5. M. Sh. Tsalenko, "Philosophy and mathematics of the modeling of information-processing operations (in the example of relational models of databases)," Semiotika Inf., No. 13, 150-183 (1979).
6. V. N. Red'ko, "Fundamentals of composition programming," Programmirovanie, No. 3, 3-13 (1979).

A GENERALIZED URN PROBLEM AND ITS APPLICATIONS

B. Artur, Yu. M. Ermol'ev, and Yu. M. Kaniovskii

UDC 519.245:519.87:519.862.7

We investigate growth processes describing the distribution of newly adopted resources in accordance with a well-defined rule based on random samples.

1. THE MEANINGFUL FORMULATION OF THE PROBLEM

We consider an infinitely large number of managers adopting some new technology which occurs in two types, A and B. We assume that each manager is guided by the following considerations: he analyzes which technology has been adopted by r randomly selected managers and if not less than m of them use A, then he also selects A, otherwise he selects B. Such a decision making rule may seem to be irrational, however, it may be logical in the following situations: if the manager has little information on the comparative gains resulting from the application of the technologies A and B; if he fears risks; if he can gather information only by means of questioning randomly selected managers already using the specific technology and they communicate to him only the type of the technology. It is interesting to find out how the fraction of the managers using only one of the technologies varies, whether one of them covers the weight of the market or whether the proportion of those adopting A (B) tends to a certain limit, and how the number of managers selecting the technology A (B) is growing.

It is natural to describe processes of this kind by means of a generalized urn scheme [1]. We imagine an urn of infinite capacity, containing white and black balls. If at each step a certain number of balls are removed from the urn and then, on the basis of a well-defined decision making rule R, one adds to the sample a white or a black ball and together with this additional ball the sample is returned to the urn, then we shall say that there is given a generalized urn scheme with decision making rule R. It is necessary to investigate the behavior of the fraction of the balls of each color.

We consider three examples of decision making rules.

R_1. One removes a random sample of r balls from the urn, where r is some odd integer. If more than half of the balls turn out to be white, then one adds a white ball, otherwise one adds a black ball. The sample and the additional ball are returned to the urn.

R_2. In a random manner one selects a sequence of balls from the urn. If m white balls appear before m black balls, then the sample, together with a white ball is returned to the urn; otherwise the sample is returned together with a black ball. Here m is a given number.

R_3. One selects in a random manner r balls from the urn, where r is some odd integer. If more than half of the balls turn out to be white, then one adds a black ball, otherwise one adds a white ball. The sample and the additional ball are returned to the urn.

We note that the rule R_3 is the reciprocal of the rule R_1. In the spatial case when $r = 1$ in R_1 or $m = 1$ in R_2, these processes reduce to the known Eggenberger—Polya urn scheme [1].

Translated from Kibernetika, No. 1, pp. 49-56, January-February, 1983. Original article submitted June 4, 1982.

0011-4235/83/1901-0061$07.50

2. TRANSITION PROBABILITIES RELATED TO THE GENERALIZED URN SCHEME

We consider a generalized urn scheme with the rule R. By the urn function we mean the probability of the event that a white ball will be added to the urn. This probability depends on the rule R, the total number n of balls in the urn and the number n_1 of white balls in the urn. We denote it by $p(n, n_1)$. Let $x \in [0, 1]$; we consider $p_n(x) = p(n, [nx])$, where $[nx]$ is the integer part of the number nx. It presents interest to investigate what happens when the number of balls in the urn grows indefinitely. It may turn out that for $n \to \infty$ we have $p_n(x) \to p(x)\ \forall x \in [0, 1]$ in the sense of the usual pointwise convergence. In this case we call $p(\cdot)$ the transition probability corresponding to the generalized urn scheme with the decision making rule R. We note that the function $p(\cdot)$ obtained in this manner will be a Borel function. We introduce concepts that will be needed in the sequel.

Let $p(\cdot)$ be a Borel function $p(\cdot)\colon [0, 1] \to [0, 1]$, being the transition probability in some generalized urn scheme. Then $p(\cdot)$ possesses the s-property if there exists a nonempty set $Q \subset [0,1]$ such that $p(x) = x\ \forall x \in Q$, Q consists of a finite number of connected components and there exists a point $\theta \in Q \cap (0, 1)$, for which

$$p(x) \leqslant x \text{ for } x \in (0, \theta),\ p(x) \geqslant x \text{ for } x \in (\theta, 1). \tag{1}$$

Similarly, we say that a transition probability possesses the inverse property s if there exists a nonempty set $Q \subset [0, 1]$ such that $p(x) = x\ \forall x \in Q$, Q consists of a finite number of connected components and there exists a point $\theta \in Q \cap (0, 1)$, for which

$$p(x) \geqslant x \text{ for } x \in (0, \theta),\ p(x) \leqslant x \text{ for } x \in (\theta, 1). \tag{2}$$

If in the relations (1), (2) the nonstrict inequalities are replaced by strict ones, i.e., one requires that $Q \cap (0, 1)$ consists of a unique point θ, then we obtain the definition of transition probabilities possessing the strong s-property and the strong inverse property s. A transition probability will be said to be linear if $p(x) \equiv x\ \forall x \in [0,1]$.

In order to elucidate the meaning of the given definitions, we consider Fig. 1, where curve 1 corresponds to a transition probability possessing the strong s-property, curve 2 to a strong inverse property s and curve 3 to the linear case.

We note that in the definitions it is not mentioned that the function $p(\cdot)$ is a transition probability for a generalized urn scheme with some rule of decision making. However, in the cases when $p(\cdot)$ is a transition probability for a generalized urn scheme with the rule R, we shall say that R possesses the s-property, the inverse property s, or that it is linear, if the corresponding transition probability possesses that property.

We find the transition probabilities for the rules R_1–R_3. For R_1, the probability that in a random sample of size r there should occur i white balls can be computed by the formula

$$p_i = \frac{C_{n_1}^i C_{n-n_1}^{r-i}}{C_n^r},\ 1 \leqslant i \leqslant r \leqslant n,\ n_1 \leqslant n,$$

where $C_n^k = \frac{n!}{k!\,(n-k)!}$. From here

$$p(n, n_1) = \sum_{i=\frac{r+1}{2}}^{r} p_i = \sum_{i=\frac{r+1}{2}}^{r} \frac{C_{n_1}^i C_{n-n_1}^{r-i}}{C_n^r}.$$

From this relation, making use of Stirling's formula ([2], p. 371) we find that for all $x \in [0, 1]$ we have

$$p(x) = \sum_{i=\frac{r+1}{2}}^{r} C_r^i x^i (1-x)^{r-i}. \tag{3}$$

It is easy to see that for $r > 1$ the rule R_1 possesses the strong s-property with $\theta = 1/2$ and for $r = 1$ it is linear. It can be easily seen that the rule R_2 is equivalent to R_1 with $r = 2m - 1$. Similarly, for R_3 we have

$$p(x) = \sum_{i=0}^{\frac{r-1}{2}} C_r^i x^i (1-x)^{r-i},\ x \in [0, 1]. \tag{4}$$

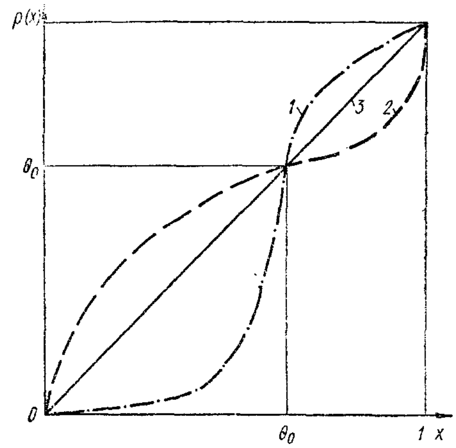

Fig. 1

i.e., the rule R_1 possesses the strong inverse property s with $\theta = 1/2$.

3. THE SIMULATION OF RANDOM PROCESSES DESCRIBED BY A GENERALIZED URN SCHEME

In the given formulations of the problems, the situation when the number of balls in the urn increases indefinitely presents interest. Consequently, if there exists $p(\cdot)$, then $p_n(\cdot)$ is close to it. Replacing $p_n(\cdot)$ by $p(\cdot)$, we obtain a convenient device for the simulation of the processes which can be described by the generalized urn scheme. It consists in the following.

There is given a Borel function $p(\cdot)$, mapping the segment [0, 1] into itself. At the initial moment t = 1, there are in the urn $m(1) \geqslant 1$ white balls and $n(1) \geqslant 1$ black balls. Let y(t) be the fraction of white balls at time $t \geqslant 1$. At time $t \geqslant 1$ one adds to the urn a white ball with probability p(y(t)) or a black one with probability $1 - p(y(t))$. One has to investigate the behavior of y(t) as $t \to \infty$.

More formally, this can be presented in the following manner. There is given a probability space (Ω, F, P) and on it there is given a family of random variables $\xi(t, x)$, independent with respect to t, being Borel functions with respect to x and such that

$$\xi(t, x) = \begin{cases} 1 \text{ with probability } & p(x), \\ 0 \text{ with probability } & 1 - p(x), \end{cases} \tag{5}$$

where $t \geqslant 1$, $x \in [0, 1]$. The total number of balls in the urn at time t is $m(1) + n(1) - 1 + t = c + t$. If m(t) is the number of white balls in the urn at time t, then $m(t + 1) = m(t) + \xi(t, y(t))$, $t \geqslant 1$. From here, since $y(t) = m(t)(c + t)^{-1}$, we find

$$y(t+1) = y(t) - (a+t)^{-1}[y(t) - \xi(t, y(t))], \quad t \geqslant 1, \; y(1) = b, \tag{6}$$

where $a = c + 1$, $b = m(1)[m(1) + n(1)]^{-1}$. The relations (5), (6) constitute the point of departure for the simulation of the processes described by the generalized urn scheme. If $p(\cdot)$ is the transition probability corresponding to the generalized urn scheme with the decision making rule R, then the Markov random process (6) is called a process simulating the generalized urn scheme with rule R.

4. THE GENERALIZED URN SCHEME WITH BALLS OF N + 1 COLORS

Let R^N be the space of N-dimensional column vectors x with coordinates $[x]_i$, i = 1, 2,...,N. We consider in R^N the simplex $X = \{x \in R^N : [x]_i \geqslant 0, i = 1, 2, \ldots, N;\ [x]_1 + [x]_2 + \ldots + [x]_N \leqslant 1\}$. We assume that on X there is given a Borel vector-function $p(\cdot)$ such that $[p(x)]_1 + [p(x)]_2 + \ldots + [p(x)]_N \leqslant 1 \; \forall x \in X$; $[p(\cdot)]_i : X \to [0, 1]$. We imagine an urn with infinite capacity with balls of N + 1 colors. At time t = 1 there are M_1 balls in the urn, and $[m_1]_i$ is the number of balls of the i-th color, i = 1, 2,...,N. Then there are $M_1 - [m_1]_1 - [m_1]_2 - \ldots - [m_1]_N$ balls of the (N + 1)-st color. At time $t \geqslant 1$ one adds to the urn one ball, whose color depends on chance and on the color composition of the balls in the urn. Let M(t) be the total number of balls in the urn at time t, i.e., $M(t) = M_1 + t - 1$ and let $m(t)$ be a vector whose i-th coordinate is equal to the number of balls of the i-th color in the urn at this moment, and let $y(t) = M(t)^{-1}m(t)$ be the vector whose i-th coordinate is the fraction of the balls of the i-th color in the urn at time t, $i = 1, 2, \ldots, N$. Then, at time t there are in the urn $M(t) - [m(t)]_1 - [m(t)]_2 - \ldots - [m(t)]_N$ balls of the $(N + 1)$-st color and their fraction is $1 - [y(t)]_1 - [y(t)]_2 - \ldots - [y(t)]_N$. A ball of the i-th color $i = 1, 2, \ldots, N$, is added with probability $[p(y(t))]_i$ at time t, while a ball of the $(N + 1)$-st color is added with probability $1 - [p(y(t))]_1 - [p(y(t))]_2 - \ldots - [p(y(t))]_N$. In analogy with Sec. 3, this process can be considered as simulating the generalized urn scheme with balls of N + 1 colors [1].

Assume that on the probability space (Ω, F, P) there is given a sequence $\xi(s, x)$ of random vectors in R^N, independent with respect to s, being Borel functions with respect to x and such that $\xi(t, x)$ has with probability 1 only one nonzero coordinate, while

$$[\xi(s, x)]_i = \begin{cases} 1 & \text{with probability } [p(x)]_i, \\ 0 & \text{with probability } 1 - [p(x)]_i, \end{cases}$$

where $x \in X, i = 1, 2, \ldots, N, s \geqslant 1$. Then

$$[m(t+1)]_i = [m(t)]_i + [\xi(t, y(t))]_i, \quad t \geqslant 1,$$

$$[m(1)]_i = [m_1]_i,$$

whence

$$[y(t+1)]_i = [y(t)]_i - [M_1 + t]^{-1}\{[y(t)]_i - [\xi(t, y(t))]_i\}, \quad t \geqslant 1, \ [y(1)]_i = M_1^{-1}[m_1]_i, \tag{7}$$

where $i = 1, 2, \ldots, N$. The relations (7) describe the evolution of the fractions of the balls of all colors in the urn, since the fraction of the balls of the $(N + 1)$-th color is $1 - [y(t)]_1 - [y(t)]_2 - \ldots - [y(t)]_N$. Writing them in vector form, we obtain the analogue of the equalities (6) for an urn with balls of N + 1 colors:

$$y(t+1) = y(t) - (M_1 + t)^{-1}[y(t) - \xi(t, y(t))], \quad t \geqslant 1, \qquad y(1) = M_1^{-1}m_1. \tag{8}$$

We note that for $s \geqslant 1, x \in X$ we have $\mathbf{M}\xi(s, x) = p(x)$. Setting $z(s, x) = \xi(s, x) - p(x)$, we write the relations (8) in the form

$$y(t+1) = y(t) - (M_1 + t)^{-1}[y(t) - p(y(t))] + (M_1 + t)^{-1}z(t, y(t)), \quad t \geqslant 1, \ y(1) = M_1^{-1}m_1. \tag{9}$$

From the definition of the vectors $z(s, x)$ there follows that for $s \geqslant 1, x \in X$, we have

$$\|z(s, x)\| \leqslant \sqrt{N+1}, \tag{10}$$

$$\mathbf{M}z(s, x) = 0, \tag{11}$$

$$\mathbf{M}z(s, x)z(s, x)' = D(x), \tag{12}$$

where the prime denotes transposition and $D(x)$ is the matrix with entries $[D(x)]_{ij}$, $i, j = 1, 2, \ldots, N$, such that

$$[D(x)]_{ij} = -[p(x)]_i[p(x)]_j \quad \text{for } i \neq j, \tag{13}$$

$$[D(x)]_{ii} = [p(x)]_i\{1 - [p(x)]_i\}. \tag{14}$$

In analogy with Sec. 2, we shall call $p(\cdot)$ the vector-function of transition probabilities.

In order to introduce the concept of vector-function of transition probabilities possessing the s-property and the inverse property s, we note that the relations (1), (2) can be written in the following equivalent form:

$$[p(x) - x](x - \theta) > 0 \quad \text{for } x \in [0, 1] \setminus Q, \tag{1'}$$

$$[p(x) - x](x - \theta) < 0 \quad \text{for } x \in [0, 1] \setminus Q. \tag{2'}$$

Starting from this, we say that the vector-function of transition probabilities p(·) possesses the s-property if there exists a nonempty set Q such that $p(x) = x\ \forall x \in Q$, Q consists of a finite number of connected components and there exists a point $\theta \in Q \cap \text{Int}\, X$ and a symmetric positive-definite matrix C such that for $x \in X \setminus Q$ we have

$$\langle C[p(x) - x],\ x - \theta\rangle > 0. \qquad (15)$$

Here <·, ·> is the standard inner product in R^N. Similarly, the vector-function of transition probabilities p(·) possesses the inverse property s if there exists a nonempty set Q such that $p(x) = x\ \forall x \in Q$, Q consists of a finite number of connected components and there exists a point $\theta \in Q \cap \text{Int}\, X$ and a symmetric positive definite matrix C such that for $x \in X \setminus Q$ we have

$$\langle C[p(x) - x],\ x - \theta\rangle < 0. \qquad (16)$$

The vector-function of transition probabilities will be said to be linear if $p(x) \equiv x$ for $x \in X$.

5. CONVERGENCE WITH PROBABILITY 1

Let ρ(x) be the distance from the point x to the set Q, i.e.,

$$\rho(x) = \inf_{y \in Q} \| y - x\|.$$

LEMMA 1. With probability 1 for $t \to \infty$ we have

$$S_t = \sum_{i=1}^{t} (M_1 + i)^{-1} z(i,\ y(i)) \to S,\ P\{\|S\| < \infty\} = 1,$$

where S is some random vector.

Proof. Taking into account the relations (10), (11) and the independence with respect to t of the random vectors z(t, x), the lemma is a special case of the statement proved in [3].

LEMMA 2. We have the relations:

$$P\left\{\pm \sum_{i=1}^{\infty} (M_1 + i)^{-1} \langle C[y(i) - \theta],\ y(i) - p(y(i))\rangle < \infty\right\} = 1, \qquad (17)$$

$$P\{\varliminf_{t\to\infty} \rho(y(t)) = 0\} = 1. \qquad (18)$$

In (17), the sign "–" corresponds to vector-functions of transition probabilities possessing the s-property, while the sign "+" corresponds to vector-functions of transition probabilities possessing the inverse property s.

Proof. The validity of this statement for vector-functions of transition probabilities with the inverse property s follows from theorem on the convergence of algorithms of stochastic optimization and estimation [3, 4]. In the case of vector-functions of transition probabilities possessing the s-property, the arguments are somewhat different and, therefore, we give here their fundamental aspects.

From the relations (7) there follows that

$$\langle C[y(i+1) - \theta],\ y(i+1) - \theta\rangle = \langle C[y(i) - \theta],\ y(i) - \theta\rangle - 2(M_1 + i)^{-1}\{\langle C[y(i) - p(y(i))],\ y(i) - \theta\rangle + \langle C[y(i) - \theta], z(i, y(i))\rangle\} + (M_1 + i)^{-2}[\langle C[y(i) - p(y(i))],\ y(i) - p(y(i))\rangle + \langle Cz(i,\ y(i)),\ z(i, y(i))\rangle],\ i \geqslant 1,\ \langle C[y(1) - \theta],\ y(1) - \theta\rangle = \text{const}.$$

Taking here the conditional mathematical expectation, by virtue of (11) we obtain:

$$\mathbf{M}\{\langle C[y(i+1) - \theta],\ y(i+1) - \theta\rangle / y(i)\} = \langle C[y(i) - \theta],\ y(i) - \theta\rangle - 2(M_1 + i)^{-1}\langle C[y(i) - p(y(i))],\ y(i) - \theta\rangle + (M_1 + i)^{-2}\{\langle C[y(i) - p(y(i))],\ y(i) - p(y(i))\rangle + \text{Sp}CD(y(i))C\},\ t \geqslant 1,$$
$$\langle C[y(1) - \theta],\ y(1) - \theta\rangle = \text{const}.$$

Taking the mathematical expectation in these relations and integrating them for $1 \leqslant i \leqslant t$, we find

$$\mathbf{M}\langle C[y(t+1)-\theta],\ y(t+1)-\theta\rangle=\langle C[y(1)-\theta],\ y(1)-\theta\rangle-2\sum_{i=1}^{t}(M_1+i)^{-1}\mathbf{M}\langle C[y(i)-$$

$$-p(y(i))],\ y(i)-\theta\rangle+\sum_{i=1}^{t}(M_1+i)^{-2}\{\mathbf{M}\langle C[y(i)-p(y(i))],\ y(i)-p(y(i))\rangle+\mathrm{Sp}CD(y(i))C\},\ t\geqslant 1. \tag{19}$$

For $x \in X$ we have $\langle C(x-\theta),\ x-\theta\rangle\leqslant\alpha<\infty$, and, therefore, discarding in (19) the nonnegative terms and taking into account the inequality (15) and the fact that for $t \geqslant 1$ we have $y(t) \in X$, we obtain

$$0\leqslant-\sum_{i=1}^{t}(M_1+i)^{-1}\mathbf{M}\langle C[y(i)-p(y(i))],$$

$$y(i)-\theta\rangle\leqslant\frac{\alpha}{2}. \tag{20}$$

From the relations (15), (20) and from the Markov inequality there follows equality (17) and from that, by means of standard arguments from the theory of stochastic optimization and estimation [3, 4], we obtain the equality (18).

The lemma is proved.

THEOREM 1. If the vector-function of transition probabilities is linear, possesses the s-property or the inverse property s, then there exists a random variable y, $P\{y\in X\}=1$ such that with probability 1 we have $y(t) \to y$ for $t \to \infty$. Moreover, if the vector-function possesses the s-property or the inverse property s, then $P\{y\in\bar{Q}\}=1$, where $\bar{Q}$ is the closure of Q.

Proof. The proof is based on Lemmas 1, 2. In case of linear vector-functions of transition probabilities, the statement follows immediately from Lemma 1. For vector-functions possessing the s-property or the inverse property s, for the proof of the convergence it is necessary to apply the arguments used in the theory of the methods of stochastic optimization and estimation [3, 4]. Since $P\{y(t)\in X\}=1$ for all $t \geqslant 1$, we also have $P\{y\in X\}=1$.

COROLLARY 1.1. If in the urn scheme with balls of two colors the transition probability possesses the inverse property s, then

1) for $p(0)>0$ $P\{y(t)\to 0\}=0$;

2) for $p(1)<1$ $P\{y(t)\to 1\}=0$;

3) for $p(0) > 0$ and $p(0)>0$ и $p(1)<1$ $P\{\{y(t)\to 0\}\cup\{y(t)\to 1\}\}=0$.

The validity of the corollary follows from Theorem 1 since the inequalities $p(0) > 0$ and $p(1) < 1$ are equivalent to $0\notin Q$ and $1\notin Q$.

COROLLARY 1.2. If in the urn scheme with balls of two colors the transition probability possesses the strong inverse property s, then

1) for $p(0)>0$ $P\{\{y(t)\to\theta\}\cup\{y(t)\to 1\}\}=1$;

2) for $p(1)<1$ $P\{\{y(t)\to 0\}\cup\{y(t)\to\theta\}\}=1$;

3) for $p(0) > 0$ and $p(1)<1$ $P\{y(t)\to\theta\}=1$.

The validity of the corollary follows from Corollary 1.1 and from the fact that in the case of a transition probability with the strong inverse property s we have $Q = \{0, \theta, 1\}$.

THEOREM 2. If the vector-function of the transition probabilities $p(\cdot)$ satisfies the s-property and the Hölder condition with the exponent $\mu\in(0, 1]$ in some N-dimensional ball $U(\theta, \varepsilon)$ with center at the point θ and of radius $\varepsilon > 0$, then $P\{y(t)\to\theta\}=0$.

Proof. By virtue of the relations (13), (14), we have

$$[D(x)]_{ii}-\sum_{\substack{j=1\\ j\neq i}}^{N}|[D(x)]_{ij}|=[p(x)]_i\{1-[p(x)]_1-[p(x)]_2-\ldots-[p(x)]_N\},\ i=1, 2, \ldots, N.$$

Since $\theta\in Q\cap\operatorname{Int}X$, we have $p(\theta) = \theta$ and $[p(\theta)]_1+[p(\theta)]_2+\ldots+[p(\theta)]_N<1$. From these two facts we have

$$[D(\theta)]_{ii}-\sum_{\substack{j=1\\ j\neq i}}^{N}|[D(\theta)]_{ij}|>0,\ i=1,2,\ldots,N.$$

Thus, for i ≠ j, i, j = 1, 2,...,N, we have

$$[D(\theta)]_{ii}[D(\theta)]_{jj}>\left\{\sum_{\substack{k=1\\ k\neq i}}^{N}|[D(\theta)]_{ik}|\right\}\left\{\sum_{\substack{k=1\\ k\neq j}}^{N}|[D(\theta)]_{jk}|\right\}.$$

The last inequality and (10) allow us to conclude, by virtue of the result in [5, p. 195], that the matrix D(θ) is positive definite. Now the required result follows from Theorem 4.1 of [3].

The theorem is proved.

COROLLARY 2.1. If in the urn scheme with balls of two colors the transition probability possesses the strong s-property and satisfies the Hölder condition in a neighborhood of the point θ, then

$$P\{\{y(t)\to 0\}\cup\{y(t)\to 1\}\}=1.$$

Physically, the result of Theorem 2 is elucidated by the fact that for a vector-function of transition probabilities possessing the s-property, the points which satisfy the inequality (15) will be unstable [3] for the system of ordinary differential equations

$$dx(t)=[p(x(t))-x(t)]\,dt.$$

We find a lower bound for the probability that the fraction of balls of the i-th color, i = 1, 2,...,N + 1, tends to 1. We note that the event B that the fraction of the balls of the (N + 1)-st color tends to 1 coincides with $\bigcap_{i=1}^{N}\{\lim_{t\to\infty}[y(t)]_i=0\}$. Let $A_i=\{[m(t)]_i=[m_1]_i+t-1\}$, let r(i, t) be vectors in R^N with coordinates $[r(i,t)]_j=[m_1]_j$ for j ≠ i, $[r(i,t)]_i=[m_1]_i+t-1$, i = 1, 2,...,N, $A_{N+1}=\{[m(t)]_i=[m_1]_i,\ i=1,2,\ldots,N\}$. Then

$$P\{\lim_{t\to\infty}[y(t)]_i=1\}\geqslant P\{A_i\},\ i=1,2,\ldots,N,$$

$$P\left\{\bigcap_{i=1}^{N}\{\lim_{t\to\infty}[y(t)]_i=0\}\right\}\geqslant P\{A_{N+1}\},$$

but

$$P\{A_i\}=\prod_{t=1}^{\infty}[p(M(t)^{-1}r(i,t))]_i=\prod_{t=1}^{\infty}\{1-\{1-[p(M(t)^{-1}r(i,t))]_i\}\},\ i=1,2,\ldots,N,$$

$$P\{A_{N+1}\}=\prod_{t=1}^{\infty}\left\{1-\sum_{i=1}^{N}[p(M(t)^{-1}m_1)]_i\right\},$$

and, therefore

$$P\{\lim_{t\to\infty}[y(t)]_i=1\}\geqslant\prod_{t=1}^{\infty}\{1-\{1-[p(M(t)^{-1}r(i,t))]_i\}\},\ i=1,2,\ldots,N, \tag{21}$$

$$P\left\{\bigcap_{i=1}^{N}\{\lim_{t\to\infty}[y(t)]_i=0\}\right\}\geqslant\prod_{t=1}^{\infty}\left\{1-\sum_{i=1}^{N}[p(M(t)^{-1}m_1)]_i\right\}. \tag{22}$$

Known results on the convergence of infinite products [2, p. 335] and the estimates (21), (22) show that the following statement holds.

THEOREM 3. If $\sum_{t=1}^{\infty}\{1-[p(M(t)^{-1}r(i,t))]_i\}<\infty$, then $P\{\lim_{t\to\infty}[y(t)]_i=1\}>0$, $i=1,2,\ldots,N$, while for $\sum_{t=1}^{\infty}\sum_{i=1}^{N}[p(M(t)^{-1}m_1)]_i<\infty$ $P\left\{\bigcap_{i=1}^{N}\{\lim_{t\to\infty}[y(t)]_i=0\}\right\}=P\{B\}>0.$

6. LIMIT THEOREMS AND RATE OF CONVERGENCE

Let $D^N[0, T]$ be the space of N-dimensional vector-functions (defined on $[0, T]$, $0 < T < \infty$, and having no discontinuities of the second kind), endowed with A. V. Skorokhod's metric [6, p. 497]. For $n \geqslant 1$ we construct in $D^N[0, T]$ the random processes

$$x_n(t) = \sqrt{s}\,[y(s) - \theta] \text{ for } n \leqslant s \leqslant ne^t < s+1 \leqslant ne^T.$$

THEOREM 4. Assume that the vector-function of transition probabilities $p(\cdot)$ possesses the inverse property s and

1) there exists a unique point $\theta \in Q \cap \operatorname{Int} X$ such that $P\{\lim\limits_{t\to\infty} y(t) = \theta\} = 1$;

2) for some $\varepsilon > 0$ the vector-function $p(\cdot)$ is differentiable on $U(\theta, \varepsilon)$ and the matrix $P(\theta) - E/2$ is stable [3], where $P(\theta)$ is the matrix with the entries $[P(\theta)]_{ij}$, $[P(\theta)]_{ij} = \left.\dfrac{\partial [p(x)]_i}{\partial [x]_j}\right|_{x=\theta}$, $i, j = 1, 2, \ldots, N$ and E is the identity matrix in R^N.

Then the random processes $x_n(\cdot)$ converge weakly in $D^N[0, T]$ as $n \to \infty$ to the stationary Gaussian Markov process $x(\cdot)$, satisfying Ito's stochastic differential equation

$$dx(t) = \left[P(\theta) - \frac{1}{2}E\right]x(t)\,dt + D(\theta)^{1/2}dw(t),$$

where $D(\theta)^{1/2}$ is the "nonnegative definite square root" of $D(\theta)$, which is a symmetric matrix, and $w(\cdot)$ is the standard N-dimensional Wiener process.

Proof. Taking into account the relations (9)-(14), the validity of the assertion of the theorem follows from Theorem 4.5 of [7, p. 83]*.

COROLLARY 4.1. If the assumptions of Theorem 4 hold, then in distribution for $n \to \infty$ we have

$$\sqrt{n}\,[y(n) - \theta] = x_n(0) \to N\left(0, \int_0^\infty \exp\left\{\left[P(\theta) - \frac{1}{2}E\right]t\right\} D(\theta) \exp\left\{\left[P(\theta) - \frac{1}{2}E\right]t\right\} dt\right),$$

where $N(0, \Xi)$ is the normal distribution in R^N with mean vector 0 and variance matrix Ξ.

Remark 4.1. The results of Theorem 4 can be used for the approximate computation of the probabilities connected with the collections of random vectors $y(t)$, $n \leqslant t \leqslant ne^T$, where $n \geqslant 1$, $0 < T < \infty$ [7, p. 95],* while those of Corollary 4.1 for the approximate computation of the probabilities referring to the random vector $y(n)$ for large n.

Remark 4.2. Theorem 4 and Corollary 4.1 characterize the rate of convergence of $y(t)$ to θ in those cases when it is known that $y(t) \to \theta$ with probability 1.

7. THE REFINEMENT OF THE CONVERGENCE THEOREMS FOR URNS WITH BALLS OF TWO COLORS

The fraction of white balls at moment t cannot be greater than $L(t) = [m(1) - 1 + t](c + t)^{-1}$ and smaller than $l(t) = m(1)(c + t)^{-1}$. Let $Q_1 = Q \setminus \{\{0\} \cup \{1\}\}$ and

$$\underline{\theta} = \min_{\theta \in Q_1} \theta > 0,\ \overline{\theta} = \max_{\theta \in Q_1} \theta < 1.$$

For arbitrary $\varepsilon \in (0, \underline{\theta})$ and $\delta \in (\overline{\theta}, 1)$ we set $t(\varepsilon) = \min t\colon l(t) < \varepsilon$, $t(\delta) = \min t\colon L(t) > \delta$. Assume that for $t \geqslant t(\varepsilon)$ we have

$$v(t) = \inf_{l(t) \leqslant x \leqslant \varepsilon} [p(x) - x],$$

while for $t \geqslant t(\delta)$ we have

$$u(t) = \inf_{\delta \leqslant x \leqslant L(t)} [x - p(x)].$$

THEOREM 5. If the transition probability possesses the inverse property s and if for some $\tau > 0$ we have $Q \setminus \{\{0\} \cup \{1\}\} \subset [\tau, 1 - \tau]$, then

*Reference [7] does not appear in Russian original — Publisher.

1) for $\sum_{t=t(\varepsilon)}^{\infty}(a+t)^{-1}v(t)=\infty \quad P\{\lim_{t\to\infty} y(t)=0\}=0;$

2) for $\sum_{t=t(\delta)}^{\infty}(a+t)^{-1}u(t)=\infty \quad P\{\lim_{t\to\infty} y(t)=1\}=0;$

3) for $\sum_{t=\max[t(\varepsilon),t(\delta)]}^{\infty}(a+t)^{-1}[v(t)+u(t)]=\infty \quad P\{\{\lim_{t\to\infty} y(t)=0\}\cup\{\lim_{t\to\infty} y(t)=1\}\}=0.$

Proof. We consider only condition 1 since in the other cases the arguments are similar.

From the inequality (2') and the equality (17) there follows the existence of a sequence t_i, $i \geqslant 1$, such that with probability 1 we have

$$\sum_{t=t_i}^{\infty}(a+t)^{-1}[y(t)-\theta][y(t)-p(y(t))]\to 0,\ i\to\infty. \tag{23}$$

If we denote by Ω_1 the event consisting in the fact that relation (23) holds, then $P\{\Omega_1\} = 1$. Let $\omega\in\Omega_1$ and assume that on ω we have $y(t) \to 0$ as $t \to \infty$. If this relation holds, then there exists $t_0 = t_0(\omega)$ such that for $t \geqslant t_0$ we have $l(t) \leqslant y(t) < \varepsilon$ and

$$\sum_{i=t}^{\infty}(a+i)^{-1}[y(t)-\theta][y(t)-p(y(t))]\geqslant(\theta-\varepsilon)\sum_{i=t}^{\infty}(a+i)\,v(i)=\infty.$$

This contradicts relation (23). Thus, for each $\omega\in\Omega_1$ one cannot have $y(t) \to 0$ and, consequently, $P\{\lim_{t\to\infty} y(t)=0\}=0.$

The theorem is proved.

Remark 5.1. Theorem 5 generalizes Corollary 1.1.

Remark 5.2. If for $x \to 0$ we have $p(x) - x \geqslant k(\ln x^{-1})^{-1}$, then condition 1 of Theorem 5 holds and if for $x \to 1$ we have $x - p(x) \geqslant k[\ln(1-x)^{-1}]^{-1}$, then condition 2 holds. Here and in the sequel, k is a nonnegative constant, not necessarily always the same.

Remark 5.3. In Theorem 5 we have not used directly the fact that the transition probability possesses the inverse property s. It is essential that $p(x) - x$ should preserve its sign, should converge sufficiently slowly to 0 for $x \to 0$ or $x \to 1$. However, if the transition probability possesses the s-property, then $p(x) \leqslant x$ as $x \to 0$ and $p(x) \geqslant 1 - x$ as $x \to 1$ and, consequently, the series which occur in the formulation of Theorem 5 converge, i.e., this theorem does not allow us to obtain any meaningful results.

In the case of an urn with balls of two colors, Theorem 3 takes the following form.

THEOREM 3'. If the transition probability possesses the s-property, then for $\sum_{t=1}^{\infty}p(m(1)\times(c+t)^{-1})<\infty$ we have $P\{\lim_{t\to\infty} y(t)=0\}>0$, while for $\sum_{t=1}^{\infty}\{1-p([m(1)-1+t](c+t)^{-1})\}<\infty$ we have $\{\lim_{t\to\infty} y(t)=1\}>0$.

Remark 3'.1. The inequalities (21), (22) hold also then when the transition probability possesses the inverse property s. However, in this case their right-hand sides are equal to zero and the inequalities are trivial.

Remark 3'.2. If for $x \to 0$ we have $p(x)\leqslant kx^{1+\mu}$, $\mu > 0$, or $1-p(1-x)\leqslant k(1-x)^{1+\varkappa}$, $\varkappa>0$, then the first or the second series, respectively, in the formulation of Theorem 3' converges.

Remark 3'.3. With the aid of the inequalities (21), (22) one can compute a lower bound $P\{y(t) \to 0\}$ and $P\{y(t) \to 1\}$. This is especially convenient if $p(x) \equiv 0$ for $0 \leqslant x \leqslant \delta_1$ or $p(x) \equiv 1$ for $\delta_2 \leqslant x \leqslant 1$. In particular, if $p(x) \equiv 0$ for $0 \leqslant x \leqslant m(1)[m(1) + n(1) + 1]^{-1}$ and $p(x) \equiv 1$ for $[m(1) + 1][m(1) + n(1) + 1]^{-1} \leqslant x \leqslant 1$, then, with nonzero probability, $y(t)$ can converge only to the points 0 and 1 and, moreover,

$$P\{y(t)\to 0\}=1-p(m(1)[m(1)+n(1)]^{-1}),\ P\{y(t)\to 1\}=p(m(1)[m(1)+n(1)]^{-1}).$$

Observing that at the points of the set Q we have $p(\theta) = \theta$, Theorem 4 can take the following form.

THEOREM 4'. Assume that the transition probability possesses the inverse property s and that

1) there exists $\theta \in Q \cap (0, 1)$ such that $P\{y(t) \to \theta\} = 1$; 2) $p(\cdot)$ is differentiable at the point θ and $p'(\theta) < 1/2$.

Then:

a) for $n \to \infty$ the random processes $x_n(\cdot)$ converge weakly in $D^1[0, T]$ to a stationary Gaussian Markov process of the form $dx(t) = [p'(\theta) - 1/2]x(t)dt + \sqrt{\theta(1-\theta)}dw(t)$, where $w(\cdot)$ is the standard one-dimensional Wiener process [6, p. 196];

b) with probability 1 we have

$$\overline{\lim_{t\to\infty}}\left(\frac{t}{2\ln\ln t}\right)^{1/2}\left[\frac{1-2p'(\theta)}{\theta(1-\theta)}\right]^{1/2}[y(t)-\theta]=1;$$

$$\underline{\lim_{t\to\infty}}\left(\frac{t}{2\ln\ln t}\right)^{1/2}\left[\frac{1-2p'(\theta)}{\theta(1-\theta)}\right]^{1/2}[y(t)-\theta]=-1.$$

Proof. Assertion a) is a special case of Theorem 4, while assertion b), taking into account (9)-(14), is a special case of Theorem 1 [8].*

By virtue of the importance in practice of the generalized urn schemes with transition probabilities possessing the strong s-property and the strong inverse property s, we formulate separately the statements proved for them.

Conclusions. If the transition probability $p(\cdot)$ possesses the strong s-property, then, with nonzero probability, the fraction $y(t)$ of the white balls can tend only to the points $0, \theta, 1$. Moreover, if in the neighborhood of the point θ, $p(\theta)$ satisfies the Hölder condition, then the probability of the fact that $y(t)$ converges to θ is equal to 0. If $p(x)$ approaches 0 sufficiently fast as $x \to 0$, for example, as $x^{1+\mu}$, $\mu > 0$, then $y(t)$ converges to 0 with positive probability, while if $1 - p(x)$ approaches 0 sufficiently fast as $x \to 1$, for example as $(1-x)^{1+\varkappa}$, $\varkappa > 0$, then $y(t)$ converges to 1 with the same probability.

If the transition probability $p(\cdot)$ possesses the strong inverse property s, then, with nonzero probability, the fraction $y(t)$ of white balls can converge only to the points $0, \theta, 1$. Moreover, if $p(x) - x$ approaches 0 in a sufficiently slow manner as $x \to 0$, for example $p(x) - x \geqslant \varepsilon_1 > 0$ or as $(\ln x^{-1})^{-1}$, then $y(t)$ converges to 0 with nonzero probability, while if $x - p(x)$ approaches 0 in a sufficiently slow manner as $x \to 1$, for example $x - p(x) \geqslant \varepsilon_2 > 0$ or as $[\ln(1-x)^{-1}]^{-1}$, then $y(t)$ converges to 1 with the same probability.

8. THE APPLICATION OF THE PROVED THEOREMS TO THE STUDY OF SIMULATING PROCESSES FOR THE DECISION MAKING RULES $R_1 - R_3$

As shown in Sec. 2, for $r > 1$ and $m > 1$ the rules R_1 and R_3 possesses the strong s-property with $\theta = 1/2$. From Eq. (3) it follows that $p(\cdot)$ is differentiable, $\lim_{x\to 0} p(x) \times (C_r^{\frac{r+1}{2}} x^{\frac{r+1}{2}})^{-1} = 1$ and $(r + 1)/2 > 2$. Therefore, from the given conclusions it follows that

$$P\left\{\lim_{t\to\infty} y(t)=\frac{1}{2}\right\}=0, \quad \text{i.e.,} \quad P\{\{\lim_{t\to\infty} y(t)=0\}\cup\{\lim_{t\to\infty} y(t)=1\}\}=1,$$

and $P\{\lim_{t\to\infty} y(t)=0\}>0$.

The rule R with $r \geqslant 1$, as shown in Sec. 2, possesses the strong s-property with $\theta = 1/2$. From formula (4) it is clear that $p(0) > 0$ and $p(1) < 1$ and, therefore, by virtue of the formulated conclusions, we have

$$P\left\{\lim_{t\to\infty} y(t)=\frac{1}{2}\right\}=1.$$

*Reference [8] does not appear in Russian original — Publisher.

We note that also in this case p(·) is differentiable and

$$p'\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^{r-1}\sum_{k=0}^{\frac{r-1}{2}}C_r^k(2k-r)<0,$$

i.e., Theorem 4' holds with $\theta(1-\theta)=1/4$.

Returning to the problem of the adoption of the technologies, considered in Sec. 1, we can draw the following conclusions. We identify a unit of technology A with a white ball. The application by the manager of the rule R_1 with $r > 1$ or R_2 with $m > 1$ leads to the fact that with nonzero probability only A or only B will be applied. Moreover, the probability with which B will fill out the market is different from zero, while the probability for the technology A can be zero. The use of the rule R_1 with $r \geqslant 1$ leads to the fact that with probability 1 the market will be filled out by both technologies in equal proportion.

LITERATURE CITED

1. N. L. Johnson and S. Kotz, Urn Models and Their Application. An Approach to Modern Discrete Probability, Wiley, New York (1977).
2. G. M. Fikhtengol'ts, Differential and Integral Calculus [in Russian], Vol. 2, Nauka, Moscow (1969).
3. M. B. Nevel'son and R. Z. Khas'minskii (Hasminskii), Stochastic Approximation and Recursive Estimation, Am. Math. Soc., Providence (1976).
4. I. I. Gikhman (Gihman) and A. V. Skorokhod (Skorohod), The Theory of Stochastic Processes, Vol. I, Springer, New York (1974).
5. Yu. M. Kaniovskii, P. S. Knopov, and Z. V. Nekrylova, Limit Theorems for Stochastic Programming Processes [in Russian], Naukova Dumka, Kiev (1980).
6. V. F. Gaposhkin and T. P. Krasulina, "On the law of the iterated logarithm in stochastic approximation processes," Teor. Veroyatn. Primen., 19, No. 4, 879-886 (1974).

DISTANCES AND CONSENSUS RANKINGS

B. G. Litvak

UDC 519.584

Expert judgments are used in many forecasting, planning, and management problems in national economy. Successful application of expert judgments is largely determined by the sophistication of the mathematical apparatus available for the analysis and processing of such information.

One of the main tasks in expert judgment framework is to rank a given set of objects. In this article we give an axiomatic definition of distances (nearness measures) between rankings and suggest how consensus rankings can be constructed. Section 1 considers a Hamming distance and its generalization to metrized rankings; Sec. 3 introduces a Spearman distance, and Sec. 5 a Euclidean distance on metrized rankings. The axiomatic systems are consistent and complete, and all the distances are uniquely defined. Sections 2, 4, and 6 focus on the construction of the consensus rankings. Section 2 proposes an algorithm to construct a metrized ranking by the branch-and-bound technique. The consensus ranking based on the Spearman distance (Sec. 4) is constructed by solving an assignment problem, while for the additive metrized ranking corresponding to the Euclidean distance an explicit formula is derived.

The consensus ranking methods proposed in this article may be used to process and analyze expert judgments.

1. HAMMING DISTANCES ON RANKINGS AND ARBITRARY RELATIONS

Consider m experts attempting to rank a set of objects by preference. The most common methods of elucidating the expert judgments in such a case involve a direct ranking of the

Translated from Kibernetika, No. 1, pp. 57-63, January-February, 1983. Original article submitted June 18, 1979.

0011-4235/83/1901-0071$07.50