ASYMPTOTIC CONDITIONS AND INFRARED DIVERGENCES IN QUANTUM ELECTRODYNAMICS

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A definition which is free of infrared divergences is proposed for the S matrix of a relativistic theory of interacting charged particles. This is achieved by a modification of the asymptotic condition and the introduction of a new space of asymptotic states. This state differs from the Fok space, but is separable and relativistically and gauge invariant. The mass operator has no nonvanishing discrete eigenvalues.

In the present paper, we shall discuss some aspects of the scattering problem in the relativistic quantum theory of interacting charged particles and photons. The main result is the description of a space of asymptotic states for such a system and the definition of an S matrix that is free of infrared divergences.*

The infrared catastrophe has frequently been discussed, the first occasion being the classical paper of Bloch and Nordsieck in 1937 [1]. The physical reasons for infrared divergences are well understood and they do not lead to any physical problems. However, the generally accepted formal treatment of the infrared catastrophe is not, in our view, completely satisfactory.

In textbooks on quantum electrodynamics, the reader must wrestle with infrared divergences and sum the probabilities of a transition from a given initial state to all final states, which include not only detectable particles but also an arbitrary number of "soft" photons (see [2]). An important role in the justification of this approach is played by the asymptotic formulas for the scattering amplitudes in the case when the artificially introduced photon mass tends to zero. The general form of these formulas was derived in the papers of Yennie et al. [3].

In the classical method just described, the cross sections and not the matrix elements are regarded as the primary objects. The initial and final states are treated as asymmetric and an S matrix is not defined at all. One is naturally led to ask whether these features are unavoidable and due to the physical nature of the problem or whether there exists an alternative approach to infrared singularities in which an S matrix can be defined. In the present paper, we attack the problem in this manner and propose a version of the asymptotic condition which is specially suited to a relativistic system of charged particles and makes possible a correct definition of an S matrix.

Our point of departure is Chung's important paper [4]. Chung surmised how one can choose states containing a charged particle and a superposition of an infinite number of photons in such a way that the matrix elements of the Feynman–Dyson S matrix between these states are finite and nonzero. Chung's generalization of the construction of these states for the case of several charged particles is too unsophisticated; in particular, it ignores the infinite Coulomb phase.

Kibble [5] made some important advances on Chung's work. He introduced a very large space of asymptotic states and showed that the Feynman–Dyson S matrix can be correctly defined in this state as a unitary operator. Kibble's space is nonseparable and contains states with an infinite number of soft photons. One can distinguish separable subspaces of Kibble's space which are mapped into one another by the S

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matrix. However, there is no stable separable subspace that is mapped into itself. This is connected with the infinite Coulomb phase contained in the S matrix. Kibble's analytic apparatus is based on the asymptotic formulas for the many-particle Green's function near the mass shell for the charged particles.* Thus, Kibble's definitions are based on the complete solution of the dynamical problem and are therefore very cumbersome.

Our approach differs from these approaches in what we modify not only the space of asymptotic states but also the very definition of the scattering operator. This enables us to compensate the Coulomb phase automatically and our space of asymptotic states is separable and is no richer in states than the Fok space. The complete procedure is suggested by the nonrelativistic theory of scattering by a long-range potential and has a simple physical interpretation. We are not forced to solve the complete equations of quantum electrodynamics in order to implement our program. Thus, we derive Chung's formulas without laborious calculations and obtain their correct generalization in the case of an arbitrary number of charged particles and photons in the initial and final states.

From the methodological point of view, the main result of our paper is a relativistically and gauge invariant definition of the S matrix and the space of asymptotic states of the charged particles.

In the present paper, we take the example of Coulomb scattering to explain the main idea of our approach. The nub of the idea is that in the definition of the wave operators we do not take $\exp\{-iH_0t\}$ but a more suitable operator $U_{as}(t)$ as the operator of the asymptotic dynamics. The choice of this operator is based on a natural physical condition, namely, the wave packets $U_{as}(t)\Psi$ at large $|t|$ must correspond to the classical motion of widely separated charged particles. The actual choice of $U_{as}(t)$ for quantum electrodynamics is discussed in Sections 2 and 3. In the next section, we introduce and discuss a space of states, different from to Fok's space, for charged particles and photons. In Section 5, we explain why this space can be used naturally as the space of asymptotic states and we give the final definition of the S matrix and compare our results with those of Chung.

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1. Nonrelativistic Coulomb Scattering

The scattering of a nonrelativistic particle by a Coulomb potential may serve to illustrate the main idea of our approach. The Hamiltonian of the system has the form

$$H = \frac{\mathbf{p}^2}{2m} + \frac{g}{r} = H_0 + V,$$

where m is the mass of the particles and g is the product of the charges of the particle and the scattering center. The asymptotic behavior of the potential V(t) in the interaction representation can be easily calculated. One must note that

$$\mathbf{r}(t) = \frac{\mathbf{p}}{m}t + \mathbf{r},\quad \mathbf{p}(t) = \mathbf{p},$$

i.e., as $|t| \to \infty$

$$V(t) = \frac{mg}{p|t|} + O\left(\frac{1}{t^2}\right).$$

The first term of this asymptotic expression cannot be integrated with respect to the time in the neighborhood of infinity and its contribution to the dynamics cannot be neglected, even for $|t| \to \infty$. In other words the asymptotic dynamics is not described by H_0 but by the explicitly time-dependent operator †

$$H_{as}(t) = H_0 + V_{as}(t) = H_0 + \frac{mg}{p|t|}.$$

The wave packets $\psi(\mathbf{r}, t)$ satisfy the asymptotic Schrödinger equation

$$i\frac{d}{dt}\psi(\mathbf{r}, t) = H_{as}(t)\psi(\mathbf{r}, t), \tag{1}$$

* Such formulas have a long history. A generating functional for the Green's functions which takes into account this asymptotic behavior rigorously was obtained by Fradkin [6].

† Here, we have used the fact that the expression for $V_{as}(t)$ is the same in both the Schrödinger and the interaction representation.

and can be represented in the form

$$\psi(\mathbf{r},t)=\frac{1}{(2\pi)^{3/2}}\int c(\mathbf{p})\exp\left\{-i\frac{p^2}{2m}t-i\frac{mg}{p}\operatorname{sign}t\ln\frac{|t|}{t_0}\right\}e^{i\mathbf{pr}}\,d\mathbf{p}. \tag{2}$$

Of course, the choice of the solution in this form corresponds to a definite choice of the "initial conditions" for Eq. (1). These conditions are determined by the following considerations. The variation with time of the distribution of coordinates and momenta in the packet (2) for $|t|\to\infty$ is governed by the equations of classical mechanics. The constant t_0 is not significant in this connection.

To justify this statement, consider, for example, the asymptotic expression for the expectation values of the coordinates and the momenta

$$\langle\hat{\mathbf{r}}(t)\rangle=(\mathbf{r}\psi(t),\psi(t))\text{ and }\langle\hat{\mathbf{p}}(t)\rangle=(\hat{\mathbf{p}}\psi(t),\psi(t)).$$

Obviously, the second quantity does not depend on the time

$$\langle\hat{\mathbf{p}}(t)\rangle=\int\mathbf{p}|c(\mathbf{p})|^2d\mathbf{p}=\langle\hat{\mathbf{p}}\rangle,$$

and the first has the following asymptotic behavior for $|t|\to\infty$:

$$\langle\hat{\mathbf{r}}(t)\rangle\simeq\frac{\langle\hat{\mathbf{p}}\rangle}{m}t-gm\left\langle\frac{\hat{\mathbf{p}}}{p^3}\right\rangle\operatorname{sign}t\ln|t|+O(1),$$

which can be readily obtained, for example, by the method of stationary phase.

These considerations show how natural the choice of the asymptotic wave packets in the form (2) is from the physical point of view. Moreover, it has been known since Dollard's paper [7] that this choice of the packets leads to the correct definition of the wave operators and the well-known expression for the S matrix. Written out more fully, the packet (2) has the form

$$\psi(t)=U_{as}(t)\psi=e^{-iH_0t}\exp\left\{-i\frac{mg}{p}\operatorname{sign}t\ln\frac{|t|}{t_0}\right\}\psi,$$

and there exists the strong limits [7]

$$U_\pm=\lim_{t\to\pm\infty}e^{iHt}U_{as}(t),$$

the S matrix

$$S=U_+^+U_-$$

giving the well-known expressions for the differential cross sections for scattering by a Coulomb potential. The choice of the parameter t_0 does not affect the expressions. In contrast, the usual definitions of the wave operators of the formal scattering theory, in which H_0 is taken as the asymptotic operator, are incorrect and perturbation theory for the usual S matrix leads to the well-known infrared catastrophe.

This example shows that the choice of the Hamiltonian of the asymptotic dynamics must be dictated by the physical nature of the problem and that it is a mistake to apply blindly the usual prescriptions of the formal theory of scattering and assume that the asymptotic dynamics is always defined by the operator H_0. This prescription can be implemented formally by choosing the, in general, nontrivial and time-dependent interaction operator $V_{as}(t)$, which coincides with the highest terms in the asymptotic expression for the operator $V(t)$ in the interaction representation. Once the operator $H_{as}(t)$ has been found, the solutions of the asymptotic Schrödinger equation must be singled out by "initial conditions" that ensure that the resulting wave packet behaves classically as $|t|\to\infty$. In the following section, we shall apply these arguments to relativistic quantum electrodynamics.

2. Construction of $V_{as}(t)$ in Quantum Electrodynamics

To be specific, we shall consider spinor electrodynamics describing a system of interacting electrons, positrons, and photons. We shall use the traditional notation $\bar{\psi}$ and ψ for the operators of the electron–positron field and A_μ for the operators of the electromagnetic field. We also take $b_i^+(p)$, $b_i(p)$, $d_i^+(p)$, $d_i(p)$, $a_\mu^+(k)$, $a_\mu(k)$; $i=1,2$; $\mu=0,1,2,3$, as the operators of creation and annihilation, respectively, of electrons, positrons, and photons. We shall adopt an explicitly covariant formalism and assume an indefinite metric in the photon Hilbert space. Finally, we shall assume that $\hbar=c=1$ and that all vectors are contravariant, i.e., the choice of a vector subscript or superscript will be dictated purely by convenience. Thus, $a^\mu = a_\mu$, $a_\mu b_\mu = a_0 b_0 - \mathbf{a}\mathbf{b} = ab$.

The field operators $\bar{\psi}$, ψ, and A_μ can be expressed in terms of the creation and annihilation operators as follows:

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int\left(\frac{m}{p_0}\right)^{1/2}\sum_n (b_n(\mathbf{p})\, w_n(\mathbf{p})\, e^{ipx} + d_n^+(\mathbf{p})v_n(\mathbf{p})e^{-ipx})\,d\mathbf{p},$$

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}}\int\left(\frac{m}{p_0}\right)^{1/2}\sum_n (b_n^+(\mathbf{p})\,\bar{w}_n(\mathbf{p})\,e^{-ipx} + d_n(\mathbf{p})\,\bar{v}_n(\mathbf{p})\,e^{ipx})\,d\mathbf{p}, \tag{3}$$

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}}\int (a_\mu^+(\mathbf{k})\,e^{-ikx} + a_\mu(\mathbf{k})\,e^{ikx})\,\frac{d\mathbf{k}}{(2k_0)^{1/2}},$$

where $w_n(\mathbf{p})$ and $v_n(\mathbf{p})$ are the corresponding spinor amplitudes.

In order to obtain an expression for the interaction operator

$$V = \int j_\mu(x)\,A_\mu(x)\,dx = -e\int :\bar{\psi}(x)\,\gamma_\mu\psi(x): A_\mu(x)\,dx \tag{4}$$

in the interaction representation, it is sufficient to substitute the explicit formulas (3) into (4) and furnish the operators of creation and annihilation with appropriate time factors, for example, $e^{ik_0t}a_\eta^+(\mathbf{k})$ and $e^{ip_0t}b_n^+(\mathbf{p})$, $k_0 = |\mathbf{k}|$, $p_0^2 = \mathbf{p}^2 + m^2$. The resulting expression for V(t) is an integral over the momenta **p**, **q**, and **k** of the fermions and photons, which are related by the equation $\mathbf{p} + \mathbf{k} = \mathbf{q}$.

Our immediate task is to investigate the asymptotic behavior of this expression for $|t| \to \infty$. In this limit, all the terms in the expression can be split into two groups. The terms of the first group contain two creation operators or two annihilation operators of charged particles. The argument of the exponential function characterizing the time dependence of these terms is proportional to the expression $(\mathbf{p}^2 + m^2)^{1/2} + ((\mathbf{p} + \mathbf{k})^2 + m^2)^{1/2} \pm k_0$, which is separated from zero for all **p** and **k**. Such terms therefore decrease sufficiently rapidly as $|t| \to \infty$. The terms of the second group have an argument of the exponential function proportional to $(\mathbf{p}^2 + m^2)^{1/2} - ((\mathbf{p} + \mathbf{k})^2 + m^2)^{1/2} \pm k_0$, which vanishes for $\mathbf{k} = 0$ for all **p**. It is clear that it is these terms of the second group that determine the desired asymptotic behavior of the operator V(t) and that the terms of the first group may be neglected.

The next simplification is based on the fact that the principal contribution to the remaining integrals in the limit $|t| \to \infty$ comes from the neighborhood of small momenta **k**. In the corresponding integrands we may therefore set $\mathbf{k} = 0$ in all the slowly varying functions, i.e., in the operators of creation and annihilation b_n and d_n and the coefficients composed of spinors. In addition, the expressions for the latter simplify considerably because of the orthogonality conditions for solutions of the Dirac equation. As a result, we obtain a simple expression for the principal term $\tilde{V}(t)$ of the asymptotic expression for the operator V(t) as $|t| \to \infty$.

We shall now write down explicitly the Schrödinger representation for $V_{as}(t)$, which coincides with $\tilde{V}(t)$ in the interaction representation. Like the total interaction, we represent the operator $V_{as}(t)$ as an integral of the product of a current-type operator and a vector potential

$$V_{as}(t) = \frac{1}{(2\pi)^{3/2}}\int J_{as}^\mu(\mathbf{k}, t)\,(a_\mu^+(-\mathbf{k}) + a_\mu(\mathbf{k}))\,\frac{d\mathbf{k}}{(2k_0)^{1/2}}, \tag{5}$$

where

$$J_{as}^\mu(\mathbf{k}, t) = -e\int p^\mu e^{i\frac{\mathbf{p}\mathbf{k}}{p_0}t}\rho(\mathbf{p})\,\frac{d\mathbf{p}}{p_0}$$

and

$$\rho(\mathbf{p}) = \sum_n (b_n^+(\mathbf{p})\,b_n(\mathbf{p}) - d_n^+(\mathbf{p})\,d_n(\mathbf{p})) = \rho_-(\mathbf{p}) - \rho_+(\mathbf{p}).$$

The corresponding expression in the interaction representation differs only by the substitution $kp \to -kp$ in the integral determining the asymptotic current.

We note that the expression for the asymptotic current does not contain spinors and that the contribution from the charged particles is given only by a distribution of charges over momenta. In this sense, the expression is universal: we would obtain a similar formula in the case of charged particles with arbitrary spin, the corresponding density $\rho(p)$ being a sum over the charged particles in the system.

The operator of the asymptotic current $J_{as}{}^{\mu}(k, t)$ has a simple physical meaning. A state of charged particles with given momenta

$$\Psi(\mathbf{p}_1 s_1, \ldots, \mathbf{p}_n s_n \,|\, \mathbf{q}_1 t_1, \ldots, \mathbf{q}_m t_m) = b^+_{s_1}(\mathbf{p}_1) \ldots b^+_{s_n}(\mathbf{p}_n)\, d^+_{t_1}(\mathbf{q}_1) \ldots d^+_{t_m}(\mathbf{q}_m)\,|0\rangle \tag{6}$$

is an eigenstate for this operator; the corresponding eigenvalue

$$j_\mu(\mathbf{k}, t \,|\, \mathbf{p}_1, \ldots, \mathbf{p}_n; \mathbf{q}_1, \ldots, \mathbf{q}_m) = \sum_{j=1}^{m} j_\mu(\mathbf{k}, t \,|\, q_j) - \sum_{j=1}^{n} j_\mu(\mathbf{k}, t \,|\, p_j),$$

where

$$j_\mu(\mathbf{k}, t \,|\, p) = e\,\frac{p_\mu}{p_0} \exp\left\{ i\,\frac{\mathbf{kp}}{p_0}\, t \right\},$$

is the classical current of point charges moving along straight lines with momenta p_i, q_j, $i = 1, \ldots, n$; $j = 1, \ldots, m$. In this sense, the operator of the asymptotic interaction is a natural relativistic generalization of the nonrelativistic asymptotic potential of the foregoing section. As in Section 2, we propose to define the asymptotic dynamics of the system by means of the operator

$$H_{as}(t) = H_0 + V_{as}(t),$$

where H_0 is the energy operator of the free fermions and photons. Our next task is to solve the Schrödinger equation with this operator.

3. Description of the Asymptotic Dynamics

The mathematical structure of the operator $V_{as}(t)$ is similar to that of the well-known Hamiltonian for the interaction of bosons with fixed fermions. As a result, the solution of the equation

$$i\,\frac{d}{dt}\,U(t) = H_{as}(t)\,U(t), \tag{7}$$

which must be satisfied by the operator $U_{as}(t)$ describing the asymptotic dynamics, can be found explicitly. In solving this equation, we shall devote our main attention to the choice of appropriate initial conditions. The point is that $V_{as}(t)$ contains the operators $a_\mu{}^+(k)$ and $a_\mu(k)$ linearly and therefore does not commute with the exact momentum of the system. This circumstance in no way invalidates our program, since we only intend to use this operator asymptotically for large $|t|$, i.e., when only small momenta k make a contribution to the integral and the lack of commutativity is not so drastic. However, we must not forget this fact in choosing the boundary conditions; in particular, the expression for $U_{as}(t)$ must not contain a contribution from integrals of $V_{as}(t)$ with respect to finite t. We shall see that this condition essentially determines $U_{as}(t)$ uniquely.

Experience of such problems suggests that it is advisable to seek the solution of Eq. (7) in the form

$$U_{as}(t) = e^{-iH_0 t} Z(t).$$

The equation for Z(t) is

$$i\,\frac{d}{dt}\,Z(t) = V^{I}_{as}(t)\,Z(t), \tag{8}$$

where

$$V^{I}_{as}(t) = e^{iH_0 t}\, V_{as}(t)\, e^{-iH_0 t}.$$

The explicit expression for the operator $V^{I}_{as}(t)$ differs from formula (5) only by the presence of the factors $e^{ik_0 t}$ and $e^{-ik_0 t}$ of $a^+_\mu(k)$ and $a_\mu(k)$.

The operator $V^{I}_{as}(t)$ possesses the following important property. The commutator

$$[V^{I}_{as}(t_1), V^{I}_{as}(t_2)] = Q(t_1, t_2)$$

commutes with $V^I_{as}(t)$ for all t, t_1, t_2. This property enables one to disentangle the T-product explicitly and, by the same token, to find the general solution of Eq. (8)

$$Z(t) = T\exp\left\{-i\int^t V^I_{as}(\tau)\,d\tau\right\},$$

namely,

$$Z(t) = \exp\left\{-i\int^t V^I_{as}(\tau)\,d\tau - \frac{1}{2}\int^t d\tau\int^\tau ds\, Q(\tau, s)\right\}. \tag{9}$$

In the last two formulas, there is a degree of freedom in the choice of the lower limit of integration. This comes about directly as a result of the aforementioned problem of the choice of the initial condition. The requirement formulated in that connection admits only a single method of calculation of the integrals in (9), namely, one must assume that

$$\int^t e^{is\tau}\,d\tau = \frac{1}{is}e^{ist}.$$

Only in this case are the forbidden terms that do not commute asymptotically with the momentum absent.

Having settled this point, we can easily obtain the explicit form of the desired operator Z(t). It is convenient to express it in the form

$$Z(t) = \exp\{R(t)\}\exp\{i\Phi(t)\},$$

where

$$R(t) = \frac{e}{(2\pi)^{3/2}}\int\frac{p_\mu}{pk}\left(a^+_\mu(\mathbf{k})e^{i\frac{kp}{p_0}t} - a_\mu(\mathbf{k})e^{-i\frac{kp}{p_0}t}\right)\rho(\mathbf{p})\,d\mathbf{p}\,\frac{d\mathbf{k}}{(2k_0)^{1/2}} \tag{10}$$

and

$$\Phi(t) = \frac{e^2}{8\pi}\int :\rho(\mathbf{p})\rho(\mathbf{q}): \frac{p\cdot q}{((pq)^2 - m^4)^{1/2}}\,\mathrm{sign}\,t\ln\frac{|t|}{t_0}\,d\mathbf{p}\,d\mathbf{q}. \tag{11}$$

The evaluation of the integrals leading to Eq. (11) is dealt with in the Appendix.

It is natural to call Φ the phase operator. This can be justified by comparing formula (11) with the formula for the nonrelativistic Coulomb phase in Section 1

$$\varphi(t) = \frac{gm}{p}\,\mathrm{sign}\,t\ln\frac{|t|}{t_0}$$

and recalling that $v(p, q) = (1 - [m^4/(pq)^2])^{1/2}$ plays the role of the relative velocity for particles with velocities p/p_0 and q/q_0. It is then seen that formula (11) gives a natural relativistic generalization of the Coulomb phase. The first factor in Z(t) is of purely relativistic origin. We shall discuss its properties and its role in the asymptotic condition in the following section.

Thus, the final expression for the operator of the asymptotic dynamics has the form*

$$U_{as}(t) = \exp\{-iH_0t\}\exp\{i\Phi(t)\}\exp\{R(t)\},$$

where the operators R(t) and Φ(t) are defined by formulas (10) and (11), respectively. We note that these operators commute. Proceeding with the generalization of the nonrelativistic formulas of Section 1, we must define the S matrix as the limit of the operator

$$S(t_1, t_2) = U^+_{as}(t_1)\exp\{-iH(t_1 - t_2)\}U_{as}(t_2)$$

as $t_1 \to \infty$ and $t_2 \to -\infty$. The expression on the right-hand side differs from the Dyson S matrix for finite times

$$S_D(t_1, t_2) = \exp\{iH_0t_1\}\exp\{-iH(t_1 - t_2)\}\exp\{-iH_0t_2\}$$

by the outer factors of the type $\exp\{R(t) + i\Phi(t)\}$. In the following sections we shall attempt to show that

* Dollard's paper [7] is not the only investigation in which an asymptotic condition is modified by replacing $\exp\{-iH_0t\}$ by an operator of the form $\exp\{-iH_0t\}Z(t)$ in the definition of the wave operators. A similar approach was also used in [8, 9, 10, 11]. However, in contrast to the operators used in all these investigations, our operator Z(t) does not commute with H_0.

the definition of the scattering operator based on the expression for $S(t_1, t_2)$ is more correct than the generally accepted limit of the operator $S_D(t_1, t_2)$ as $t_1 \to \infty$ and $t_2 \to -\infty$.

4. Space of Asymptotic States

The greater part of this section will be devoted to an investigation of the properties of the operator $W(t) = \exp\{R(t)\}$, which occurs in the expression for the operator of the asymptotic dynamics in Section 3. In the first place, we shall be interested in its effect on arbitrary state vectors. Our problem is greatly simplified by the fact that W(t) preserves the number, momenta, and spins of the charged particles. Putting it more precisely, the "infinitesimal" subspaces formed by vectors of the form

$$\Psi_{n,m}(p_1s_1,\ldots,p_ns_n|q_1l_1,\ldots,q_ml_m)\otimes\Psi_\gamma$$

where $\Psi_{n,m}(p_1s_1,\ldots|q_1l_1,\ldots)$ is the charged-particle state introduced in (6) and Ψ_γ is an arbitrary photon state, reduce the operator W(t). In each such subspace, W(t) is defined by its action on Φ_γ, which, in its turn, is defined by the operator

$$W_{n,m}(t)=\exp\left\{\frac{e}{(2\pi)^{3/2}}\int(f^\mu_{n,m}(\ldots|\mathbf{k},t)\,a_\mu^+(\mathbf{k})-f^{\mu*}_{n,m}(\ldots|\mathbf{k},t)\,a_\mu(\mathbf{k}))\frac{d\mathbf{k}}{(2k_0)}\right\}$$

where

$$f^\mu_{n,m}(\ldots|\mathbf{k},t)=\sum_{i=1}^{n}\frac{p_i^\mu}{p_ik}e^{i\frac{kp_i}{p_{i0}}t}-\sum_{i=1}^{m}\frac{q_i^\mu}{q_ik}e^{i\frac{kq_i}{q_{i0}}t}.$$

An expression of the type exp{linear form in a and a^+}, which we have just encountered, is an old friend of all field theoreticians. Let us briefly recall the well-known properties of such expressions, taking as an example the operator

$$W=\exp\left\{\sum_i(\alpha_i^*a_i-\alpha_ia_i^+)\right\},$$

defined in terms of a discrete set of boson operators a_i and a_i^+ and numerical coefficients α_i, α_i^*; $i = 1, 2, \ldots$ If W is reduced to normal form, we obtain

$$W=\exp\left\{-\frac{1}{2}\sum_i|\alpha_i|^2\right\}\exp\left\{-\sum_i\alpha_ia_i^+\right\}\exp\left\{\sum_i\alpha_i^*a_i\right\}. \tag{12}$$

Clearly, this expression is meaningless if $\sum_i |\alpha_i|^2 = \infty$. Putting it more correctly, W is not defined in the Fok space in this case. However, this does not prevent it from being a perfectly sensible unitary operator in a larger space, such as the complete infinite tensor product (in the sense of von Neumann [12]) of the spaces $\mathcal{H}_i$ of individual oscillators.

The difference between these two spaces can be best illustrated in the occupation number representation [13]. An arbitrary state vector in this representation is a functional $f = f(\{n\})$ on the space of infinite sequences of integers

$$\{n\}=(n_1,n_2,\ldots,n_i,\ldots),$$

where n_i numbers the states of the i-th oscillator. The Fok space $\mathcal{H}_0$ is formed by functionals that are nonvanishing only on sequences for which $\sum_i n_i < \infty$. The set of such sequences is countable, i.e., $\mathcal{H}_0$ is separable. In contrast, the set of all sequences is continuous, i.e., the space $\mathcal{H}$ of all functionals for which

$$\sum_{\{n\}}|f(\{n\})|^2<\infty$$

is nonseparable. This space is isomorphic with the aforementioned infinite tensor product of von Neumann:

$$\mathcal{H}\sim\prod_i\otimes\mathcal{H}_i.$$

The action of the operators a_i^+ and a_i in $\mathcal{H}$ is defined by exactly the same formulas as in the Fok space.

For example,

$$a_i^+ f(n_1, \ldots, n_i, \ldots) = \sqrt{n_i}\, f(n_1, \ldots, n_i - 1, \ldots).$$

It is clear that $\mathscr{H}_0$ is invariant under the action of a_i^+ and a_i. However, this is not necessarily true for arbitrary functions of a_i^+ and a_i. For example, the operator W is unitary in the space $\mathscr{H}$ for any choice of the numbers α_i. However, if $\sum_i |\alpha_i|^2 = \infty$, it maps every vector belonging to the Fok subspace out of the latter. It is therefore natural to consider the Fok space $\mathscr{H}_0$, the image of the Fok space $\mathscr{H}_{(\alpha)}$ under W:

$$\mathscr{H}_{(\alpha)} = W_{(\alpha)}\mathscr{H}_0.$$

Here, we use a more detailed notation $W_{(\alpha)}$ for W. Clearly, $\mathscr{H}_{(\alpha)}$ is a separable subspace of $\mathscr{H}$ which is as rich in elements as the Fok space $\mathscr{H}_0$.

There exist entire classes of numerical sequences $\{\alpha\}$ to which there correspond one and the same subspace $\mathscr{H}_{(\alpha)}$. The simple formula

$$W_{(\beta)}^+ W_{(\alpha)} = \exp\left\{\frac{1}{2}\sum_i (\beta_i^*\alpha_i - \beta_i\alpha_i^*)\right\} W_{(\alpha-\beta)}, \tag{13}$$

in which the sequence $\{\alpha-\beta\}$ is composed of the numbers $\alpha_i - \beta_i$, shows that the spaces $\mathscr{H}_{(\alpha)}$ and $\mathscr{H}_{(\beta)}$ coincide if

$$\sum_i |\alpha_i - \beta_i|^2 < \infty, \quad \sum_i |\operatorname{Im} \alpha_i\beta_i^*| < \infty, \tag{14}$$

since, in this case, the operator on the right-hand side of (13) is unitary in $\mathscr{H}_0$ and

$$\mathscr{H}_{(\alpha)} = W_{(\beta)} W_{(\beta)}^+ W_{(\alpha)} \mathscr{H}_0 = W_{(\beta)}\mathscr{H}_0 = \mathscr{H}_{(\beta)}.$$

Our assertions and formulas can be trivially generalized to the case of subscripts i of arbitrary nature. In particular, they are applicable to our operator W(t), or rather to the corresponding suboperators $W_{n,m}(t)$. In this case, the role of the numbers α_i is played by the function $f^{\mu}_{n,n}(\ldots|\mathbf{k}, t)$ and the role of the sum $\sum |\alpha_i|^2$ by the integral

$$\int f^{\mu}_{n,m}(\ldots|\mathbf{k}, t) f^{\mu *}_{n,m}(\ldots|\mathbf{k}, t) \frac{d\mathbf{k}}{2k_0} \tag{15}$$

(there is no summation over n and m). This integral diverges at the lower limit,* so that the operators $W_{n,m}(t)$ are not defined in the Fok space for photons and, by the same token, W(t) is not defined in the Fok space $\mathscr{H}_f$ for photons and charged particles. However, it can be defined as a unitary operator in the nonseparable space

$$\mathscr{H} = \mathscr{H}_{f0} \otimes \mathscr{H}_n$$

where $\mathscr{H}_{f0}$ is the Fok space for charged particles, and $\mathscr{H}_n$ is the nonseparable von Neumann space for photons. The Fok space $\mathscr{H}_f$ is a subspace of $\mathscr{H}$.

In $\mathscr{H}$, we consider the separable subspace

$$\mathscr{H}_{as} = W^+(t)\mathscr{H}_f,$$

calling it the space of asymptotic states for reasons that will become apparent in the next section. Alternatively, the space $\mathscr{H}_{as}$ can be defined by the formula

$$\mathscr{H}_{as} = \exp\{-R_f\}\mathscr{H}_f,$$

in which the formally antisymmetric operator R_f is defined by a formula of the type

$$R_f = \frac{e}{(2\pi)^{3/2}} \int (f_\mu(\mathbf{k}, p) a_\mu^*(\mathbf{k}) - f_\mu^*(\mathbf{k}, p) a_\mu(\mathbf{k})) \rho(p)\, dp \frac{d\mathbf{k}}{(2k_0)^{1/2}},$$

---

*Of course, this integral also diverges at the upper limit. The theory of renormalization must, of course, be invoked to tackle this ultraviolet divergence. In the spirit of this theory, we shall assume that, wherever necessary, cutoff factors have been introduced so that, for example, $\exp\{iHt\}$ is defined as a unitary operator in the Fok space. The infrared divergences of the integral (15) still remain, since they are not eliminated by renormalizations.

and the form factor $f_\mu(\mathbf{k}, p)$ may be an arbitrary function for which the integrals

$$\int\left(f_\mu(\mathbf{k}, p)-\frac{p_\mu}{pk}e^{i\frac{kp}{p_0}t}\right)\left(f^*_\mu(\mathbf{k}, p)-\frac{p_\mu}{pk}e^{-i\frac{kp}{p_0}t}\right)\frac{d\mathbf{k}}{2k_0}, \quad (16)$$

$$\int\left|f^*_\mu(\mathbf{k}, p)\frac{q_\mu}{qk}e^{i\frac{kq}{q_0}t}-f_\mu(\mathbf{k}, p)\frac{q_\mu}{qk}e^{-i\frac{kq}{q_0}t}\right|\frac{d\mathbf{k}}{2k_0}, \quad (17)$$

converge for all p and q. For if this is the case, conditions of the type (14) hold for pairs of operators $W^+_{n,m}(t)$ and $(\exp\{-R_f\})_{n,m}$ for all n and m, i.e., the images of the Fok space under the isometric action of the operators W^+ and $\exp\{-R_f\}$ in $\mathscr{H}$ coincide.*

We shall now show that the space $\mathscr{H}_{as}$ has the following properties:

1) it does not depend on t;

2) it is relativistically and gauge invariant;

3) it contains a Lorentz and gauge invariant subspace $\mathscr{H}'_{as}$ with nonnegative metric.

The first property is obvious, since the time does not occur at all in the alternative definition of $\mathscr{H}_{as}$. The second property can be more precisely formulated as follows: the space $\mathscr{H}_{as}$ reduces the representation of the Poincaré group $a, \Lambda \to U(a, \Lambda)$ and the group of gauge transformations $\lambda(k) \to U(\lambda)$ defined naturally in the whole $\mathscr{H}$. To prove this, we note that the operators $a_\mu(k)$ and $\rho(p)$ transform under this representation of the Poincaré group in accordance with the following formulas:

$$\sqrt{k_0}U(a, \Lambda)a_\nu(k)U^+(a, \Lambda) = \sqrt{(\Lambda k)_0}(\Lambda^{-1})_\nu{}^\mu a_\mu(\Lambda k)e^{-ika},$$

$$p_0U(a, \Lambda)\rho(p)U^+(a, \Lambda) = (\Lambda p)_0\,\rho(\Lambda p).$$

It follows that $U(a, \Lambda)W^+(t)U^+(a, \Lambda)$ is an operator of the type $\exp\{-R_f\}$, where the function

$$f^\mu_{a,\Lambda}(k, p) = \frac{p_\mu}{pk}e^{i\frac{kp}{p_0}s+ika}, \quad s = \frac{p_0}{(\Lambda p)_0}t,$$

satisfies conditions of the type (16) and (17). Thus,

$$U(a, \Lambda)\mathscr{H}_{as} = U(a, \Lambda)W^+(t)\mathscr{H}_F = U(a, \Lambda)W^+(t)U^+(a, \Lambda)\mathscr{H}_F = \mathscr{H}_{as}.$$

Moreover, only the photon operators are affected by a gauge transformation

$$U(\lambda)a_\nu(k)U^+(\lambda) = a_\nu(k) + k_\nu\lambda(k),$$

so that the operators $W^+(t)$ and $U(\lambda)W^+(t)U^+(\lambda)$ differ only by a factor of the form

$$\exp\left\{\frac{e}{(2\pi)^{3/2}}\int\left(\lambda(k)e^{-i\frac{kp}{p_0}t}-\lambda^*(k)e^{i\frac{kp}{p_0}t}\right)\rho(p)\,dp\,\frac{d\mathbf{k}}{(2k_0)^{1/2}}\right\},$$

which commutes with $W^+(t)$ and is properly defined on the Fok space. Therefore, $\mathscr{H}_{as}$ is also invariant under the action of the operators of $U(\lambda)$.

The subspace $\mathscr{H}'_{as}$ with nonnegative metric is formed by vectors Ψ satisfying the additional condition

$$k_\mu a_\mu\Psi = 0,$$

whose relativistic and gauge invariance is manifest. The existence of such a subspace $\mathscr{H}'_F$ of the Fok space $\mathscr{H}_F$ and the fact that its metric is nonnegative are well known. We shall show that among the operators of the type $\exp\{-R_f\}$ there exist operators that commute with the additional condition. The image of $\mathscr{H}'_F$ in $\mathscr{H}_{as}$ under the action of such operators on $\mathscr{H}'_F$ gives us the space $\mathscr{H}'_{as}$.

In order to satisfy the aforementioned commutation condition, the function $f_\mu(k, p)$ must be transverse, i.e., satisfy the condition $k_\mu f_\mu(k, p) = 0$. At the first glance, the condition of transversality contradicts the requirement that $f_\mu(k, p)$ have a nontransverse singularity of the type p_μ/pk for small k. However, we are rescued by the existence of a light-like vector $c_\mu(k)$ such that

$$k_\mu c_\mu = 1; \quad c_\mu c_\mu = 0,$$

*A space similar to $\mathscr{H}_{as}$ was introduced by Blanchard [14] in connection with an investigation of a nonrelativistic Pauli–Fierz model.

whose components are conveniently taken to be

$$c_0(k) = \frac{1}{2k_0}, \quad \mathbf{c} = -\frac{1}{2}\frac{\mathbf{k}}{k_0^2}.$$

The transverse form factor $f_\mu(k, p)$ can be taken to be the expression

$$f_\mu(k, p) = \left(\frac{p_\mu}{pk} - c_\mu\right)\varphi(k, p), \tag{18}$$

where $\varphi(k, p) = 1$ in the neighborhood of $k = 0$. Conditions of the type (14) are satisfied if $f_\mu(k, p)$ is chosen in this manner.

We shall mention that the subspace $\mathcal{H}_F'$, like $\mathcal{H}_{as}'$, is not yet a physical state space. The group of gauge transformations operates in $\mathcal{H}_{as}'$ nontrivially. The physical state vectors are in a one-to-one correspondence with the classes of vectors in $\mathcal{H}_{as}'$ forming cyclic subspaces under the gauge group. The metric in the space of such classes $\mathcal{H}_{as, ph}$ is positive definite (see [15]).

This completes the proof of the properties of $\mathcal{H}_{as}$. In the following and final section we shall discuss the role of this space in the scattering theory of a system of charged particles.

5. Definition of the S Matrix

We are now in a better position to say exactly what we mean by the scattering operator. Consider the operator

$$\tilde{S}_D(t_1, t_2) = \exp\{-i\Phi(t)\}S_D(t_1, t_2)\exp\{i\Phi(t)\},$$

which differs from the Dyson S matrix for finite times by the "phase" factors $\exp\{\pm i\Phi(t)\}$. In the product

$$S(t_1, t_2) = W^+(t_1)\tilde{S}_D(t_1, t_2)W(t_2)$$

the cofactors realize the following sequence of mappings:

$$\mathcal{H}_{as} \xrightarrow{W} \mathcal{H}_F \xrightarrow{\tilde{S}} \mathcal{H}_F \xrightarrow{W^+} \mathcal{H}_{as}$$

(see footnote p. 750), so that $S(t_1, t_2)$ is defined as an operator in $\mathcal{H}_{as}$. The S matrix

$$S = \lim_{\substack{t_1 \to \infty \\ t_2 \to -\infty}} S(t_1, t_2) \tag{19}$$

also acts in $\mathcal{H}_{as}$. In this sense, it is natural to call $\mathcal{H}_{as}$ the space of asymptotic states.

The importance of the operators $W^+(t_1)$ and $W(t_2)$ in the definition of the S matrix is that they force us to introduce and exploit the space $\mathcal{H}_{as}$ of asymptotic states. In the actual calculations of the matrix elements of the limit operator S between the states of $\mathcal{H}_{as}$, one can forget the operators $W^+(t_1)$ and $W(t_2)$. For the formal transition to the limit in the coefficient functions of the operators R(t)

$$\lim_{t \to \pm\infty} \frac{1}{pk}\exp\left\{ikp\frac{t}{p_0}\right\} = \pm i\pi\delta(kp) = 0$$

shows that these operators disappear in the limit $|t| \to \infty$ and, hence, the operators W(t) are transformed into identity operators. The role of the operators $\exp\{\pm i\Phi(t)\}$ consists of eliminating the infinite phase factors which would arise if we calculated the matrix elements of the Feynman S matrix between the states of $\mathcal{H}_{as}$. It is precisely the inclusion of these phase operators in the definition of the S matrix that enables us to get by with a single separable space for the initial and final states.

The S matrix is relativistically and gauge invariant. For the operators $W^+(t_1)$ and $W(t_2)$ effectively disappear in the limits $t_1 \to \infty$ and $t_2 \to -\infty$ and the phase operators acquire an explicitly covariant form and do not contain photon operators. If S is written formally in the form

$$S = \exp\{-i\Phi(\infty)\}S_F\exp\{i\Phi(\infty)\},$$

where S_F is the Feynman S matrix, the individual cofactors on the right-hand side are explicitly invariant. This formula can be given a meaning if one bears in mind the passage to the limit from a theory with a finite interaction radius. Let us ascribe a mass λ to the photon and let $S_{F,\lambda}$ be the corresponding Feynman

S matrix. Then

$$S=\lim_{\lambda\to 0}\exp\{i\Phi(\lambda)\}\,S_{F,\lambda}\exp\{i\Phi(\lambda)\},$$

where

$$\Phi(\lambda)=-\frac{e^2}{8\pi}\int \ln\lambda t_0\frac{pq}{((pq)^2-m^4)^{1/2}}:\rho(\mathbf{p})\rho(\mathbf{q}):d\mathbf{p}d\mathbf{q}.$$

The action of the operator $\exp\{i\Phi(\lambda)\}$ on a state of the form (6) reduces to multiplication by the phase factor

$$\exp\Big\{-i\frac{e^2}{4\pi}\ln\lambda t_0\sum_{i\neq j}(v^{-1}(p_i,p_j)+v^{-1}(q_i,q_j)-v^{-1}(p_i,q_j))\Big\}.$$

The cancellation of such factors with diverging phases in the Feynman S matrix $S_{F,\lambda}$ is known in both the nonrelativistic case [16] as well as in the case of quantum electrodynamics [3, 17]. It should be noted that we have obtained an expression for the phase factors without solving the complete dynamical problem.

Thus, our proposed alteration of the asymptotic condition in the relativistic theory of interacting charged particles leads to a modification of the space of asymptotic states and the definition of an S matrix. Besides the charged particles, the asymptotic states must contain an infinite number of photons, whose low-frequency spectrum is determined by the state of the charges. The redefinition of the S matrix reduces to the separation of the operator "phase" factors. The first property has a relativistic character; the second is already required in the case of nonrelativistic scattering by a Coulomb potential.

The arguments that led us to our basic assumption are not proofs. For example, we have not verified the existence of a limit in the definition (19) of the S matrix. At the present level of our mathematical understanding of relativistic field theory, "proof" almost of necessity means "verify for perturbation theory." Thus, we should at least verify that the matrix elements $\langle\Psi|S|\Psi'\rangle$ of the S matrix between arbitrary states Ψ and Ψ' in $\mathcal{H}'_{as}$ do not contain infrared divergences in all orders of the expansion in e^2. Such a program can only be implemented by laborious calculations. Fortunately, Chung has already carried out an appreciable part of these calculations. Let us therefore compare his and our assumptions.

Chung considered in detail the scattering of a particle by an external field. In this case, there is a single charge in the initial state and in the final state; the phase operator vanishes on such states, i.e., the S matrix on these states coincides with the Feynman S matrix. In accordance with our assumption, we must consider the matrix element of this S matrix between states of the form

$$\Psi_{as}(p)=\exp\{-R_f\}\,b_i^+(p)|0\rangle,$$

where f is the form factor (18). The states considered by Chung can, with a slightly different notation, be expressed in the form

$$\Psi_{ch}(p)=D(a_\mu^*,p)\,b_i^*(p)|0\rangle=N^{-1/2}\exp\Big\{\frac{e}{(2\pi)^{3/2}}\int\sum_{n=1}^{2}F^n(k,p)\,\varepsilon_\mu^n a_\mu^*(\mathbf{k})\frac{d\mathbf{k}}{(2k_0)^{1/2}}\Big\}\,b_i^*(\mathbf{p})|0\rangle,$$

where $\varepsilon_\mu^n(k)$, $n=1, 2$, are transverse polarization vectors ($\varepsilon_0^n=0$, $\varepsilon^n k=0$) and the form factor $F^n(k, p)$ has the form

$$F^n(k,p)=\frac{pe^n}{pk}\varphi(k,p)$$

with the same cutoff factor $\varphi(k, p)$ as $f_\mu(k, p)$, and N is a diverging normalizing factor

$$N=\exp\Big\{\frac{e^2}{(2\pi)^3}\int\sum_n|F^n(k,p)|^2\frac{d\mathbf{k}}{2k_0}\Big\}.$$

Chung showed that the matrix element $\langle\Psi_{ch}(p)|S|\Psi_{ch}(p')\rangle$ has no infrared divergences in all orders of perturbation theory. We shall show that our states $\Psi_{as}(p)$ are equivalent to Chung's states $\Psi_{ch}(p)$, i.e., his assertion also holds for our states.

The state $\Psi_{as}(p)$ can be expressed in the form

$$\Psi_{as}(p)=\exp\Big\{\frac{e}{(2\pi)^{3/2}}\int(f_\mu(k,p)\,a_\mu^*(\mathbf{k})-f_\mu^*(k,p)\,a_\mu(\mathbf{k}))\frac{d\mathbf{k}}{(2k_0)^{1/2}}\Big\}\,b_i^*(\mathbf{p})|0\rangle,$$

since the state $b_i^+(p)|0\rangle$ is an eigenstate of the operator $\rho(\mathbf{p})$.

The operation of a gauge transformation $U(\lambda)$ on $\Psi_{as}(p)$ reduces to the substitution

$$f_\nu(k,p) \to f_\nu(k,p) + k_\nu \lambda(k) = f'_\nu(k,p).$$

We can choose λ such that $f'_0(k, p) = 0$ for small k. Since f_μ is transverse, the form factor f'_μ is also transverse and is therefore a linear combination of ε^1_μ and ε^2_μ:

$$f'_\nu(k,p) = c^1(k,p)\,e^1_\nu(k) + c^2(k,p)\,e^2_\nu(k),$$

where

$$c^n(k,p) = f'_\nu e^n_\nu = f_\nu e^n_\nu = \frac{p_\nu e^n_\nu}{pk}\varphi(k,p) = F^n(k,p),$$

since k_μ and c_μ are orthogonal to ε^n_μ, n = 1, 2. Thus, the state $\Psi_{as}(p)$ is equivalent to a state of the form

$$\Psi'_{as}(p) = \exp\left\{\frac{e}{(2\pi)^{3/2}}\int\sum_n \left(F^n(k,p)\,e^n_\mu a^+_\mu(\mathbf{k}) - F^{n*}(k,p)\,e^n_\mu a_\mu(\mathbf{k})\right)\frac{d\mathbf{k}}{(2k_0)^{1/2}}\right\} b^+_i(\mathbf{p})|0\rangle.$$

This state is identical with a Chung state, as is easily seen by reducing the operator occurring in its definition to a normal form in accordance with a formula of the type (12) and then omitting the factor containing photon annihilation operators.

Similarly, one can verify the equivalence of the states proposed by Chung for systems of several charged particles and the corresponding asymptotic states constructed in accordance with our prescription. However, in this case, one must remember that the phase operator is nontrivial. Some calculations taking this fact into account have been made by Storrow [18]. Chung did not make a special study of states that, apart from charged particles, also contain hard photons. An uncritical generalization of his suggestions to this case can lead to incorrect assumptions. For example, the states

$$\Psi_{ch}(p,k) = D(a_\nu^+, p)\, b_i^+(p)\, a_\nu^+(k)\,|0\rangle$$

and

$$\Psi_{as}(p,k) = \exp\{-R_i\}\, b_i^+(p)\, a_\nu^+(k)\,|0\rangle$$

are not equivalent, since one cannot neglect the photon annihilation operators in the reduction of the states $\Psi_{as}(p, k)$ in accordance with the above method. One can show that the matrix elements of the S matrix between states of the second kind vanish as the photon momentum decreases but that states of the first kind lead to matrix elements that diverge as $k \to 0$. Thus, it is the second definition, a special case of our general definition of asymptotic states, which is correct.

In conclusion, we should like to point out that the space $\mathcal{H}_{as}$ does not contain single-particle states of charged particles. Putting it more precisely, the representation of the Poincaré group $a, \Lambda \to U(a, \Lambda)$ acting in the physical Hilbert space $\mathcal{H}_{as\ ph}$ does not contain discrete irreducible terms with nonvanishing mass. In other words, one can say that the relativistic concept of a charged particle does not exist. This is a well-known assertion (see [19]) and our paper proves that it is a natural assertion from the point of view of scattering theory. It would be interesting if one could find a relativistically invariant complete set of commuting operators in $\mathcal{H}_{as}$ which could be used to define a natural basis of asymptotic states.

APPENDIX

In this appendix, we wish to calculate the asymptotic behavior at large $|t|$ of the integral

$$\frac{1}{2i}\int^t d\tau \int^\tau ds\,[V^I_{as}(s), V^I_{as}(\tau)],$$

which defines the phase operator $\Phi(t)$. Using the explicit expression for $V^I_{as}(t)$, we rewrite this integral in the form

$$\frac{e^2}{(2\pi)^3}\int dp\,dq\,\frac{pq}{p_0 q_0}\rho(\mathbf{p})\rho(\mathbf{q})\int^t d\tau\int^\tau ds\int\frac{d\mathbf{k}}{2k_0}\sin\left(k\left(\frac{q}{q_0}\tau - \frac{p}{p_0}s\right)\right).$$

We note that

$$\rho(\mathbf{p})\rho(\mathbf{q}) = :\rho(\mathbf{p})\rho(\mathbf{q}): + \delta(\mathbf{p}-\mathbf{q})[\rho_+(\mathbf{p}) + \rho_-(\mathbf{p})],$$

so that the integral splits up into two terms. The second of these is proportional to t and contributes to the renormalization of the mass of the charged particles. Let us consider in more detail the first term, for which we integrate successively over **k**, s, and τ. We have

$$\int \frac{d\mathbf{k}}{2k_0} \sin\left(k\left(\frac{q}{q_0}\tau - \frac{p}{p_0}s\right)\right) = 2\pi^2\delta\left(\left(\frac{q}{q_0}\tau - \frac{p}{p_0}s\right)^2\right)\varepsilon(\tau - s).$$

The argument of the δ-function has one root s smaller in modulus than τ; the difference of the roots is proportional to $((pq)^2 - m^4)^{1/2}$. Thus, integrating over s, we obtain the expression

$$\frac{e^2}{8\pi}\int dp\,dq \frac{pq}{((pq)^2 - m^4)^{1/2}} : \rho(\mathbf{p})\rho(\mathbf{q}) : \int^t \frac{d\tau}{|\tau|},$$

which defines the phase operator (11).

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