SPACES OF PIECEWISE-CONTINUOUS ALMOST-PERIODIC FUNCTIONS

AND OF ALMOST-PERIODIC SETS ON THE LINE. I

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In connection with the study of piecewise-continuous almost-periodic functions we introduce the notion of a countable almost-periodic number set. We investigate various properties of it: In particular, we prove that the space of almost-periodic sets is closed with respect to the operation of free union.

1. Among the various generalizations of almost-periodic functions, the specialists in differential equations with impulse action (and with discontinuous dynamical systems) most often include the class of piecewise-continuous almost-periodic functions, considered for the first time by Wexler [1] in connection with the determination of a piecewise continuous almost-periodic solution of the impulse system

$$dx/dt = A(t)x,$$
$$\Delta x|_{t_k} = c_k, \quad t_k > t_{k-1}.$$

Wexler [1] gave the following definition of almost-periodicity for a piecewise-continuous function x(t): R → R with discontinuities at the points $\{t_k\} = \{t_k\}_{k=-\infty}^{+\infty}$:

H_0. For each $\varepsilon > 0$ there exists a relatively dense set of the points τ such that

$$|x(t+\tau) - x(t)| < \varepsilon \quad \forall t \in R: |t - t_k| > \varepsilon \quad \forall k \in \mathbb{Z}.$$

Let us observe that the discontinuity at the point t_s is removable and therefore, strictly spreaking, the object of determination is the pair $X = (x(t), \{t_k\})$, where the following condition has been imposed on the sequence $\{t_k\}$ in [1]:

H_1. $\lim_{k\to\pm\infty} t_k = \pm\infty$ and the sequences $\{t_k^j\} = \{t_{k+j} - t_k\}$ are almost-periodic (with respect to k) equipotentionally with respect to $j \in \mathbb{Z}$.

Moreover, the piecewise-continuous almost-periodic functions, considered in [1], have the additional properties H_1 and H_2, following from the boundedness of the almost-periodic function A(t) and of the almost-periodic sequence $\{c_k\}$:

H_2. $\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 : |x(t') - x(t'')| < \varepsilon,$

$$\forall t', t'' : |t' - t''| < \delta(\varepsilon), [t', t''] \cap \{t_k\} = \varnothing.$$

H_3. x(t) is bounded: $|x(t)| \le m \in R \ \forall t \in R$.

As we will see in the sequel, none of the above conditions is a consequence of the others. The existence of the limits $\lim_{t\to t_k \pm 0} x(t) = x(t_k \pm 0)$ follows from H_2. By definition, we let the function x(t) be continuous on the left.

Properties H_0, H_1, and H_2 with the additional condition $t_k > t_{k-1}$ were isolated for the first time and taken as axioms for the definition of a new class of piecewise-continuous almost-periodic functions in [2]. In [2, 3] certain results that characterize this class of functions have been obtained. In [4, 5] two different (it can be shown that they are non-

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equivalent) definitions of the piecewise-continuous almost-periodic functions $(x(t), \{t_k\})$ are proposed, which are however equivalent to the definition of [2] for $\inf_k t_k^1 > 0$.

The abundance of definitions of piecewise-continuous almost-periodic functions and the small number of results about them (even the closedness with respect to addition has not been proved) have inspired the authors to write this article. In particular, special attention will be paid to piecewise-continuous almost-periodic functions that satisfy (in the sense of Wexler) axioms H_0, H_1, H_2, and H_3. We denote the class of these functions by APW (R).

The first part of this article is devoted to the analysis of condition H_1.

2. Let T be a countable set of real numbers such that it contains arbitrarily large positive and negative numbers and for each $m > 0$ the set $\{t \in T: |t| \leq m\}$ is finite. We denote the family of these sets by $\mathfrak{A}$. It follows from the definition that each number $t \in T$ occurs in T only a finite number of times, called the multiplicity of t. The number

$$\rho(T_1, T_2) = \inf_{\varphi} \sup_{t \in T_1} |\varphi(t) - t|,$$

where the infimum is taken over all bijections $\varphi: T_1 \to T_2$; $T_1, T_2 \in \mathfrak{A}$, defines a distance in $\mathfrak{A}$. It is easy to show that the space $(\mathfrak{A}, \rho)$ is complete. If $T \in \mathfrak{A}$, then for $\tau \in R$ the set $T + \tau$, whose elements are precisely the elements of T increased by τ, also belongs to $\mathfrak{A}$, $\rho(T + \tau, T) \leq \tau$, and

$$\rho(T_1 + \tau, T_2 + \tau) = \rho(T_1, T_2) \quad \forall T_1, T_2, T \in \mathfrak{A}, \quad \tau \in R.$$

The mapping $\theta_s: \mathfrak{A} \times R \to \mathfrak{A}$, defined by the equality $\theta_s(T) = T + s$, is continuous with respect to the totality of arguments and, obviously, defines a dynamical system on $(\mathfrak{A}, \rho)$ and is such that

$$\rho(\theta_s(T_1), \quad \theta_s(T_2)) = \rho(T_1, T_2) \quad \forall T_1, T_2, \tau. \tag{1}$$

3. We make small digression from the treatment. Let (M, d) be a complete metric space, in which a continuous dynamical system φ_s: $M \times R \to M$ with the following property is defined: $\exists C \geq 1$ such that

$$d(\varphi_s(m_1), \varphi_s(m_2)) \leqslant Cd(m_1, m_2) \quad \forall s, m_1, m_2. \tag{2}$$

Besides the usual definition of a Bohr-almost-periodic motion (for each $\varepsilon > 0$ there exists a relatively dense set of ε-shifts of the trajectory), in this system we can give also the definition of the Bochner-almost-periodic motion $\varphi_s(m)$: Each sequence $\{\alpha_n\}$ must have a subsequence $\{\alpha_{n_k}\}$ such that $\lim_{k \to +\infty} \varphi_{\alpha_{n_k}}(m) = m_1$, for a certain $m_1 \in M$.

<u>THEOREM 1</u>. Under the above conditions each Bohr-almost-periodic motion is equivalent to a Bochner-almost-periodic motion. The set $H(m) = \text{Closure}\{\varphi_s(m), s \in R\}$ is compact if and only if $\varphi_s(m)$ is an almost-periodic motion, and moreover, $H(m_1) = H(m) \; \forall \; m_1 \in H(m)$.

<u>Proof</u> (cf. [6, 7]). Let $\varphi_s(m)$ be a Bochner-almost-periodic motion and $\{m_k\} \subset H(m)$. Then there exists a sequence $\{\alpha_k\}$ of real numbers such that $d(m_k, \varphi_{\alpha_k}(m)) < 1/k$. We choose a subsequence $\{\alpha_{k_p}\} \subset \{\alpha_k\}$ such that $\lim_{p \to \infty} \varphi_{\alpha_{k_p}}(m) = \overline{m} \in H(m)$. Then $\lim_{p \to \infty} d(m_{k_p}, \overline{m}) \leqslant \lim_{p \to \infty} (d(m_{k_p}, \varphi_{\alpha_{k_p}}^{(m)}) + d(\overline{m}, \varphi_{\alpha_{k_p}}^{(m)})) = 0$ and, therefore, H(m) is compact. The compactness of H(m) ensures in the usual manner that $\varphi_s(m)$ is a Bochner-almost-periodic motion. Further, if $m^* \in H(m_1)$, then $\lim_{k \to \infty} d(\varphi_{\alpha_k}(m_1), m^*) = 0$ for a certain sequence $\{\alpha_k\}$. We choose $\{\beta_k\}$ such that $d(\varphi_{\alpha_k}(m_1), \varphi_{\beta_k}(m)) < 1/k$. Then $\lim_{k \to \infty} d(\varphi_{\beta_k}(m), m^*) \leqslant \lim_{k \to \infty} (d(\varphi_{\beta_k}(m), \varphi_{\alpha_k}(m_1)) + d(\varphi_{\alpha_k}(m_1), m^*)) = 0$ and $m^* \in H(m)$; $H(m_1) \subseteq H(m)$. On the other hand, if $m_1 \in H(m)$ and $\lim_{k \to +\infty} d(\varphi_{\alpha_k}(m), m_1) = 0$, then, by (2), $d(\varphi_{-\alpha_k} \times (m_1), m) \leqslant Cd(m_1, \varphi_{\alpha_k}(m)) \to 0$ and $m \in H(m_1)$. Arguments, similar to the preceding ones, show that $H(m) \subseteq H(m_1)$, and as a consequence $H(m) = H(m_1)$.

Thus, if the motion $\varphi_s(m)$ is Bochner-almost-periodic, then the closure of its trajectory is a compact minimal set and therefore, by the first Birkhoff theorem [7, p. 69], the motion $\varphi_s(m)$ is recurrent. Since the dynamical system (M, φ_s) is Lyapunov-stable by virtue of (2) [7], it follows by the Markov theorem [7, p. 98] that the recurrent motion is Bohr-almost-periodic.

Conversely, if the motion $\varphi_s(m)$ is Bohr-almost-periodic, then it is recurrent [7, p. 86], and, by the second Birkhoff theorem [7, p. 70] in a complete space, the closure of the trajectory of a recurrent motion is a compact minimal set. By the same token, the theorem is proved.

4. The space $(\mathfrak{A}, \theta_s)$ satisfies all the conditions of Sec. 3, and Theorem 1 is valid for it, and, by (1), the definitions of almost periodicity reduce to the following one:

a set $T \in \mathfrak{A}$ is Bohr-almost-periodic if for each $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that each interval of length $T(\varepsilon)$ contains a number τ such that $\rho(T, T + \tau) < \varepsilon$;

a set $T \in \mathfrak{A}$ is Bochner-almost-periodic if each sequence $\{h_k\}$ has a subsequence $\{h_{k_n}\}$ such that $T = h_{k_n} \to T_1 \in \mathfrak{A}$ as $n \to \infty$.

COROLLARY 1. If $T_0, T_1, \ldots, T_m$ are almost-periodic sets, then the set $T_0 \sqcup \ldots \sqcup T_m$ is also almost-periodic, where $A \sqcup B$ denotes the free union of A and B.

Proof. It is sufficient to consider the case n = 2. At first we observe that the following statements are valid for $P, Q, R, S \in \mathfrak{A}$ and $h \in R$: a) $P \sqcup Q \in \mathfrak{A}$; b) $(P \sqcup Q) + h = (P + h) \sqcup (Q + h)$; and c) $\rho(P \sqcup Q,\ R \sqcup S) \leqslant \max(\rho(P, R);\ \rho(Q, S))$. Since T_1 and T_2 are almost-periodic sets, each sequence of real numbers $\{h_k\}$ has a subsequence $\{h_{k_n}\}$ such that $T_i + h_{k_n} \to T_i^*$ as $n \to \infty$, i = 1, 2. We set $T_{12} = T_1 \sqcup T_2$ and $T_{12}^* = T_1^* \sqcup T_2^*$. Then $\rho(T_{12} + h_{k_n}, T_{12}^*) = \rho((T_1 + h_{k_n}) \sqcup (T_2 + h_{k_n}),\ T_1^* \sqcup T_2^*) \leqslant \max(\rho(T_1 + h_{k_n}, T_1^*); \rho(T_2 + h_{k_n}, T_2^*)) \to 0$, and, therefore, the set T_{12} is almost-periodic.

Definition. A set $T \in \mathfrak{A}$ will be said to be strongly almost-periodic if its elements can be numbered $T = \{t_n\}_{n=-\infty}^{+\infty}$ such that the set of sequences $\{t_n^j\}$ is almost-periodic with respect to n equipotentionally with respect to j. We will call such a numbering of elements of T an almost-periodic representation of the almost periodic set T.

Example. The set of integers $\mathbb{Z} \subset R$ with the natural representation $\mathbb{Z} = \{n\}$ is strongly almost-periodic: The sequences $\{t_n^j\} = \{j\}$ are periodic with period $1\ \forall j$. Each strongly almost-periodic set has an infinite number of almost-periodic representations, e.g., $\mathbb{Z} = \{n + (-1)^{n+1}2\}$, but, as we show below, there always exists a representation $\{t_n\}$ of it (unique up to shift of numbers) such that $t_n \geq t_{n-1}\ \forall n$.

THEOREM 2 [8]. The set T is strongly almost-periodic with an almost-periodic representation $\{t_n\}$ if and only if

$$t_n = na + c_n, \tag{3}$$

where $\{c_n\}$ is an almost-periodic sequence and $\alpha \neq 0$.

Let $i(\alpha, \beta)$ denote the number of elements of the strongly almost-periodic set T in (α, β). Theorem 2 ensures the existence of the limit

$$\lim_{q \to \infty} q^{-1} i(\alpha, \alpha + q) = 1/a,$$

uniformly with respect to α. We will call the number a in (3) the growth index of T.

5. THEOREM 3. A set is Bohr (Bochner)-almost-periodic if and only if it is strongly almost-periodic. Moreover, an almost-periodic representation of an almost-periodic set can be obtained by numbering its elements in increasing order (with regard for multiplicity).

Proof. Let $\mathfrak{B}$ denote the set of all increasing sequences of real numbers $\{t_n\}$ that are unbounded above and below and do not have finite limit points and suppose that $t_0 \geq 0$ is a number such that either β_1) $t_0 = 0$ or β_2) $t_0 > 0$ and $(0, t_0) \not\ni t_i \forall i$. We introduce in $\mathfrak{B}$ a distance d between its elements $\{t_n^{(1)}\}$ and $\{t_n^{(2)}\}$ (it may be equal to $+\infty$) by the following relation:

$$d(\{t_n^{(1)}\}, \{t_n^{(2)}\}) = \inf_{\varphi} \sup_{n} |\varphi(t_n^{(1)}) - t_n^{(1)}|,$$

where the infimum is taken over all order-preserving bijections $\varphi: \{t_n^{(1)}\} \to \{t_n^{(2)}\}$ (each such bijection obviously has the following form: $\forall n\ \varphi(t_n^{(1)}) = t_{n+m_0}^{(2)}$).

LEMMA 1. The spaces $(\mathfrak{A}, \rho)$ and $(\mathfrak{B}, d)$ are isometric, and an isomorphism $\lambda: \mathfrak{A} \to \mathfrak{B}$ is given by the numbering of the elements $T \in \mathfrak{A}$ in increasing order (with regard for multiplicity) and by the choice of an element of T that satisfies β_1) and β_2) at t_0.

Proof. It is obvious that the mapping λ is one-to-one and $\rho(T_1, T_2) \le d(\lambda(T_1), \lambda(T_2))$. Now let the number $\rho(T_1, T_2)$ be finite. We prove that then $\rho_{12} \triangleq \rho(T_1, T_2) = d(\lambda(T_1), \lambda(T_2))$. By virtue of the above-given definitions, for each $\varepsilon > 0$ there exists a bijection $\varphi: \lambda(T_1) \to \lambda(T_2)$ such that $\rho_{12} \leqslant \sup_n |\varphi(t_n^{(1)}) - t_n^{(1)}| \leqslant \rho_{12} + \varepsilon$. Starrting from φ, we construct an order preserving bijection $\psi: \lambda(T_1) \to \lambda(T_2)$ $(\psi(t_n^{(1)}) \leqslant \psi(t_{n+1}^{(1)}) \Leftrightarrow t_n^{(1)} \leqslant t_{n+1}^{(1)}\ \forall n)$, such that

$$\rho_{12} \leqslant \sup_n |\psi(t_n^{(1)}) - t_n^{(1)}| \leqslant \rho_{12} + \varepsilon \tag{4}$$

[by definition of ρ_{12}, the lower bound in (4) is obvious].

If we set $\varphi(t_i^{(1)}) = t_{\varphi^*(i)}^{(2)}$ then the bijection $\varphi: \lambda(T_1) \to \lambda(T_2)$ is defined if and only if the bijection $\varphi^*: \mathbb{Z} \to \mathbb{Z}$ is defined. Let us set $P_0 = \{i > 0 : \varphi^*(i) < \varphi^*(0)\}$, and $M_0 = \{k > 0 : \varphi^*(k) > \varphi^*(0)\}$. The set P_0 is finite since for $i \in P_0$

$$\varphi(t_0^{(1)}) = t_{\varphi^*(0)}^{(2)} \geqslant t_{\varphi^*(i)}^{(2)} = \varphi(t_i^{(1)}),$$

$$0 \leqslant \varphi(t_0^{(1)}) - \varphi(t_i^{(1)}) \leqslant \varphi(t_0^{(1)}) - \varphi(t_i^{(1)}) + t_i^{(1)} - t_0^{(1)} = (\varphi(t_0^{(1)}) - t_0^{(1)}) - (\varphi(t_i^{(1)}) - t_i^{(1)}) < 2(\rho_{12} + \varepsilon),$$

and $\lambda(T_2) \notin \mathfrak{B}$ when P_0 is infinite. Similarly, for $k \in M_0$ we have $\varphi(t_k^{(1)}) \geqslant \varphi(t_0^{(1)})$ and M_0 is finite. We assume that

$$L \overset{\text{def}}{=} M_0 \cup P_0 \cup \{0\} = \{k_n < k_{n-1} < \ldots < k_1 < 0 < i_1 < \ldots < i_m\}.$$

We replace the restriction $\varphi^*: L \to \varphi^*(L)$ by the order-preserving mapping $\bar{\bar{\varphi}}^*: L \to \varphi^*(L)$, [$\bar{\bar{\varphi}}^*(k_n)$ is the minimum number in $\varphi^*(L)$, etc.]. If we set $\bar{\varphi}^*: \mathbb{Z} \to \mathbb{Z}$, where $\bar{\varphi}^*|_L = \bar{\bar{\varphi}}^*, \bar{\varphi}^*|_{\mathbb{Z} \setminus L} = \varphi$; and $\bar{\varphi}: \lambda(T_1) \to \lambda(T_2)$, where $\bar{\varphi}(t_s^{(1)}) = t_{\bar{\varphi}^*(s)}^{(2)}$, then $\bar{\varphi}$ and $\bar{\varphi}^*$ are bijections and

$$|\bar{\varphi}(t_p^{(1)}) - t_p^{(1)}| < \rho_{12} + \varepsilon \quad \forall p. \tag{5}$$

It is sufficient to prove this relation only for $p \in L$. Let us consider the case $m < n$ (the remaining cases $m = n$ and $m > n$ are considered analogously). If $m < n$, then $\bar{\varphi}^*(P_0 \cup \{0\}) \subset \varphi^*(M_0)$ and, therefore, for $\bar{\varphi}(t_{i_k}^{(1)}) \geqslant t_{i_k}^{(1)}$ we have $0 \leqslant \bar{\varphi}(t_{i_k}^{(1)}) - t_{i_k}^{(1)} = t_{\bar{\varphi}^*(i_k)}^{(2)} - t_{i_k}^{(1)} = t_{\varphi^*(k_r)}^{(2)} - t_{i_k}^{(1)} = \varphi(t_{k_r}^{(1)}) - t_{i_k}^{(1)} \leqslant \varphi(t_{k_r}^{(1)}) - t_{k_r}^{(1)} < \rho_{12} + \varepsilon$; but if $\bar{\varphi}(t_{i_k}^{(1)}) < t_{i_k}^{(1)}$, then

$$0 < t_{i_k}^{(1)} - \bar{\varphi}(t_{i_k}^{(1)}) = t_{i_k}^{(1)} - t_{\bar{\varphi}^*(i_k)}^{(2)} = t_{i_k}^{(1)} - t_{\varphi^*(k_p)}^{(2)} \leqslant t_{i_k}^{(1)} - t_{\varphi^*(i_k)}^{(2)} = t_{i_k}^{(1)} - \varphi(t_{i_k}^{(1)}) < \rho_{12} + \varepsilon.$$

Further, $\bar{\varphi}^*(\{k_n, \ldots, k_{n-m+1}\}) = \varphi^*(P_0)$, and therefore for $\bar{\varphi}(t_\delta^{(1)}) \geqslant t_\delta^{(1)}$, $\delta \in \{k_n, \ldots, k_{n-m+1}\}$, we have

$$0 \leqslant \bar{\varphi}(t_\delta^{(1)}) - (t_\delta^{(1)}) = \varphi(t_{i_q}^{(1)}) - t_\delta^{(1)} \leqslant \varphi(t_\delta^{(1)}) - t_\delta^{(1)} < \rho_{12} + \varepsilon;$$

but if $\bar{\varphi}(t_\delta^{(1)}) < t_\delta^{(1)}$, then

$$0 < t_\delta^{(1)} - \bar{\varphi}(t_\delta^{(1)}) = t_\delta^{(1)} - t_{\bar{\varphi}^*(\delta)}^{(2)} = t_\delta^{(1)} - t_{\varphi^*(i_d)}^{(2)} = t_\delta^{(1)} - \varphi(t_{i_d}^{(1)}) \leqslant t_{i_d}^{(1)} - \varphi(t_{i_d}^{(1)}) < \rho_{12} + \varepsilon;$$
$$\bar{\varphi}^*(\{k_{n-m}, \ldots, k_1\}) \subset \varphi^*(M_0 \cup \{0\}).$$

The following estimate is valid for $\bar{\varphi}(t_{k_\delta}^{(1)}) \leqslant t_{k_\delta}^{(1)}$

$$0 \leqslant t_{k_\delta}^{(1)} - \bar{\varphi}(t_{k_\delta}^{(1)}) = t_{k_\delta}^{(1)} - t_{\varphi^*(k_\alpha)}^{(2)} = t_{k_\delta}^{(1)} - \varphi(t_{k_\alpha}^{(1)}) \leqslant t_0^{(1)} - \varphi(t_0^{(1)}) < \rho_{12} + \varepsilon.$$

Further, for $\bar{\varphi}^*(k_\delta) \subset \varphi^*(M_0 \cup \{0\})$ there exists a $\mu \ge \delta$ such that $\varphi^*(k_\mu) \geqslant \bar{\varphi}^*(k_\delta)$, since, otherwise, $\forall \mu \ge \delta$

$$\varphi^*(k_\mu) < \bar{\varphi}^*(k_\delta), \quad \bar{\varphi}^*(\{k_n, \ldots, k_\delta\}) = \{\varphi^*(s) : s \in L, \quad \varphi^*(s) \leqslant \bar{\varphi}^*(k_\delta)\} \supset \varphi^*(P_0 \cup \{0\} \cup \{k_n, \ldots, k_\delta\}).$$

Therefore, for $\bar{\varphi}(t_{k_\delta}^{(1)}) > t_{k_\delta}^{(1)}$ we have

$$0 < \bar{\varphi}(t_{k_\delta}^{(1)}) - t_{k_\delta}^{(1)} = t_{\bar{\varphi}^*(k_\delta)}^{(2)} - t_{k_\delta}^{(1)} \leqslant t_{\varphi^*(k_\mu)}^{(2)} - t_{k_\mu}^{(1)} = \varphi(t_{k_\mu}^{(1)}) - t_{k_\mu}^{(1)} < \rho_{12} + \varepsilon,$$

and, by the same token, (5) is proved.

Now let $j_0 = \bar{\varphi}^*(0)$, and set $\psi(t_s^{(1)}) = t_{j_0+s}^{(2)}$, and prove (4) for this ψ. For example, let $s > 0$ and $j_0 + s = \bar{\varphi}^*(\alpha)$. If $\psi(t_s^{(1)}) \geq t_s^{(1)}$, then

a1) for $\alpha \leqslant s$ $0 \leqslant \psi(t_s^{(1)}) - t_s^{(1)} = \bar{\varphi}(t_\alpha^{(1)}) - t_s^{(1)} = t^{(2)}_{\bar{\varphi}^*(\alpha)} - t_s^{(1)} \leqslant \bar{\varphi}(t_\alpha^{(1)}) - t_\alpha^{(1)} < \rho_{12} + \varepsilon$;

a2) for $\alpha > s$ there exists a positive number $\beta \leq s$ such that $\bar{\varphi}^*(\beta) \geqslant \bar{\varphi}^*(\alpha)$, otherwise, considering $\bar{\varphi}^*(N) = \{m \in \mathbb{Z} : m > j_0\}$, we would have $\bar{\varphi}^*(\beta) < \bar{\varphi}^*(\alpha)$ $\forall 0 \leqslant \beta \leqslant s$ and $\{j_0, j_0 + 1, \ldots, j_0 + s\} \supset \bar{\varphi}^*\{0, 1, \ldots, s, \alpha\}$. Therefore $0 \leqslant \psi(t_s^{(1)}) - t_s^{(1)} = \bar{\varphi}(t_\alpha^{(1)}) - t_s^{(1)} \leqslant \bar{\varphi}(t_\beta^{(1)}) - t_\beta^{(1)} < \rho_{12} + \varepsilon$. But if $\psi(t_s^{(1)}) < t_s^{(1)}$, then

b1) for $\alpha \geq s$ we have $\alpha \geqslant s$ $0 < t_s^{(1)} - \psi(t_s^{(1)}) = t_s^{(1)} - \bar{\varphi}(t_\alpha^{(1)}) \leqslant t_\alpha^{(1)} - \bar{\varphi}(t_\alpha^{(1)}) < \rho_{12} + \varepsilon$;

b2) for each $\alpha < s$ there exists a $\beta \geq s$ such that $\bar{\varphi}^*(\beta) \leqslant \bar{\varphi}^*(\alpha)$, otherwise, since $\bar{\varphi}^*(N) = \{m \in \mathbb{Z} : m > j_0\}$, we would have $\bar{\varphi}^*(\beta) > \bar{\varphi}^*(\alpha)$ $\forall \beta \geqslant s$ and $\{j_0, j_0 + 1, \ldots, j_0 + s\} \subset \bar{\varphi}^*\{0, 1, \ldots, s - 1\}$. Therefore,

$$0 < t_s^{(1)} - \psi(t_s^{(1)}) = t_s^{(1)} - \bar{\varphi}(t_\alpha^{(1)}) \leqslant t_\beta^{(1)} - \bar{\varphi}(t_\beta^{(1)}) < \rho_{12} + \varepsilon.$$

Analogous investigation of the case $s < 0$ completes the proof of estimate (4) and Lemma 1.

LEMMA 2. The set T is strongly almost periodic with an almost-periodic representation $\{t_n\}$ if and only if for each $\eta > 0$ the set Ω_η of all numbers ω such that

$$|t_n^{h_\omega} - \omega| < \eta \tag{6}$$

for a certain $h_\omega \in \mathbb{Z}$ and all $n \in \mathbb{Z}$ is relatively dense in R.

Lemma 2 was proved for the first time in [1] for $t_{n+1} > t_n$. Its proof, given below, is a modification of the arguments of [1] for the general case.

Proof. Necessity. By virtue of Theorem 2, the necessity of the denseness of Ω_η is obvious [if $h_\omega \eta$ is the period for the almost-periodic sequence $\{c_n\}$ from (3), then (6) is fulfilled with $\omega = h_\omega a$].

Sufficiency. Let us set $H_\eta = \{h_\omega, \omega \in \Omega_\eta\}$. We arrange the integers from H_η in the strictly increasing sequence $\{h_i\}_{i=-\infty}^{+\infty}$. It is obvious that if we set $h_0 = 0$, $h_i = -h_{-i}$ $\forall i$. We prove the relative denseness of $\{h_i\}_{i=-\infty}^{+\infty}$ (to this end it is sufficient to show that $\lim\limits_{i\to\pm\infty} h_i = \pm\infty$ and the sequence $\{h_{i+1} - h_i\}$ is bounded above); $\lim\limits_{i\to\pm\infty} h_i = \pm\infty$, since in the contrary case ($|h_i| \leq m_0$, by virtue of the symmetry of H_η) we would have $-c \leq t_0^{h_\omega} \leq c$ $\forall \omega \in \Omega_\eta$, and by (6), $|\omega| \leq c + \eta$, which contradicts the relative denseness of Ω_η.

We set $\Omega_i = \{\omega : h_\omega = h_i\}$ and prove that for all $j \geq i$

$$\omega_j - \omega_i \geqslant -2\eta \tag{7}$$

($\omega_k \in \Omega_k$). Since

$$t^{h_j}_{-h_i+s} - t^{h_i}_{-h_i+s} = t_{h_j-h_i+s} - t_s, \tag{8}$$

and with respect to (j, i) we can always find an $\bar{s} = s(i, j)$ such that $t_{h_j-h_i+\bar{s}} \geq t_{\bar{s}}$ (otherwise, $t_{h+s} < t_s$ $\forall s \in \mathbb{Z}$, where $h = h_j - h_i \geq j - i \geq 0$ and $\lim\limits_{n\to+\infty} t_{nh+s} < +\infty$), it follows that $t^{h_j}_{-h_i+\bar{s}} - t^{h_i}_{-h_i+\bar{s}} \geqslant 0$. Since

$$-\eta < t_r^{h_k} - \omega_k < \eta \tag{9}$$

(for arbitrary k and r), we have $\omega_j - \omega_i \geqslant (t^{h_j}_{-h_i+\bar{s}} - \eta) - (t^{h_i}_{-h_i+\bar{s}} + \eta) \geqslant -2\eta$.

Further, if $\overline{\lim\limits_{i\to\infty}}(h_{i+1} - h_i) = +\infty$, then, by virtue of (8) (for $s = 0$, $j = i + 1$) we can assert that for each L there exists an i_0 such that $t^{h_{i_0+1}}_{-h_{i_0}} - t^{h_{i_0}}_{-h_{i_0}} \geqslant L + 6\eta$. But then, by (9), $\omega_{i+1} - \omega_i \geq L + 4\eta$, which contradicts the relative denseness of $\Omega_\eta = \bigcup\limits_{i=-\infty}^{+\infty} \Omega_i$ [in an interval

$(\omega_{i_0} + 2\eta, \omega_{i_0+1} - 2\eta)$ of length at least L that does not contain ω_k for any $k \in \mathbb{Z}$ by virtue of (7)].

Thus, the sequence $\{h_{i+1} - h_i\}$ is bounded and the set $H_\eta = \{h_i\}$ is relatively dense. Each integer $h_i \in H_\eta$ is a 2η-almost-period, common for all sequences $\{t_h{}^j\}$, $j \in \mathbb{Z}$ (since $t^j_{n+h_i} - t^j_n = t^{h_i}_{n+j} - t^{h_i}_n \; \forall j, n$). By the same token, the proof of Lemma 2 is complete.

Now we complete the proof of Theorem 3. Let the set T be Bochner (Bohr)-almost-periodic. Then $\lambda(t)$ is an almost-periodic representation of it. Indeed, for each $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that on each segment of length $T(\varepsilon)$ we can find a τ such that $\rho(T, T + \tau) < \varepsilon$. But then $|-t_{n+j_0} + t_n - \tau| = d(\lambda(T), \lambda(T + \tau)) = \rho(T, T + \tau) < \varepsilon$ [here $\lambda(T) = \{t_n\}$, $\lambda(T + \tau) = \{t_{n-j_0(\tau)} + \tau\}$). By virtue of Lemma 2, we see that T is a strongly almost-periodic set with an almost periodic representation $\lambda(T)$.

But if the set T is strongly almost periodic, then the conditions of Lemma 2 are fulfilled and for each $\eta > 0$ there exists a relatively dense set Ω_η such that for $\omega \in \Omega_\eta$

$$(T, T + \omega) \leqslant \sup_n |\varphi(t_n + \omega) - t_n - \omega| < \eta, \quad \varphi(t_n + \psi) = t_{n+h_\omega},$$

and, therefore, T is a Bohr-almost-periodic set. Theorem 3 is completely proved.

COROLLARY 2. If $T_1, \ldots, T_n$ are almost periodic sets with growth indices $a_1, \ldots, a_n$, respectively, then $T = T_1 \sqcup \ldots \sqcup T_n$ is also an almost-periodic set with growth index a, where $1/a = 1/a_1 + \ldots + 1/a_n$.

Corollary 2 follows from Corollary 1 and the remark to Theorem 2 with regard for the fact that $i(\alpha, \beta) = i_1(\alpha, \beta) + \ldots + i_n(\alpha, \beta)$.

COROLLARY 3 (see Theorem 1). If T_0 is an almost-periodic set with an almost-periodic representation $T_0 = \{an + c_n{}^{(0)}\}$, then $T \in H(T_0)$ if and only if there exists an almost-periodic representation $T = \{an + c_n + \theta\}$, where $\theta \in [0, a)$, $\{c_n\} \in H(\{c_n{}^{(0)}\})$.

Proof. 1. Let $T = \{an + c_n + \theta\}$, where $\{c_n\} \in H(\{c_n{}^{(0)}\})$ (i.e., $\lim_{k\to\infty} \sup_n |c_n - c^{(0)}_{n+m_k}| = 0$). for a certain sequence of integers $\{m_k\}$). But $T_0 + \theta - am_k = \{an + c^{(0)}_n + \theta - am_k\} = \{an + c^{(0)}_{n+m_k} + \theta\}$ and, therefore, $T_0 + \theta - am_k \to T$ as $k \to +\infty$. 2. If $T \in H(T_0)$, then for a certain sequence of real numbers $\{h_k\}_{k=1}^{+\infty}$ we have $T_0 + h_k \to T$ as $k \to +\infty$. We set $h_k = m_k a + \theta_k$. Then $T_0 + h_k = \{an + c^{(0)}_n + h_k\} = \{an + c^{(0)}_{n-m_k} + \theta_k\}$. Since the sequence $\{c_n{}^{(0)}\}$ is Bochner-almost-periodic, the sequence $\{m_k\}$ has a subsequence $\{m_{k_j}\}$ such that $\lim_{j\to\infty} \sup_n |c^{(0)}_{n-m_{k_j}} - c_n| = 0$ and $\{c_n\} \in H(\{c^{(0)}_n\})$. In this connection, we can assume that $\theta_{k_j} \to \theta \in [0, a]$. But then $T_0 + h_{k_j} \to \{an + c_n + \theta\}$, and, therefore, $T = \{an + c_n + \theta\}$ (if $\theta = a$, then $T = \{an + c_{n-1}\}$).

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