MAPPINGS CONJUGATE TO CONVEX MANY-VALUED MAPPINGS

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A great number of extremal problems of practical importance are described in terms of many-valued (point-set) mappings. Among problems of such type are optimal control problems for discrete processes, the problem of finding the minimax for connected sets, on which is based the solution of many nonantagonistic games with information transfer, a whole series of problems connected with the investigation of a family of mathematical programming problems, etc. Pshenichnyi [1] introduced the concept of a conjugate transformation for convex many-valued mappings and pointed out the fruitfulness of using such a transformation for investigating and solving extremal problems given by convex many-valued mappings. But in order for conjugate mappings to become a convenient and effective mathematical tool for the investigation and solution of convex extremal problems, it is necessary to develop a technique for computing the conjugate mappings for complex convex many-valued mappings obtained as a result of operations over other convex many-valued mappings. The creation of a certain foundation for such a technique for computing the conjugate mappings is the first and foremost aim of the present paper. However, it should be noted that several results of the paper (Theorems 6, 8, 9) are of independent interest. In particular, by using the method of proof of Theorem 2 of [1], from Theorem 8 we easily obtain a generalization of the marginal value theorem to the case of arbitrary locally convex spaces.

Basic Definitions and Notation. Auxiliary Results

As in [1] all problems are investigated in real locally-convex separable linear topological spaces X, Y, Z.

Let f be some function on X with values from $[-\infty, +\infty]$. It is obvious that for a convex function f the set

$$\operatorname{dom} f = \{x : x \in X, f(x) < +\infty\}$$

is a convex set, while the convexity of the set

$$\det f = \{(x, \alpha) : (x, \alpha) \in X \times R, \alpha \geqslant f(x)\}$$

is equivalent to the convexity of function f. By $\partial f(x_0)$ we denote the subdifferential of the convex function f at point x_0; however, if φ is a convex function of the product of spaces X and Y, then the subdifferential of φ at the point (x_0, y_0) is denoted by $\partial_{X,Y}\varphi(x_0, y_0)$, whereas $\partial_X \varphi(x_0, y)$ is the subdifferential of function $\varphi(x, y)$ at point x_0 with respect to the argument x for a fixed y.

By M(X, Y) we denote the collection of all many-valued mappings of space X into space Y; here, in contrast to [2], we allow that an empty set can be a value of a many-valued mapping. We stipulate that for this case, for any $\xi \subset Y$, $\lambda \in R$,

$$\xi + \varnothing = \varnothing;$$
$$\lambda\varnothing = \varnothing.$$

Let $a \in M(X, Y)$. We say that mapping a is convex if for any $x_1, x_2 \in X$

$$a(\lambda x_1 + (1-\lambda)x_2) \supseteq \lambda a(x_1) + (1-\lambda)a(x_2),$$
$$0 \leqslant \lambda \leqslant 1.$$

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Two sets are closely related with each convex many-valued mapping:

$$\operatorname{dom} a = \{x : x \in X,\ a(x) \neq \varnothing\},$$
$$\operatorname{graf} a = \{(x, y) : (x, y) \in X \times Y,\ y \in a(x)\},$$

while with each pair $(x_0, y_0) \in$ graf a, the cones of admissible directions

$$K_{y_0}(a, x_0) = \{\bar{y} : \bar{y} = \lambda(y - y_0),\ y \in a(x_0),\ \lambda > 0\},$$
$$K_a(x_0, y_0) = \{(\bar{x}, \bar{y}) : (\bar{x}, \bar{y}) = \lambda(x - x_0, y - y_0),$$
$$(x, y) \in \operatorname{graf} a,\ \lambda > 0\}.$$

It is easy to see that graf a and dom a are convex sets in $X \times Y$ and X, respectively. We also note that the giving of set graf a uniquely defines the mapping a since if ξ is a convex set in $X \times Y$, then the mapping $a(x) = \{y : (x, y) \in \xi\}$ is convex and graf $a = \xi$.

In what follows we denote by W(X, Y) the collection of all convex many-valued mappings of space X into Y. Let $a \in W(X, Y)$, $y^* \in Y^*$. We set

$$w_a(x, y^*) = \begin{cases} \inf\limits_y \{ < y, y^* > : y \in a(x)\}, & x \in \operatorname{dom} a; \\ +\infty, & x \bar{\in} \operatorname{dom} a. \end{cases}$$

The function $w_a(x, y^*)$ is a convex function and

$$\operatorname{dom} w_a(x, y^*) = \operatorname{dom} a.$$

The many-valued mapping $a^*_{x_0} : Y^* \to X^*$, defined by the rule

$$a^*_x(y^*) = \partial w_a(x_0, y^*),$$

is said to be conjugate to a at point x_0.

THEOREM 1. $(-x^*, y^*) \in K^*_a(x_0, y_0)$ if and only if $y^* \in K^*_{y_0}(a, x_0)$, while $x^* \in a^*_{x_0}(y^*)$.

THEOREM 2. Let φ be a continuous convex function on $X \times Y$ and let there exist a point (x_1, y_1) such that $\varphi(x_1, y_1) < 0$. In addition, let the lower bound of $<y, y^*>$ with respect to y on the set $\varphi(x_0, y) \leq 0$ be achieved at a point y_0. Then for the mapping

$$a(x) = \{y : \varphi(x, y) \leqslant 0\}$$

$$a^*_{x_0}(y^*) = \{x^* : (-x^*, y^*) \in \bigcup_{\gamma \leqslant 0} \gamma \partial_{x, y} \varphi(x_0, y_0)\}.$$

Both theorems are due to Pshenichnyi [1]. The first one of them permits us to reduce the finding of a conjugate mapping to describing a cone dual to $K_a(x_0, y_0)$, which in many cases essentially simplifies the tasks. The second theorem describes the conjugate mapping for a very important class of convex many-valued mappings and is a corollary of the first.

Operations over Many-Valued Mappings

Definition 1. A mapping $a \in M(X, Y)$ is called the sum of mappings $a_1 + a_2 \in M(X, Y)$ and is denoted $a_1 + a_2$ if

$$a(x) = a_1(x) + a_2(x).$$

By the product of a mapping $b \in M(X, Y)$ on $\lambda \in R$ we mean the mapping λb, where

$$(\lambda b)(x) = \lambda b(x).$$

It is easily seen that relative to the operation of addition the set M(X, Y) is an Abelian semigroup, while the operation of multiplication by a number from R is commutative, associative, and distributive. We note that

$$\operatorname{dom} a_1 + a_2 = \operatorname{dom} a_1 \bigcap \operatorname{dom} a_2,$$

$$\operatorname{dom} \lambda b = \operatorname{dom} b.$$

The set M(X, Y) can be partially ordered. For this it is sufficient to set $a > b$ if for all $x \in X$, $a(x) \supseteq b(x)$. If ξ is an arbitrary subset of mappings, then by

$$\bigvee_{a \in \xi} a, \quad \bigwedge_{a \in \xi} a$$

we denote, respectively, the upper and the lower bounds of set ξ, i.e., the following mappings:

$$\left(\bigvee_{a \in \xi} a\right)(x) = \bigcup_{a \in \xi} a(x);$$

$$\left(\bigwedge_{a \in \xi} a\right)(x) = \bigcap_{a \in \xi} a(x).$$

It is obvious that in the complete distributive lattice M(X, Y) the zero element is the mapping $a(x) \equiv \emptyset$, while the unit element is $b(x) \equiv Y$.

With each mapping $a \in M(X, Y)$ there uniquely corresponds the mapping $a^{-1} \in a$, defined inverse to M(Y, X) as follows:

$$a^{-1}(y) = \{x : y \in a(x)\}.$$

It is understood that $a^{-1}(a\ (\operatorname{dom} a)) = \operatorname{dom} a$ and, in addition, $(a^{-1})^{-1} = a$.

Definition 2. If $a \in M(X, Y)$, $b \in M(Y, Z)$, then by the product of mappings a and b we mean the mapping $b \circ a \in M(X, Z)$, where

$$(b \circ a)(x) = b(a(x)).$$

We note that $(b \circ a)^{-1} = a^{-1} \circ b^{-1}$.

Let A be a linear operator mapping space X into space Y, and let B be a linear operator mapping space Z into space U. Then with every $a \in M(Y, Z)$ we can associate a mapping $b \in M(X, U)$ defined as follows:

$$b(x) = Ba(Ax). \tag{1}$$

THEOREM 3. 1. The set W(X, Y) is closed relative to the operations of addition and multiplication by an element of R and to the operation of intersection of mappings.

2. If $a \in W(X, Y)$, then $a^{-1} \in W(Y, X)$.

3. If $a \in W(X, Y)$, $b \in W(Y, Z)$, then $b \circ a \in W(X, Z)$.

4. If $a \in W(Y, Z)$, A is a linear operator from X into Y, B is a linear operator from Z into U, then the mapping defined by formula (1) is a convex many-valued mapping of space X into U.

To prove the theorem it is enough to be convinced that in each of the items the graph of the many-valued mapping – the result of the operation – is a convex set.

Computation of Conjugate Mappings

THEOREM 4. Let $a \in W(X, Y)$ and $\lambda \in R$. Then

$$(\lambda a)^*_{x_0}(y^*) = |\lambda| a^*_{x_0}(\operatorname{sign} \lambda y^*).$$

Proof. For all $x \in X$ and $\lambda \in R$

$$w_{\lambda a}(x, y^*) = \inf_y \{ < y, y^* > : y \in \lambda a(x)\} = \inf_y \{\lambda < y, y^* > : y \in a(x)\} =$$

$$= |\lambda| \inf_y \{ < y, \operatorname{sign} \lambda y^* > : y \in a(x)\} = |\lambda| w_a(x, \operatorname{sign} \lambda y^*).$$

The theorem's validity now follows at once from the definition of a conjugate mapping.

THEOREM 5. Let a, $b \in W(X, Y)$. Assume that there exists a point from $a \cap \operatorname{dom} b$ at which either the function $w_a(x, y^*)$ or the function $w_b(x, y^*)$ is continuous. Then

$$(a+b)^*_{x_0}(y^*) = (a^*_{x_0} + b^*_{x_0})(y^*).$$

Proof. For all $x \in \operatorname{dom} a \cap \operatorname{dom} b$

$$w_{a+b}(x, y^*) = \inf_y \{ <y, y^*> : y \in a(x) + b(x)\} = \inf_{y_1, y_2} \{<y_1, y^*> + <y_2, y^*> : y_1 \in a(x), y_2 \in b(x)\} =$$

$$= \inf_{y_1} \{ <y_1, y^*> : y_1 \in a(x)\} + \inf_{y_2} \{<y_2, y^*> : y_2 \in b(x)\} = w_a(x, y^*) + w_b(x, y^*).$$

Consequently,

$$w_{a+b}(x, y^*) = \begin{cases} w_a(x, y^*) + w_b(x, y^*), & x \in \operatorname{dom} a \operatorname{dom} b; \\ +\infty, & x \bar{\in} \operatorname{dom} a \operatorname{dom} b. \end{cases}$$

By the theorem's hypothesis there exists a point from $\operatorname{dom} w_a(x, y^*) \cap \operatorname{dom} w_b(x, y^*)$ at which one of the functions, say $w_a(x, y^*)$, is continuous. This allows us to take advantage of Theorem 1 from [3] and to complete the proof of the theorem.

THEOREM 6. Let $a \in W(X, Y)$ and let the lower bound of $\langle x, x^* \rangle$ with respect to x on the set $a^{-1}(y_0)$ be achieved at point x_0. Then

$$(a^{-1})^*_{y_0}(x^*) = -(b^*_{x_0})^{-1}(-x^*),$$

where $b^*_{x_0}$ is the restriction of the mapping $a^*_{x_0}$ to the cone $K^*_{y_0}(a, x_0)$.

Proof. The lower bound of $<x, x^*>$ with respect to x on the set $a^{-1}(y_0)$ is achieved at point x_0; therefore, $x^* \in K^*_{x_0}(a^{-1}, y_0)$. By Theorem 1 $-y^* \in (a^{-1})^*_{y_0}(x^*)$ if and only if $(y^*, x^*) \in K_{a^{-1}}(y_0, x_0)$. But since

$$\operatorname{graf} a^{-1} = \{(y, x) : x \in a^{-1}(y)\} = \{(y, x) : y \in a(x)\} = \{(y, x) : (x, y) \in \operatorname{graf} a\},$$

then $K_{a^{-1}}(y_0, x_0) = \{(\bar{y}, \bar{x}) : (\bar{x}, \bar{y}) \in K_a(x_0, y_0)\}$ and $(x^*, y^*) \in K^*_a(x_0, y_0)$. Consequently,

$$(x^*, y^*) \in K^*_a(x_0, y_0)$$

if and only if $y^* \in K^*_{y_0}(a, y_0)$, while $-x^* \in a^*_{x_0}(y^*)$. Therefore, $(a^{-1})^*_{y_0}(x^*) = \{-y^* : -x^* \in a^*_{x_0}(y^*), y^* \in K^*_{y_0}(a, x_0)\} =$ $-\{y^* : -x^* \in b^*_{x_0}(y^*)\} = -(b^*_{x_0})^{-1}(-x^*)$.

THEOREM 7. Let a, $b \in W(X, Y)$ and let there exist a point in $\operatorname{graf} a \cup \operatorname{graf} b$, an interior point either for graf a or for graf b. Assume that the lower bound of $<y, y^*>$ with respect to y on the set $a(x_0) \cap b(x_0)$ is achieved at point y_0 and that there exists a point in $a(x_0) \cap b(x_0)$, an interior point either for $a(x_0)$ or for $b(x_0)$. Then

$$(a \wedge b)^*_{x_0}(y^*) = \{x^* : x^* = x^*_1 + x^*_2, x^*_1 \in a^*_{x_0}(y^*_1),$$
$$x^*_2 \in b^*_{x_0}(y^*_2), y^*_1 \in K^*_{y_0}(a, x_0), y^*_2 \in K^*_{y_0}(b, x_0), y^* = y^*_1 + y^*_2\}. \tag{2}$$

Proof. Let $c = a \wedge b$ and $x^* \in c^*_{x_0}(y^*)$. Since $\operatorname{graf} c = \operatorname{graf} a \cap \operatorname{graf} b$ and there exists a point of graf c which is interior either for graf a or for graf b,

$$K^*_c(x_0, y_0) = K^*_a(x_0, y_0) + K^*_b(x_0, y_0). \tag{3}$$

The lower bound of $<y, y^*>$ with respect to y on the set $c(x_0)$ is achieved at point y_0; therefore, $y^* \in K^*_{y_0}(c, x_0)$ and by Theorem 1

$$(-x^*, y^*) \in K^*_c(x_0, y_0).$$

Using (3) we obtain

$$(-x^*, y^*) = (-x^*_1 y^*_1) + (-x^*_2, y^*_2),$$

where $(-x_1^*, y_1^*) \in K_a^*(x_0, y_0), (-x_2^*, y_2^*) \in K_b^*(x_0, y_0)$. But by the same theorem

$$x_1^* \in a_{x_0}^*(y_1^*),\ y_1^* \in K_{y_0}^*(a, x_0),$$
$$x_2^* \in b_{x_0}^*(y_2^*),\ y_2^* \in K_{y_0}^*(b, x_0). \tag{4}$$

Consequently, $x^* = x_1^* + x_2^*$, $y^* = y_1^* + y_2^*$, where x_1^*, x_2^*, y_1^*, y_2^* satisfy (4). Thus, we have shown that $c_{x_0}^*(y^*)$ is included in the right-hand side of (2).

Now let x_0^* belong to the right-hand side of (2), i.e., there exist x_1^*, x_2^*, y_1^*, y_2^* which satisfy (4) and $x_1^* + x_2^* = x_0^*$, $y_1^* + y_2^* = y^*$. We note that $y^* \in K_{y_0}^*(c, x_0)$, since by virtue of the last hypothesis of the theorem

$$K_{y_0}^*(c, x_0) = K_{y_0}^*(a, x_0) + K_{y_0}^*(b, x_0).$$

Therefore, by Theorem 1

$$(-x_1^*, y_1^*) \in K_a^*(x_0, y_0),\ (-x_2^*, y_2^*) \in K_b^*(x_0, y_0),$$
$$c_{x_0}^*(y^*) = \{x^* : (-x^*, y^*) \in K_c^*(x_0, y_0)\}.$$

From equality (3) we obtain

$$(-x_1^*, y_1^*) + (-x_2^*,\ y_2^*) = (-x_0^*, y^*) \in K_c^*(x_0, y_0).$$

Consequently, $x_0^* \in c_{x_0}^*(y^*)$. By the same token we have proved the inclusion of the right-hand side of (2) in $c_{x_0}^*(y^*)$, which proof completes the proof of the whole theorem.

Let $a \in W(X, Z)$, φ_0 be a real-valued convex continuous function on $X \times Z$. We set

$$f(x) = \begin{cases} \inf_z \{\varphi_0(x, z) : z \in a(x)\}, & x \in \operatorname{dom} a; \\ +\infty, & x \bar{\in} \operatorname{dom} a. \end{cases} \tag{5}$$

It is easy to verify that f is a convex function. We compute the subdifferential $\partial f(x_0)$ of function f at the point x_0, assuming that the lower bound of $\varphi_0(x_0, z)$ with respect to z on the set $a(x_0)$ is achieved at a point z_0. Let $Y = Z \times R$ such that y is a pair (z, α). We set

$$\varphi(x, y) = \varphi_0(x, z) - \alpha$$

and we introduce the following convex many-valued mappings into consideration:

$$b(x) = a(x) \times R;$$
$$c(x) = \{y : \varphi(x, y) \leqslant 0\};$$
$$d = b \wedge c.$$

If $y^*(z^*, \alpha^*)$, then

$$w_d(x, y^*) = \inf_{(z,\alpha)} \{<z, z^*> + \alpha^*\alpha : z \in a(x),$$
$$\varphi_0(x, z) \leqslant \alpha\} = \begin{cases} \inf_z \{<z, z^*> + \alpha^*\varphi_0(x, z) : z \in a(x)\}, & \alpha^* \geqslant 0; \\ -\infty, \alpha^* < 0. \end{cases}$$

In particular, if $y_0^* = (0^*, 1)$, then $w_d(x, y_0^*) = f(x)$. Consequently, the computation of $\partial f(x_0)$ is reduced to the computation of $d_{x_0}^*(y_0^*)$. Since $d = b \wedge c$, we make use of the preceding theorem. We convince ourselves that all the hypotheses of Theorem 7 are fulfilled. Since

$$\operatorname{graf} b = \operatorname{graf} a \times R,\ \operatorname{graf} c = \det \varphi_0,$$

we have that

$$(x, z, \alpha) \in \operatorname{int}(\operatorname{graf} c) \cap \operatorname{graf} b.$$

is valid for any triple $(x, z) \in \operatorname{graf} a$, $\alpha \in \varphi_0(x, z)$ by virtue of the continuity of φ_0. In exactly the same way we can prove that there exists a point from $d(x_0)$ which is an interior point for $c(x_0)$. Since $f(x_0) = \varphi_0(x_0, z_0)$, where $z_0 \in a(x_0)$, and $y_0^* = (0^*, 1)$, the lower bound of $<y, y_0^*>$ with respect to y on the set $d(x_0)$ is achieved at the point $y_0 = (z_0, \varphi_0(x_0, z_0))$. Consequently, by Theorem 7,

$$\partial f(x_0)=\{x^*:x^*=x_1^*+x_2^*, x_1^*\in b_{x_0}^*(y_1^*), x_2^*\in c_{x_0}^*(y_2^*);\ y_1^*\in K_{y_0}^*(b,x_0), y_2^*\in K_{y_0}^*(c,x_0),\ y_0^*=y_1^*+y_2^*\}. \tag{6}$$

Let us compute the dual cones and the conjugate mappings occurring in (6). Since the function φ_0 is continuous, by Theorem 2.2 of [4]

$$K_{y_0}^*(c,x_0)=\bigcup_{\gamma\leqslant 0}\gamma\partial_y\varphi(x_0,y_0)=\bigcup_{\gamma\leqslant 0}\gamma(\partial_z\varphi_0(x_0,z_0)\times\{-1\}).$$

It is obvious that

$$K_{y_0}^*(b,x_0)=K_{z_0}^*(a,x_0)\times\{0\}.$$

Now let $y_1^*=(z_1^*,0)$, $y_2^*=\gamma_0(z_2^*,-1)$, and $y_0^*=y_1^*+y_2^*$, where

$$z_1^*\in K_{z_0}^*(a,x_0),\ z_2^*\in\partial_z\varphi_0(x_0,z_0),\ \gamma_0\leqslant 0.$$

Then $0^*=z_1^*+\gamma_0 z_2^*$, $1=-\gamma_0$. Consequently, $\gamma_0=-1$, $z_1^*-z_2^*=0^*$. Since the function φ_0 is continuous

$$\partial_z\varphi_0(x_0,z_0)=\mathrm{pr}_{Z^*}\partial_{x,z}\varphi_0(x_0,z_0).$$

Therefore, by Theorem 2 we obtain

$$c_{x_0}^*(y_2^*)=\{x^*:(-x^*,y_2^*)\in-\partial_{x,y}\varphi(x_0,y_0)\}=$$

$$=\{x^*:(-x^*,-z_2^*,1)\in-(\partial_{x,z}\varphi_0(x_0,z_0)\times\{-1\})\}=\{x^*:(x^*,z_2^*)\in\partial_{x,z}\varphi_0(x_0,z_0)\}.$$

We obtain the next theorem from formula (6) by keeping in mind that $b_{x_0}^*(y_1^*)=a_{x_0}^*(z_1^*)$ and $z_1^*=z_2^*$.

THEOREM 8. Let $a\in W(X,Z)$, φ_0 be a real-valued convex continuous function on $X\times Z$. If for $x=x_0$ the lower bound in the right-hand side of formula (5) is achieved at the point z_0, then for the function f defined by equality (5),

$$\partial f(x_0)=\{x^*:x^*=x_1^*+x_2^*,\ x_1^*\in a_{x_0}^*(z^*),(x_2^*,z^*)\in\partial_{x,z}\varphi_0(x_0,z_0);\ z^*\in K_{z_0}^*(a,x_0)\}.$$

THEOREM 9. Let $a\in W(X,Y)$, $b\in W(Y,Z)$, and let $w_b(y,z^*)$ be a continuous function of y on dom b. If the lower bound of $w_b(y,z^*)$ with respect to y on the set $a(x_0)$ is achieved at point y_0, then

$$(b\circ a)_{x_0}^*(z^*)=(d_{x_0}^*\circ b_{y_0}^*)(z^*),$$

where $d_{x_0}^*$ is the restriction of mapping $a_{x_0}^*$ to the cone $K_{y_0}^*(a,x_0)$.

Proof. Let $c=b\circ a$. Then for any $x\in$ dom c,

$$w_c(x,z^*)=\inf_z\{<z,z^*>:z\in(b\circ a)(x)\}=\inf_z\{<z,z^*>:z\in\bigcup_{y\in a(x)}b(y)\}.$$

Since set c(x) is convex,

$$w_c(x,z^*)=\inf_y\{\inf_{z\in b(y)}<z,z^*>:y\in a(x)\}=\inf_y\{w_b(y,z^*):y\in a(x)\}.$$

The theorem's hypotheses ensure the applicability of Theorem 8 in the case at hand. Therefore,

$$c_{x_0}^*(z^*)=\{x^*:x^*\in a_{x_0}^*(y^*),y^*\in b_{y_0}^*(z^*)\cap K_{y_0}^*(a,x_0)\}=\{x^*:x^*\in d_{x_0}^*(y^*),y^*\in b_{y_0}^*(z^*)\}=(d_{x_0}^*\circ b_{y_0}^*)(z^*).$$

THEOREM 10. Let $a\in W(Y,Z)$, $A:X\to Y$, $B:Z\to U$ be linear continuous operators. If $w_a(y,B^*u^*)$ is a continuous function of y on dom a, then for mapping b, which is defined by formula (1),

$$b_{x_0}^*(u^*)=A^*a_{Ax_0}^*(B^*u^*).$$

Proof. We set $c=a\circ A$. Since

$$w_b(x,u^*)=\inf_u\{<u,u^*>:u\in Bc(x)\}=\inf_z\{<Bz,u^*>:z\in c(x)\}=\inf_z\{<z,B^*u^*>:z\in c(x)\}=w_c(x,B^*u^*),$$

$b_{x_0}^*(u^*)=c_{x_0}^*(B^*u^*)$. But by virtue of the theorem's hypothesis and of Theorem 9,

$$c_{x_0}^*(B^*u^*)=\{x^*:x^*=A^*y^*,\ y^*\in a_{Ax_0}^*(B^*u^*)\},$$

which proves the theorem.

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