1$, $v(\tau) \in \Omega_V$, and reports it to the opponent P. On this interval, P chooses his control $u(\tau)$, $0 \le \tau \le \varepsilon_1$, $u(\tau) \in \Omega_U$, starting already from the knowledge of not only $z(\tau)$, but also v(z). Here Ω_U and Ω_V are the sets of all possible measurable functions with values in U and V respectively. At the instant $t = \varepsilon_1$ E chooses $\varepsilon_2 > 0$ and $v(\tau)$, $\varepsilon_1 \le \tau \le \varepsilon_1 + \varepsilon_2$, and reports them to player P. The latter chooses $u(\tau)$, $\varepsilon_1 \le \tau \le \varepsilon_1 + \varepsilon_2$, from $z(\tau)$ and the control $v(\tau)$, and so forth until the instant at which the game is ended.

Strategies introduced in this way are called ε strategies. Although the control is chosen by player P not only on the basis of the local information, but also from the knowledge of the future behavior of the opponent E, this knowledge can be arbitrarily small, in the sense that the length of the time interval on which E reports his future control can be made arbitrarily small.

We shall say that the game which starts at the point z_0 can be finished with a value c, if for any ε-strategy $v(\tau)$ chosen by E, player P finds an ε-strategy $u(\tau)$ which is such that

$$I(z_0, u(\tau), v(\tau)) \leqslant c,$$

i.e.,

$$z(t_1) = z(z_0, u(\tau), v(\tau), t_1) \in M \text{ and } p(z(t_1)) \leqslant c.$$

We want to find out what pay-off can be secured by Player P if the game commences at the instant t = 0 at the point z_0.

We now set up definitions and lemmas which will be used to prove a fundamental theorem which gives an answer to this question.

Definition 1. The operator T_ε, $\varepsilon \geq 0$, matches each set $X \subset E^n$ by the set of $T_\varepsilon(X) \subset E^n$ of points $z \in E^n$ which are such that for any control $v(\tau) \in \Omega_u$, $0 \leq \tau \leq \varepsilon$, there exists $u(\tau) \in \Omega_u$, $0 \leq \tau \leq \varepsilon$, and such that for these controls the trajectory of the system (1.1) with the start at the point $z(0) = z_0$ falls into the set X exactly at the instant $t = \varepsilon$ (it is assumed that the control $v(\tau)$, $0 \leq \tau \leq \varepsilon$ is known to player P). If $X = \emptyset$, then $T_\varepsilon(X) = \emptyset$.

LEMMA 1. The operator T_ε has the properties:

1) $T_0(X) = X$;

2) $T_\varepsilon(X) \subset T_\varepsilon(X')$, if $X \subset X'$;

3) $T_{\varepsilon_1} T_{\varepsilon_2}(X) \subset T_{\varepsilon_1+\varepsilon_2}(X)$;

4) $T_\varepsilon(X)$ is closed, if X is a closed set, and if $z_0 \in T_\varepsilon(X)$ for any $\varepsilon > \varepsilon_0$, then $z_0 \in T_{\varepsilon_0}(X)$;

5) for a collection of sets X_α, $\alpha \in A$, $\bigcap_\alpha T_\varepsilon(X_\alpha) \supset T_\varepsilon\left(\bigcap_\alpha X_\alpha\right)$;

6) if $\{X_i\}$ is a sequence of closed sets enclosed by one another, $X_{i+1} \subset X_i$, then

$$\bigcap_{i=1}^{\infty} T_\varepsilon(X_i) = T_\varepsilon\left(\bigcap_{i=1}^{\infty} X_i\right).$$

Proof. Properties 1 and 2 follow directly from the definition of the operator T_ε. Property 5 is a consequence of property 2. We shall prove 3. Let $z_0 \in T_{\varepsilon_1} T_{\varepsilon_2}(X)$ and let $v(\tau)$, $0 \leq \tau \leq \varepsilon_1 + \varepsilon_2$ be an arbitrary control of player E. According to the definition of T_{ε_1}, for any $v_1(\tau)$, $0 \leq \tau \leq \varepsilon_1$ (we take $v_1(\tau) = v(\tau)$, $0 \leq \tau \leq \varepsilon_1$) there exists $u_1(\tau)$, $0 \leq \tau \leq \varepsilon_1$ which is such that

$$z(\varepsilon_1) = z(z_0, u_1(\tau), v_1(\tau), \varepsilon_1) \in T_{\varepsilon_2}(X).$$

This means that for any $v_2(\tau)$, $\varepsilon_1 \leqslant \tau \leqslant \varepsilon_1 + \varepsilon_2$ (we take $v_2(\tau) = v(\tau)$, $\varepsilon_1 \leqslant \tau \leqslant \varepsilon_1 + \varepsilon_2$) there exists $u_2(\tau)$ such that the trajectory of the system (1.1), beginning at the instant $t = \varepsilon_1$ at the point $z(\varepsilon_1)$ and corresponding to the controls $u_2(\tau)$, $v_2(\tau)$, $\varepsilon_1 \leqslant \tau \leqslant \varepsilon_1 + \varepsilon_2$, falls into X exactly at the instant $t = \varepsilon_1 + \varepsilon_2$.

Thus, if $z_0 \in T_{\varepsilon_1} T_{\varepsilon_2}(X)$, then for an arbitrary control $v(\tau)$, $0 \leqslant \tau \leqslant \varepsilon_1 + \varepsilon_2$, there exists a control $u(\tau)$ of player P which can be chosen in the form

$$u(\tau) = \begin{cases} u_1(\tau), & 0 \leqslant \tau \leqslant \varepsilon_1, \\ u_2(\tau), & \varepsilon_1 \leqslant \tau \leqslant \varepsilon_1 + \varepsilon_2, \end{cases} \tag{1.3}$$

so that $z(t) = z(z_0, u(\tau), v(\tau), t)$ falls into X exactly at the instant $t = \varepsilon_1 + \varepsilon_2$. This however, means that $z_0 \in T_{\varepsilon_1+\varepsilon_2}(X)$. Property 3 has thus been proved. We omit the proof of properties 4 and 6. We only note they are based on the use of the properties already proved and Theorem 1 [7].

Definition 2. The rational partition $\omega = \{\tau_i\}$ of the interval [0, t] is the term given to its arbitrary partition by points τ_i such that $\tau_0 = 0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_{m-1} \leqslant t$ (τ_{m-1} can be equal to t if and only if t is rational), m is arbitrary, and all τ_i are rational; $|\omega| = t$, where t is the length of the interval to be partitioned.

Each rational partition ω can be matched with the operator

$$T_\omega(X) = T_{\varepsilon_1} T_{\varepsilon_2} \dots T_{\varepsilon_m}(X),$$

where $\varepsilon_i = \tau_i - \tau_{i-1}$, $\varepsilon_m = t - \tau_{m-1}$.

Definition 3. We shall say that the partition $\omega' = \{\tau'_j\}$ of the interval [0, t], is finer than $\omega = \{\tau_i\}$ (we denote it by $\omega' \prec \omega$), if all points τ_i are amongst τ'_j, i.e., $\tau_i = \tau'_{j(i)}$.

LEMMA 2. If $\omega' \prec \omega$, then $T_{\omega'}(X) \subset T_\omega(X)$.

Proof. Let $\omega = \{\tau_i\}$, $\omega' = \{\tau'_j\}$ and $\omega' \prec \omega$. This means that for each τ_i there exists τ'_j such that $\tau_i = \tau'_j$. We denote by j(i) the smallest number j such that $\tau_i = \tau'_j$. Then $\varepsilon_i = \tau_i - \tau_{i-1} = \tau'_{j(i)} - \tau'_{j(i-1)} = \sum_{j=j(i-1)+1}^{j(i)} \varepsilon'_j$.

In view of property 3 of Lemma 1

$$T_{\varepsilon_i}(X) \supset T_{\varepsilon_{j(i-1)+1}} T_{\varepsilon_{j(i-1)+2}} \dots T_{\varepsilon_{j(i)}}(X).$$

Then obviously

$$T_\omega(X) = T_{\varepsilon_1} T_{\varepsilon_2} \dots T_{\varepsilon_m}(X)$$

$$\supset T_{\varepsilon_{j(0)+1}} \dots T_{\varepsilon_{j(1)}} T_{\varepsilon_{j(1)+1}} \dots T_{\varepsilon_{j(2)}} \dots T_{\varepsilon_{j(k)+1}} \dots T_{\varepsilon_{j(k+1)}} \dots T_{\varepsilon_{j(m-1)+1}} \dots T_{\varepsilon_{j(m)}}(X) = T_{\omega'}(X).$$

The lemma has been proved.

Definition 4. $\widetilde{T}_t(X) = \bigcap_{|\omega|=t} T_\omega(X)$.

LEMMA 3.

1) $\widetilde{T}_0(X) = X$;

2) $\widetilde{T}_t(X) \subset \widetilde{T}_t(X')$, if $X \subset X'$;

3) if X is a closed set, then $\widetilde{T}_t(X)$ is also closed;

4) if X is closed and $z \in \widetilde{T}_t(X)$ for all $t > t_0$, then $z \in \widetilde{T}_{t_0}(X)$. The proof of the lemma follows directly from the definitions introduced above and from the properties of the operator T_ε.

LEMMA 4. If X is a closed set, then

$$\widetilde{T}_{\theta_1+\theta_2}(X) = \widetilde{T}_{\theta_1} \widetilde{T}_{\theta_2}(X)$$

for $\theta_1, \theta_2 \geq 0$, $\theta_1 + \theta_2 \leq t_1$, i.e., the operators form a semi-group.

Proof. We assume that θ_1 is a rational number. Each partition ω of the interval $[0, \theta_1 + \theta_2]$ is matched with the partition ω' obtained from ω by adding another partition point $\tau = \theta_1$:

$$\omega' = \{\tau_0^1 = 0, \tau_1^1, \dots, \tau_{m_1-1}^1, \tau = \theta_1, \tau_1^2, \dots, \tau_{m_2-1}^2\}.$$

It is obvious that $\omega' \prec \omega$.

It is easy to verify that

$$\bigcap_{|\omega|=\theta_1+\theta_2} T_\omega(X) = \bigcap_{|\omega'|=\theta_1+\theta_2} T_{\omega'}(X). \tag{1.4}$$

Indeed, since the set of partitions ω' is narrower than the set of all possible partitions ω, we have

$$\bigcap_{|\omega|=\theta_1+\theta_2} T_\omega(X) \subset \bigcap_{|\omega'|=\theta_1+\theta_2} T_{\omega'}(X). \tag{1.5}$$

Let $z_0 \in \bigcap_{|\omega'|=\theta_1+\theta_2} T_{\omega'}(X)$.

This means that $z_0 \in T_{\omega'}(X)$ for any ω', and since $\omega' \prec \omega$, we have $z_0 \in T_\omega(X)$. From the fact that for each ω there exists a corresponding ω', it follows that

$$z_0 \in \bigcap_{|\omega|=\theta_1+\theta_2} T_\omega(X).$$

In view of arbitrariness of z_0,

$$\bigcap_{|\omega|=\theta_1+\theta_2} T_\omega(X) \supset \bigcap_{|\omega'|=\theta_1+\theta_2} T_{\omega'}(X). \tag{1.6}$$

A comparison of (1.5) and (1.6) gives Eq. (1.4). Using (1.4) and property 6 of the operator T_ε, we obtain

$$\begin{aligned}\widetilde{T}_{\theta_1+\theta_2}(X) &= \bigcap_{|\omega|=\theta_1+\theta_2} T_\omega(X) = \bigcap_{|\omega'|=\theta_1+\theta_2} T_{\omega'}(X) \\ &= \bigcap_{|\omega_1|=\theta_1} \bigcap_{|\omega_2|=\theta_2} T_{\omega_1} T_{\omega_2}(X) \supset \bigcap_{|\omega_1|=\theta_1} T_{\omega_1} \bigcap_{|\omega_2|=\theta_2} T_{\omega_2}(X) = \widetilde{T}_{\theta_1}\widetilde{T}_{\theta_2}(X),\end{aligned} \tag{1.7}$$

where ω_1 is a rational partition of the interval $[0, \theta_1]$, and ω_2 is a partition of the interval $[\theta_1, \theta_1 + \theta_2]$.

We shall prove a converse insertion. Since each partition

$$\omega_2 = \{\tau_0^2 = \theta_1, \tau_1^2, \dots, \tau_{m_2-1}^2\}$$

is given by a finite collection of rational numbers, the set of such partitions is countable. They can be renumbered:

$$\bar{\omega}_2^1, \bar{\omega}_2^2, \bar{\omega}_2^3, \dots, \bar{\omega}_2^k, \dots, |\bar{\omega}_2^i| = \theta_2.$$

We shall construct the partition ω_2^k. In the role of partition points we take all partition points $\bar{\omega}_2^j$, $j \leqslant k$. It is obvious that $\omega_2^k \prec \bar{\omega}_2^j$, $j = 1, 2, \dots, k$, $|\omega_2^k| = \theta_2$ and, in addition, $\omega_2^{k+1} \prec \omega_2^k$. We shall show that

$$\widetilde{T}_{\theta_2}(X) = \bigcap_{|\omega_2|=\theta_2} T_{\omega_2}(X) = \bigcap_{k=1}^{\infty} T_{\omega_2^k}(X). \tag{1.8}$$

Indeed, if $z_0 \in \widetilde{T}_{\theta_2}(X)$, then $z_0 \in \bigcap_{k=1}^{\infty} T_{\omega_2^k}(X)$, since the set of partitions ω_2^k, is only a part of the set of all possible partitions. Let now $z_0 \in \widetilde{T}_{\omega_2^k}(X)$ for each ω_2^k, and let $\omega_2 = \bar{\omega}_2^i$ be an arbitrary partition obtained by enumeration of the numbers i. Then $\omega_2^i \prec \bar{\omega}_2^i = \omega_2$.

Consequently, for each ω_2,

$$\bigcap_{k=1}^{\infty} T_{\omega_2^k}(X) \subset T_{\omega_2^i}(X) \subset T_{\omega_2}(X).$$

With this Eq. (1.8) has been proved.

Using property 5 of the operator T_ε and (1.8), we obtain

$$\begin{aligned}\widetilde{T}_{\theta_1}\widetilde{T}_{\theta_2}(X) &= \bigcap_{|\omega_1|=\theta_1} T_{\omega_1} \bigcap_{|\omega_2|=\theta_2} T_{\omega_2}(X) \\ &= \bigcap_{|\omega_1|=\theta_1} T_{\omega_1}\left(\bigcap_{k=1}^{\infty} T_{\omega_2^k}(X)\right) = \bigcap_{|\omega_1|=\theta_1} \bigcap_{k=1}^{\infty} T_{\omega_1} T_{\omega_2^k}(X) \supset \bigcap_{|\omega'|=\theta_1+\theta_2} T_{\omega'}(X) = \widetilde{T}_{\theta_1+\theta_2}(X).\end{aligned} \tag{1.9}$$

A comparison of the relationships (1.7) and (1.9) completes the proof of Lemma 4 for a rational θ_1. If θ_1 is an irrational number, then it can be represented as a limit of a sequence of rational numbers $\{\theta_1^i\} \to \theta_1$, $\theta_1^i > \theta_1$, $\theta_1^i > \theta_1^{i+1}$, for which, in view of what has been already proved,

$$\widetilde{T}_{\theta_1^i+\theta_2}(X) = \widetilde{T}_{\theta_1^i}\widetilde{T}_{\theta_2}(X).$$

Then by property 4 of the operator T_ε, we have

$$\widetilde{T}_{\theta_1+\theta_2}(X) = \widetilde{T}_{\theta_1}\widetilde{T}_{\theta_2}(X).$$

Lemma 4 has been proved.

We consider the set $N(c) = \{z : z \in M, \ p(z) \leqslant c\}$. Let c^* be the smallest value of c for which $N(c) \neq \emptyset$. It is obvious that N(c) is closed and for $c_1 \geq c_2 \geq c^*$, we have $N(c_2) \subset N(c_1)$, i.e., as c increases, the set N(c) does not contract. We denote

$$c(z_0) = \inf_{z_0 \in \widetilde{T}_{t_1}(N(c))} c, \tag{1.10}$$

i.e., a minimum c which is such that $z_0 \in \widetilde{T}_{t_1}(N(c))$. If $z_0 \bar{\in} \widetilde{T}_{t_1}(N(c))$ for any $c \geq c^*$, then

$$c(z_0) = +\infty.$$

We shall show that the infimum is attained on the right side. We consider the sequence $\{c_i\} \to c\{z_0\}, c_{i+1} < c_i$. By the definition of $c(z_0)$,

$$z_0 \in \bigcap_{i=1}^{\infty} \widetilde{T}_{t_1}(N(c_i)) = \bigcap_{|\omega|=t_1} \bigcap_{i=1}^{\infty} T_\omega(N(c_i)),$$

i.e., $z_0 \in \bigcap_{i=1}^{\infty} T_\omega(N(c_i))$ for any ω. By property 6 of the operator T_ε,

$$\bigcap_{i=1}^{\infty} T_\omega(N(c_i)) = T_\omega\left(\bigcap_{i=1}^{\infty} N(c_i)\right) = T_\omega(N(c(z_0))).$$

Thus, $z_0 \in T_\omega(N(c(z_0)))$ for any ω,

$$z_0 \in \widetilde{T}_{t_1}(N(c(z_0))).$$

We now return to the game under consideration. The following theorem gives an answer to the question: what can player P secure, if the game begins at a point z_0.

Fundamental Theorem. 1. If the point z_0 is such that $c(z_0) < +\infty$ and $c(z_0) \leq c < +\infty$, then whatever ε strategy is chosen by player E, the game commencing at the instant $t = 0$ at the point z_0 can be finished with the price c. If, however, $c^* \leq c < c(z_0)$, then player E has an ε strategy which is such that for any action of the opponent P the game is not completed with the price c.

2. If $c(z_0) = +\infty$, then for any action of player P, player E has an ε strategy $\nu(\tau)$, $0 \leq \tau \leq t_1$ which is such that E prevents player P leading the object onto the set M and completing the game with a finite price.

Proof. 1. Let $c(z_0) \leq c$ and suppose that player E arbitrarily chose $\varepsilon_1 > 0$ and $v(\tau)$, $\tau \in [0, \varepsilon_1]$ and reported to P. Since in this case

$$z_0 \in \widetilde{T}_{t_1}(N(c)) \subset T_{\varepsilon_1} \widetilde{T}_{t_1 - \varepsilon_1}(N(c)),$$

by the definition of the operator T_ε on the interval $[0, \varepsilon_1]$, for any $v(\tau)$, there exists $u(\tau)$ such that

$$z(\varepsilon_1) = z(z_0, u(\tau), v(\tau), \varepsilon_1) \in \widetilde{T}_{t_1 - \varepsilon_1}(N(c)). \tag{1.11}$$

Let E choose $\varepsilon_2 > 0$ and $v(\tau)$, $\varepsilon_1 \leq \tau \leq \varepsilon_1 + \varepsilon_2$, and report them to player P. From Eq. (1.11) it follows that $z(\varepsilon_1) \in T_{\varepsilon_2} \widetilde{T}_{t_1 - \varepsilon_1 - \varepsilon_2}(N(c))$. This, however, means that there exists $u(\tau)$, $\varepsilon_1 \leqslant \tau \leqslant \varepsilon_1 + \varepsilon_2$, and $z(\varepsilon_1 + \varepsilon_2) = z(\varepsilon_0, u(\tau), v(\tau), \varepsilon_1 + \varepsilon_2) \in \widetilde{T}_{t_1 - (\varepsilon_1 + \varepsilon_2)}(N(c))$, and so forth. We obtain $z(t_1) = z(z_0, u(\tau), v(\tau), t_1) \in N(c)$, and then $z(t_1) \in M$ and $p(z(t_1)) \leq c$.

Let now $c^* \leq c < (z_0)$. We shall show that in this case player E can choose an ε strategy $v(\tau)$ such that the game will not be finished with the value c. From the definition of $c(z_0)$

$$z_0 \bar{\in} \widetilde{T}_{t_1}(N(c)) = \bigcap_{|\omega|=t_1} T_\omega(N(c)).$$

Consequently, there exists a partition ω such that $z_0 \bar{\in} T_\omega(N(c)) = T_{\tau_1 - \tau_0} T_{\tau_2 - \tau_0} \ldots T_{t_1 - \tau_{m-1}}(N(c)) = T_{\varepsilon_1} T_{\varepsilon_2} \ldots T_{\varepsilon_m}(N(c))$. This means that on the interval $[0, \varepsilon_1]$ there exists a control of player E such that

$$z(\varepsilon_1) = z(z_0, u(\tau), v(\tau), \varepsilon_1) \bar{\in} T_{\varepsilon_2} \ldots T_{\varepsilon_m}(N(c)), \tag{1.12}$$

regardless whatever control $u(\tau)$ was applied by player P. From relation (1.12), it follows that on the interval $[\varepsilon_1, \varepsilon_1 + \varepsilon_2]$ there exists $v(\tau)$ such that for any $u(\tau)$

$$z(\varepsilon_1 + \varepsilon_2) = z(z_0, u(\tau), v(\tau), \varepsilon_1 + \varepsilon_2) \bar{\in} T_{\varepsilon_3} \ldots T_{\varepsilon_m}(N(c))$$

and so forth, with $z(t_1) \bar{\in} N(c)$. Thus, if $c^* \leq c < (z_0)$, player E has an ε strategy such that $z(t_1) \bar{\in} N(c)$, regardless how the opponent P acted.

2. Since $z_0 \bar{\in} \widetilde{T}_{t_1}(N(c))$ for any c, then obviously for any $u(\tau)$ we have $z(t_1) \bar{\in} M$ and the game is not finished with a finite value, i.e., $I(z_0, u, v) = +\infty$. The theorem has been proved.

We note that in the proof of the thoerem the choice of an ε strategy is pointed out for player E which prevents P completing the game with a value $c < c(z_0)$. At the same time, on each step E chooses its control, proceeding from the phase position of the object at the instant when the control is being chosen, and reports it to the opponent P. Player P knows the control $v(\tau)$ not on the entire interval [0, t], but only in the nearest future, i.e., on a certain time interval of length ε.

COROLLARY. A set of points from which the game can be finished with the value c coincides with the set $\tilde{T}_{t_1}(N(c))$. The proof follows directly from the theorem.

2. Linear Games of Fixed Duration

Let a game be described by the system of linear differential equations

$$\dot{z} = Az + u + v, \tag{2.1}$$

where $z = (z^1, \ldots, z^n)$ is the vector of phase coordinates, $u = (u^1, \ldots, u^n)$ is the control of player P, $v = (v^1, \ldots, v^n)$ is the control of player E; $u \in U$, $v \in V$. The sets U and V are compact, and U is convex.

It is obvious that all conditions formulated in Sec. 1, which must be satisfied by the right sides of the system, are satisfied.

We assume that the set M is closed and convex, while p(z) is a convex continuous function specified on the whole M. In this case the set $N(c) = \{z: z \in M, p(z) \leq c\}$ will be convex and closed, and for $c_1 \geq c_2 \geq c^*$ $N(c_2) \subset N(c_1)$.

The instant t_1 of ending the game is fixed.

In the previous section we reduced the study of the game to the construction of operators $\tilde{T}_t$ which map a set of the space E^n into other sets of the same space. In the general case of a game with prescribed duration, it is very difficult to construct these operators efficiently. The problem is simplified, if $\tilde{T}_t$ coincides with T_t. This is the case when the operator T_t itself has the property

$$T_{\theta_1}T_{\theta_2}(X) = T_{\theta_1+\theta_2}(X),\ \theta_1, \theta_2 \geqslant 0,\ \theta_1 + \theta_2 \leqslant t_1.$$

Indeed, then for an arbitrary rational partition ω of the interval [0, t]

$$T_\omega(X) = T_{\varepsilon_1}T_{\varepsilon_2}\ldots T_{\varepsilon_m}(X) = T_{\varepsilon_1+\ldots+\varepsilon_m}(X) = T_t(X)$$

and

$$\tilde{T}_t(X) = \bigcap_{|\omega|=t} T_\omega(X) = T_t(X).$$

Next we shall show that for certain linear games the operators T_t form a semigroup.

We know that if the controls $u(\tau)$, $v(\tau)$, $0 \leq \tau \leq t$, and the initial point $z(0) = z_0$, are given, then the corresponding trajectory of the system (2.1) is

$$z(t) = z(z_0, u(\tau), v(\tau), t) = \Phi(t)z_0 + \int_0^t \Phi(t-\tau)\,(u(\tau) + v(\tau))\,d\tau, \tag{2.2}$$

where $\Phi(t)$ is the solution of the equation $\dot{\Phi} = A\Phi$, $\Phi(0) = I$; I is a unit matrix. From this, z_0 can be found in the form

$$z_0 = \Phi(-t)z(t) - \int_0^t \Phi(-\tau)(u(\tau) + v(\tau))\,d\tau. \tag{2.3}$$

We shall determine the operator T_ε for our game. It matches any set $X \subset E^n$ with the set of initial conditions $T_\varepsilon(X)$ which is such that for any $z_0 \in T_\varepsilon(X)$, and for any $v(\tau)$, $0 \leq \tau \leq \varepsilon$, there exists a control $u(\tau)$ of player P such that

$$z(\varepsilon) = z(z_0, u(\tau), v(\tau), \varepsilon) \in X$$

(with the condition that $v(\tau)$, $0 \leq \tau \leq \varepsilon$ is known to the opponent P). Moreover, if $X = \emptyset$, then $T_\varepsilon(X) = \emptyset$.

It is obvious that the operator T_ε possesses all the properties formulated in Lemma 1 of Sec. 1.

We use the following notation:

$$N_X(\varepsilon, v(\tau)) = \{z_0 : z_0 = \Phi(-\varepsilon) z - \int_0^\varepsilon \Phi(-\tau)\ (u(\tau) + v(\tau))\, d\tau,$$

$$z \in X,\ u(\tau) \in \Omega_u\},$$

i.e., this is the set of starting points from which the set X can be reached exactly at the instant $t = \varepsilon$ for a given control $v(\tau)$, $0 \le \tau \le \varepsilon$.

LEMMA 1. If X is a convex closed set, then $N_X(\varepsilon, v(\tau))$ is also closed and convex.

We omit the proof, noting only that Theorem 1 [7] is used for the proof of closedness, while convexity follows at once from convexity of the set X, Ú.

LEMMA 2.

$$T_\varepsilon(X) = \bigcap_{v(\tau) \in \Omega_v} N_X(\varepsilon, v(\tau)). \tag{2.4}$$

Proof. Let $z_0 \in T_\varepsilon(X)$. Then, for any $v(\tau) \in \Omega_v$, $0 \le \tau \le \varepsilon$, there exists a control $u(\tau)$, $0 \le \tau \le \varepsilon$, of player P such that $z(\varepsilon) = z(z_0, u(\tau), v(\tau), \varepsilon) \in X$.

Consequently, z_0 is representable in the form

$$z_0 = \Phi(-\varepsilon) z(\varepsilon) - \int_0^\varepsilon \Phi(-\tau)(u(\tau) + v(\tau))\, d\tau$$

for any $v(\tau) \in \Omega_V$, where $z(\varepsilon) \in X$, $u(\tau) \in \Omega_U$, i.e.,

$$z_0 \in \bigcap_{v(\tau) \in \Omega_v} N_X(\varepsilon, v(\tau)).$$

We obtain

$$T_\varepsilon(X) \supset \bigcup_{v(\tau) \in \Omega_v} N_X(\varepsilon, v(\tau)). \tag{2.5}$$

We shall prove a converse insertion. Let $z_0 \in \bigcap_{v(\tau) \in \Omega_v} N_X(\varepsilon, v(\tau))$. This signifies that for any $v(\tau)$, $0 \le \tau \le \varepsilon$, z_0 is representable in the form

$$z_0 = \Phi(-\varepsilon) z(\varepsilon) - \int_0^\varepsilon \Phi(-\tau)(u(\tau) + v(\tau))\, d\tau,$$

where $z(\varepsilon) \in X$, $u(\tau) \in \Omega_U$, i.e., for any $v(\tau) \in \Omega_V$ there exists $u(\tau) \in \Omega_U$ such that

$$z(\varepsilon) = \Phi(\varepsilon) z_0 + \int_0^\varepsilon \Phi(\varepsilon - \tau)(u(\tau) + v(\tau))\, d\tau \in X.$$

By the definition of the operator T_ε, T_ε $z_0 \in T_\varepsilon(X)$, i.e.,

$$T_\varepsilon(X) \subset \bigcap_{v(\tau) \in \Omega_v} N_X(\varepsilon, v(\tau)). \tag{2.6}$$

Comparing (2.5) and (2.6), we obtain (2.4).

COROLLARY. If the set X is convex, then $T_\varepsilon(X)$ is also convex.

Indeed, $T_\varepsilon(X)$ is an intersection of convex sets.

We shall now use convexity of the sets $T_\varepsilon(X)$ for their analytical description. We note that if the set X is convex and closed, then it is uniquely determined by its support function $W_X(\psi) = \sup_{z \in X}(\psi, z)$. In fact, the set X is then specified by a system of linear inequalities $(\psi, z) \le W_X(\psi)$ for all $\psi \in E^n$. If $X = \emptyset$, we put $W_X(\psi) \equiv -\infty$.

Since the sets N(c), U, V are specified, we can consider their support functions $W_{N(c)}(\psi)$, $W_U(\psi)$, $W_V(\psi)$ as known.

Let X be an arbitrary convex closed set. We calculate the support function for $N_X(\varepsilon, v, (\tau))$:

$$W_{N_X(\varepsilon,v(\tau))}(\psi) = \sup_{z\in N_X(\varepsilon,v(\tau))} (\psi, z) = \sup_{\substack{z\in X\\ u(\tau)\in\Omega_u}} \Big[(\Phi^*(-\varepsilon)\psi, z) + \int_0^\varepsilon (-\Phi^*(-\tau)\psi,$$

$$u(\tau) + v(\tau))\,d\tau\Big] = W_X(\Phi^*(-\varepsilon)\psi) + \int_0^\varepsilon W_U(-\Phi^*(-\tau)\psi)\,d\tau + \int_0^\varepsilon (-\Phi^*(-\tau)\psi, v(\tau))\,d\tau.$$

Since the set $T_\varepsilon(X) = \bigcap_{v(\tau)\in\Omega_v} N_X(\varepsilon, v(\tau))$, it is obvious that $z \in T_\varepsilon(X)$, if for all $\psi \in E^n$ the system of inequalities

$$(\psi, z) \leqslant \inf_{v(\tau)\in\Omega_v} \{W_X(\Phi^*(-\varepsilon)\psi) + \int_0^\varepsilon [W_u(-\Phi^*(-\tau)\psi) - (\Phi^*(-\tau)\psi, v(\tau))]d\tau\}$$

$$= W_X(\Phi^*(-\varepsilon)+) + \varphi(\varepsilon,\psi)$$

is satisfied. Here $\varphi(\varepsilon, \psi) = \int_0^\varepsilon W_U(-\Phi^*(-\tau)\psi)\,d\tau - \int_0^\varepsilon W_V(\Phi^*(-\tau)\psi)\,d\tau$ is bounded for any $\psi \in E^n$ in view of compactness of the sets U and V.

Thus, we have proved that the set $T_\varepsilon(X)$ is convex and is described by the set of linear inequalities

$$(\psi, z) \leqslant W_X(\Phi^*(-\varepsilon)\psi) + \varphi(\varepsilon, \psi), \tag{2.7}$$

which must be satisfied for all ψ.

Unfortunately, a nonconvex function stands on the right side of (2.7), and therefore in the general case it is not the support function of the set $T_\varepsilon(X)$. We now note that [8] if we are given a system of linear inequalities

$$(\psi, z) \leqslant W(\psi), \ \forall \psi,$$

where $W(\psi)$ is a positively homogeneous function in ψ, then the support function of a convex set of points z defined by this system is given by the expression

$$\widetilde{W}(\psi) = \inf_{\Sigma\psi_i=\psi} \Sigma W(\psi_i), \tag{2.8}$$

where the lower bound is taken over all finite expansions of the vector ψ into a sum of the vectors ψ_i. Accordingly, the support function of the set $T_\varepsilon(X)$ can be written

$$W_X^\varepsilon(\psi) = W_{T_\varepsilon(X)}(\psi) = \inf_{\Sigma\psi_i=\psi} \Sigma[W_X(\Phi^*(-\varepsilon)\psi_i) + \varphi(\varepsilon, \psi_i)].$$

<u>THEOREM 1.</u> If X is an arbitrary convex closed set, then one of the following conditions holds.

1. A = 0, i.e., a game with a simple motion is considered and $T_{\varepsilon_1}T_{\varepsilon_2}(X) \neq \varnothing$.
2. The function

$$\varphi(\varepsilon, \psi) + W_X(\Phi^*(-\varepsilon)\psi)$$

is convex with respect to ψ for any ε; then

$$T_{\varepsilon_1}T_{\varepsilon_2}(X) = T_{\varepsilon_1+\varepsilon_2}(X).$$

To prove the theorem, it is sufficient to show that the support functions of the sets $T_{\varepsilon_1}T_{\varepsilon_2}(X)$ and $T_{\varepsilon_1+\varepsilon_2}(X)$ are equal.

We shall first consider the case where condition 2 is satisfied. In this case, the expression

$$W_X(\Phi^*(-\varepsilon)\psi) + \varphi(\varepsilon, \psi)$$

is a convex function, and therefore it follows from (2.8) that it is in fact a support function for $T_\varepsilon(X)$.

We have

$$W_X^{\varepsilon_1+\varepsilon_2}(\psi) \equiv W_{T_{\varepsilon_1+\varepsilon_2}(X)}(\psi) = W_X(\Phi^*(-\varepsilon_1-\varepsilon_2)\psi) + \varphi(\varepsilon_1+\varepsilon_2, \psi).$$

Further,

$$W_X^{\varepsilon_1,\varepsilon_2}(\psi) \equiv W_{T_{\varepsilon_1}T_{\varepsilon_2}(X)}(\psi) = W^{\varepsilon_1}_{T_{\varepsilon_2}(X)}(\psi)$$

$$= W_{T_{\varepsilon_2}(X)}(\Phi^*(-\varepsilon_1)\psi) + \varphi(\varepsilon_1,\psi) = W_X(\Phi^*(-\varepsilon_2)\Phi^*(-\varepsilon_1)\psi) + \varphi(\varepsilon_2,\Phi^*(-\varepsilon_1)\psi) + \varphi(\varepsilon_1,\psi)$$

$$= W_X(\Phi^*(-\varepsilon_1-\varepsilon_2)\psi) + \int\limits_{\varepsilon_1}^{\varepsilon_1+\varepsilon_2}[W_U(-\Phi^*(-\tau)\psi) - W_V(\Phi^*(-\tau)\psi)]\,d\tau + \int\limits_0^{\varepsilon_1}[W_U(-\Phi^*$$

$$(-\tau)\psi) - W_V(\Phi^*(-\tau)\psi)]\,d\tau = W_X(\Phi^*(-\varepsilon_1-\varepsilon_2)\psi) + \varphi(\varepsilon_1+\varepsilon_2,\psi).$$

A comparison of the expression obtained shows that $W_X^{\varepsilon_1+\varepsilon_2}(\psi) = W_X^{\varepsilon_1,\varepsilon_2}(\psi)$. This proves the required result.

Let now A = 0. Then $\Phi(\varepsilon) = I$ (a unit matrix). We take into account this fact in the calculations which will be carried out in a general form. Since $\varphi(\varepsilon,\psi) + W_X(\Phi^*(-\varepsilon)\psi)$ is no longer necessarily a convex function, then

$$W_X^{\varepsilon_1+\varepsilon_2}(\psi) = \inf_{\Sigma\psi_i=\psi}\Sigma[W_X(\Phi^*(-\varepsilon_1-\varepsilon_2)\psi_i) + \varphi(\varepsilon_1+\varepsilon_2,\psi_i)],$$

$$W_X^{\varepsilon_1,\varepsilon_2}(\psi) = \inf_{\Sigma\psi_i=\psi}\Sigma[W_{T_{\varepsilon_2}(X)}(\Phi^*(-\varepsilon_1)\psi_i) + \varphi(\varepsilon_1,\psi_i)]. \tag{2.9}$$

Since according to the properties of the operator $T_\varepsilon(X)$ $T_{\varepsilon_1}T_{\varepsilon_2}(X) \subset T_{\varepsilon_1+\varepsilon_2}(X)$, we have $W_X^{\varepsilon_1+\varepsilon_2}(\psi) \geqslant W_X^{\varepsilon_1,\varepsilon_2}(\psi)$. However, this can easily be verified from Eqs. (2.8) and the expression

$$\varphi(\varepsilon_2,\Phi^*(-\varepsilon_1)\psi) + \varphi(\varepsilon_1,\psi) = \varphi(\varepsilon_1+\varepsilon_2,\psi), \tag{2.10}$$

which is also easily checked.

We shall now show that $W_X^{\varepsilon_1+\varepsilon_2}(\psi) \leqslant W_X^{\varepsilon_1,\varepsilon_2}(\psi)$. For this we make a few preliminary observations. If $W_X(\psi)$ is a support function of the set X, then we denote

$$K_X = \{\psi\colon W_X(\psi) < +\infty\}.$$

It is easy to see that K_X is convex cone. From the expressions for $W_X^\varepsilon(\psi)$ it is not difficult to see now that

$$K_{T_\varepsilon(X)} = \Phi^*(\varepsilon)K_X.$$

Exactly in the same way as (2.9)

$$K_{T_{\varepsilon_1}T_{\varepsilon_2}(X)} = \Phi^*(\varepsilon_1)K_{T_{\varepsilon_2}(X)} = \Phi^*(\varepsilon_1+\varepsilon_2)K_X = K_{T_{\varepsilon_1+\varepsilon_2}(X)}.$$

Thus, the functions $W_X^{\varepsilon_1,\varepsilon_2}(\psi)$ and $W_X^{\varepsilon_1+\varepsilon_2}(\psi)$ assume infinite values for the same sets of vectors ψ. Therefore, in all the subsequent discussions all vectors ψ and their expansions can be taken from the sets $\Phi^*(\varepsilon_1 + \varepsilon_2)K_X$, since outside these sets $W_X^{\varepsilon_1,\varepsilon_2}(\psi)$ and $W_X^{\varepsilon_1+\varepsilon_2}(\psi)$ are simultaneously equal to infinity.

Further, the following properties are valid.

1. $W_X^\varepsilon(\psi) \leqslant W_X(\psi) + \varphi(\varepsilon,\psi)$.

This follows from the expression for $W_X^\varepsilon(\psi)$.

2. For any $\delta > 0$, we can find an expansion $\Sigma\psi_i = \psi$, such that

$$0 \leqslant \Sigma[W_X(\Phi^*(-\varepsilon)\psi_i) + \varphi(\varepsilon,\psi_i)] - W_X^\varepsilon(\psi) \leqslant \delta$$

and

$$\varphi(\varepsilon,\psi) \geqslant \Sigma\varphi(\varepsilon,\psi_i).$$

The first statement follows from the definition of a lower bound. The second follows from the fact that from the first property we can always find an expansion such that

$$\sum_i[W_X(\Phi^*(-\varepsilon)\psi_i) + \varphi(\varepsilon,\psi_i)] \leqslant W_X(\Phi^*(-\varepsilon)\psi) + \varphi(\varepsilon,\psi),$$

and $W_X(\psi)$ is convex.

Proceeding from (2.9), we now choose for a given δ an expansion of the vector ψ such that

$$W_X^{\varepsilon_1,\varepsilon_2}(\psi) \geqslant \sum_i [W_{T_{\varepsilon_2}(X)}(\Phi^*(-\varepsilon_1)\psi_i) + \varphi(\varepsilon_1,\psi_i)] - \delta.$$

Let k be the number of nonzero vectors in the expansion $\Sigma\psi_i = \psi$. We choose for each ψ_i an expansion $\Sigma\psi_{ij} = \psi_i$, such that

$$W_{T_{\varepsilon_2}(X)}(\Phi^*(-\varepsilon_1)\psi_i) \geqslant \sum_j [W_X(\Phi^*(-\varepsilon_2)\Phi^*(-\varepsilon_1)\psi_{ij}) + \varphi(\varepsilon_2, \Phi^*(-\varepsilon_1)\psi_{ij})] - \frac{\delta}{k}$$

and

$$\varphi(\varepsilon_2, \Phi^*(-\varepsilon_1)\psi_{ij}) \geqslant \sum_j \varphi(\varepsilon_2, \Phi^*(-\varepsilon_1)\psi_{ij}). \tag{2.11}$$

Then

$$W_X^{\varepsilon_1,\varepsilon_2}(\psi) \geqslant \sum_i \Big\{\sum_j [W_X(\Phi^*(-\varepsilon_1-\varepsilon_2)\psi_{ij})\ \varphi(\varepsilon_2, \Phi^*(-\varepsilon_1)\psi_{ij})] + \varphi(\varepsilon_1,\psi_i)\Big\} - 2\delta$$

$$= \sum_i\sum_j [W_X(\Phi^*(-\varepsilon_1-\varepsilon_2)\psi_{ij}) + \varphi(\varepsilon_2, \Phi^*(-\varepsilon_1)\varphi_{ij}) + \varphi(\varepsilon_1,\psi_{ij})] + \sum_i [\varphi(\varepsilon_1,\psi_i) - \sum_j \varphi(\varepsilon_1,\psi_{ij})] - 2\delta.$$

Taking into account the first expression in (2.9) and (2.10) and the fact that $\sum_i\sum_j \psi_{ij} = \sum_i \psi_i = \psi$, we obtain

$$W_X^{\varepsilon_1,\varepsilon_2}(\psi) \geqslant W_X^{\varepsilon_1+\varepsilon_2}(\psi) + \sum_i [\varphi(\varepsilon_1,\psi_i) - \sum_j \varphi(\varepsilon_1,\psi_{ij})] - 2\delta. \tag{2.12}$$

Since we are dealing with a simple motion and hence $\Phi(t) = I$, we obtain

$$\varphi(\varepsilon,\psi) = \int_0^\varepsilon [W_U(-\psi) - W_V(\psi)]\,d\tau = \varepsilon[W_U(-\psi) - W_V(\psi)].$$

It now follows from (2.11) that

$$\varepsilon_2[W_U(-\psi_i) - W_V(\psi_i)] \geqslant \varepsilon_2 \sum_j [W_U(-\psi_{ij}) - W_V(\psi_{ij})].$$

Multiplying this inequality by $\varepsilon_1/\varepsilon_2$, we obtain

$$\varphi(\varepsilon_1,\psi_i) \geqslant \sum_j \varphi(\varepsilon_1,\psi_{ij}).$$

It now follows from (2.12) that

$$W_X^{\varepsilon_1,\varepsilon_2}(\psi) \geqslant W_X^{\varepsilon_1+\varepsilon_2}(\psi) - 2\delta.$$

Since δ is arbitrary, we have $W_X^{\varepsilon_1,\varepsilon_2}(\psi) \geqslant W_X^{\varepsilon_1+\varepsilon_2}(\psi)$, which was to be proved.

Thus, $W_X^{\varepsilon_1,\varepsilon_2}(\psi) = W_X^{\varepsilon_1+\varepsilon_2}(\psi)$, i.e., $T_{\varepsilon_1}T_{\varepsilon_2}(X) = T_{\varepsilon_1+\varepsilon_2}(X)$. This completes the proof of the theorem.

We can now formulate certain sufficient conditions for a possibility of finishing the game.

THEOREM 2. Let a convex closed set be given and one of the conditions of Theorem 1 be fulfilled. Then, for player P to be able to lead the trajectory of the object (2.1) from a starting point z_0 into N(c) for any ε-strategy of player E, it is necessary and sufficient to satisfy the inequality

$$(\psi, z) \leqslant W_{N(c)}(\Phi^*(-t_1)\psi) + \varphi(t_1,\psi)$$

for all ψ.

The proof follows from the fundamental theorem of Sec. 1, Theorem 1 of this section and the fact that, under the conditions of Theorem 2, $\tilde{T}_t(N(c)) = T_t(N(c))$.

COROLLARY. If $U = U_1 - \alpha V$, where U_1 is a convex compact set, $\alpha \geq 1$, then the second condition of Theorem 1 is satisfied.

Indeed, in this case,

$$W_U(\psi) = W_{U_1}(\psi) + \alpha W_V(-\psi)$$

and therefore

$$\varphi(\varepsilon, \psi) = \int_0^\varepsilon [W_U(-\Phi^*(-\tau)\psi) - W_V(\Phi^*(-\tau)\psi)]\, d\tau$$

$$= \int_0^\varepsilon [W_{U_1}(-\Phi^*(-\tau)\psi) + (\alpha - 1) W_V(\Phi^*(-\tau)\psi)]\, d\tau,$$

from which it follows that $\varphi(\varepsilon, \psi)$ is a function that is convex with respect to ψ.

We also note that condition 2 of Theorem 1 is satisfied for games such as "isotropic rockets" [11] and "boy and crocodile." For such games these conditions lead to the already known sufficient conditions in the case of a prescribed completion time, showing at the same time that they are also necessary conditions.

LITERATURE CITED

1. W. H. Fleming, "A note on differential games of prescribed durations," Contributions on the Theory of Games, Annals of Mathematical Studies, Vol. 3, (1957).
2. W. H. Fleming, "The convergence problem for differential games," Journal of Mathematical Analysis and Applications, 3 (1961).
3. W. H. Fleming, "The convergence problem for differential games," Adv. in Game Theory, Annals of Mathematical Studies (1964).
4. M. Sagaidak, "A differential game," Theory of Optimal Solutions [in Russian], No. 2, Izd. IK AN USSR, Kiev (1968), pp. 14-22.
5. B. N. Pshenichnyi, "The structure of differential games," Dokl. Akad. Nauk SSSR, 184, No. 2, 285-287, Moscow.
6. B. N. Pshenichnyi, "The structure of differential games," Theory of Optimal Solutions [in Russian], No. 1, Izd. IK AN USSR, Kiev (1968).
7. A. F. Filippov, "Certain optimal control problems," Vestnik MGU, Seriya Matematiki, Mekhaniki i Astronomii, No. 2, 25-32, Moscow (1959).
8. S. Karlin, Mathematical Methods in the Theory of Games, Programming and Economics [Russian translation], Izd. Mir, Moscow (1964).
9. L. S. Pontryagin, "Linear differential games, I," Dokl. Akad. Nauk SSSR, 174, No. 6, Moscow (1967).
10. L. S. Pontrayagin, "Linear differential games, II," Dokl. Akad. Nauk SSSR, 175, No. 4, Moscow (1967).
11. L. S. Pontryagin, "On the theory of differential games," Usp. Matemat. Nauk, 21, No. 4, Moscow (1966).