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A Theoretical Analysis of Vertical Flow Equilibrium

Y. C. YORTSOS
Petroleum Engineering Program, Department of Chemical Engineering, University of Southern California, Los Angeles, CA 90089-1211, U.S.A.

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Abstract. The assumption of Vertical Equilibrium (VE) and of parallel flow conditions, in general, is often applied to the modeling of flow and displacement in natural porous media. However, the methodology for the development of the various models is rather intuitive, and no rigorous method is currently available. In this paper, we develop an asymptotic theory using as parameter the variable $R_{\mathrm{L}} = L/H\sqrt{k_{\mathrm{V}}/k_{\mathrm{H}}}$. It is rigorously shown that the VE model is obtained as the leading order term of an asymptotic expansion with respect to $1/R_{\mathrm{L}}^2$. Although this was numerically suspected, it is the first time that it is theoretically proved. Using this formulation, a series of special cases are subsequently obtained depending on the relative magnitude of gravity and capillary forces. In the absence of strong gravity effects, they generalize previous works by Zapata and Lake (1981), Yokoyama and Lake (1981) and Lake and Hirasaki (1981), on immiscible and miscible displacements. In the limit of gravity-segregated flow, we prove conditions for the fluids to be segregated and derive the Dupuit and Dietz (1953) approximations. Finally, we also discuss effects of capillarity and transverse dispersion.

Key words: Porous media, vertical equilibrium, asymptotics, displacement processes.

Nomenclature

- C concentration, dimensionless
- D dispersion tensor $[L^2T^{-1}]$
- f fractional flow
- g gravity acceleration $[LT^{-2}]$
- H reservoir thickness $[L]$
- h dimensionless front location
- k mean permeability $[L^2]$
- K permeability $[L^2]$
- L reservoir length $[L]$
- M viscosity ratio, dimensionless
- N_{CT} transverse capillary number
- N_{G} gravity number

N_{TD} transverse dispersion number
P dimensionless pressure
q flow velocity $[LT^{-1}]$
R_{L} VE parameter
S saturation
T time $[T]$
t dimensionless time
u dimensionless horizontal velocity
v dimensionless vertical velocity
X horizontal coordinate $[L]$
x dimensionless horizontal coordinate
y dimensionless vertical coordinate
w dimensionless vertical velocity

Greek
α dispersivity $[L]$
γ interfacial tension $[MT^{-2}]$
δ permeability ratio, dimensionless
ϵ aspect ratio, dimensionless
Θ_{G} gravity number
κ dimensionless permeability
λ dimensionless mobility
μ viscosity $[ML^{-1}T^{-1}]$
Π dimensionless pressure
ρ density $[ML^{-3}]$
ϕ porosity, dimensionless
ψ normalized mobility, dimensionless

Subscripts
a air
c capillary
H horizontal
L longitudinal
o oil
or residual oil
r relative
T total

V vertical
w water
wr residual water
0 leading order

1. Introduction

The description of displacement processes in oil reservoirs or water aquifers is often greatly simplified when the reservoir is narrow and long and the flow almost parallel. This is typically the case in many applications. Approximations under such conditions have been postulated by many researchers. In general, a Vertical Equilibrium (VE) is typically assumed (Figure 1). Depending on the strength of gravity, the various approaches can be classified in two categories: one in which viscous forces and heterogeneity are predominant on the distribution of phases, and another in which the phases completely segregate due to gravity.

The first category is intended to capture primarily the effects of viscous forces and their interaction with heterogeneity (Figure 2). It has been studied by several authors including Coats *et al.* (1971), Yokoyama and Lake (1981), Zapata and Lake (1981), and more recently by Pande and Orr (1989) and Lake *et al.* (1990). A useful discussion of VE using physical arguments can be found in Lake (1989). Since gravity is unimportant, the term vertical here is meant to denote the direction along the transverse coordinate, in which case Transverse Equilibrium (TE) would be a most appropriate terminology. In most of these studies a two-layer description is taken, using rather intuitive, although correct, in retrospect, arguments. Extensive numerical simulation has verified the validity of the various approaches, particularly as it regards the dimensionless parameter $R_{\mathrm{L}} = L/H\sqrt{k_{\mathrm{V}}/k_{\mathrm{H}}}$, which must take large enough values for the VE to be applicable. Along the same lines must be considered the work by Lake and Hirasaki (1981) on tracer dispersion in stratified systems, as well the various phenomenological viscous fingering models, such as Koval (1963), Todd and Longstaff (1972) and Fayers (1984). While they have only an empirical basis, the numerical evidence is in many cases supportive of their applicability.

The second category emphasizes gravity in addition to viscous forces and it should be more applicable to systems of higher permeability. Not surprising, the original contributions in this direction were made in connection with groundwater aquifers, where the so-called Dupuit assumption was introduced (see Bear, 1972). Viscous, two-phase flow was studied by Dietz (1953), and elaborated by Le Fur and Sourieau (1963), Beckers (1965) and others. A complete segregation of the immiscible phases is assumed, a sharp macroscopic interface separating the two regions (Figure 3). Recently, Fayers and Muggeridge (1990) extended this approach to tilted reservoirs with dip.

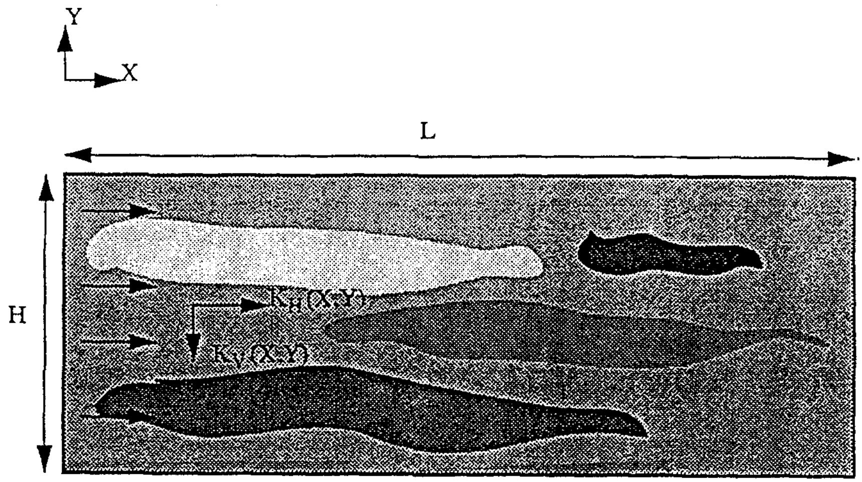

Fig. 1. Schematic of heterogeneous reservoir for VE.

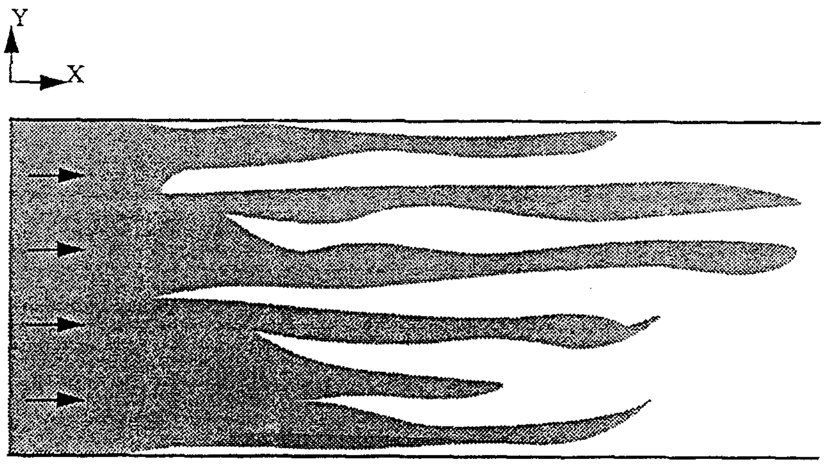

Fig. 2. Schematic of viscous fingering in VE.

While the two classes seem to derive from the same conditions, no effort has been taken to treat them in a uniform fashion. In fact, it is not entirety clear which are the relevant parameters in the parameter space that demarkate the two regimes and where do the various approximations hold. At present, most of the available evidence is numerical. While this may be sufficient under certain conditions, a rigorous derivation would be nonetheless desirable to clearly identify the various approximations and assumptions. This is particularly the case for layered systems, where presently available formalisms are awkward and difficult to extend to many layers.

The objective of this paper is to provide a unified approach based on a rigorous asymptotic expansion of the flow equations in systems where the VE is expected to

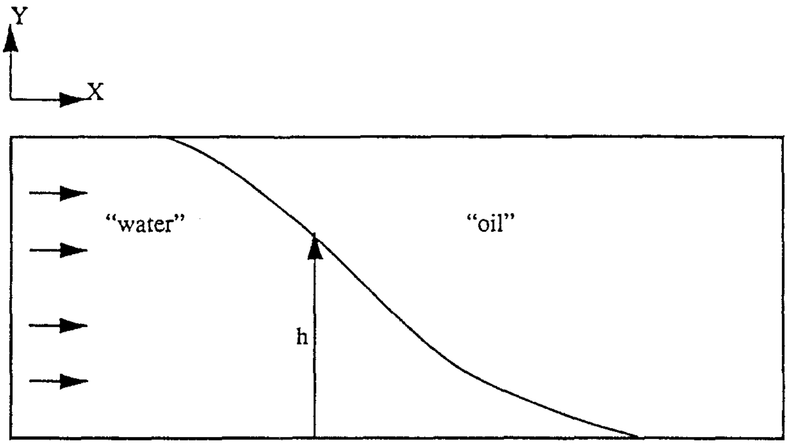

Fig. 3. Schematic of gravity-segregated displacement.

apply. These include isotropic reservoirs which are narrow and long, or anisotropic reservoirs with a large k_V/k_H ratio. First, the fundamental asymptotic analysis is developed for a model immiscible, two-phase displacement. In the absence of gravity or capillarity, an extension of the classical Buckley–Leverett equation, that also includes cross-flow (viscous mixing) terms, is obtained. This equation is then applied to a layered system of arbitrary number of layers. A hyperbolic non-linear system results that describes the interaction between the layers. Weak effects of capillarity and gravity are next introduced. The problem for miscible displacement is formulated in a subsequent section. Tracer dispersion in a layered system is analyzed leading to the results of Lake and Hirasaki (1981). Finally, we consider strong gravity and capillary effects that lead to segregated flow. The conditions for the latter are specified and we derive generalized expressions for the Dupuit and Dietz approximations. Flow segregation due to gravity–capillary equilibrium is also discussed.

2. Asymptotic Analysis

Consider the constant rate immiscible displacement of 'oil' by 'water' in a two-dimensional reservoir of thickness H and length L, and denote $\epsilon \equiv H/L$. The latter does not necessarily have to be small. For simplicity, the reservoir has no dip (for an extension to the case with dip see Yortsos, 1992). We take in general an anisotropic, heterogeneous system, with different permeabilities in the principal directions taken to coincide with the 'horizontal' (H or X) and the 'vertical' (V or Y) directions, respectively

$$K_H = k_H \kappa_H(X, Y); \quad K_V = k_V \kappa_V(X, Y). \tag{1}$$

Here, $K_{\rm H}$, $K_{\rm V}$ denote the two permeabilities, which can be further normalized by their mean values $k_{\rm H}$, $k_{\rm V}$. The spatial dependence is thus on the normalized permeabilities $\kappa_i > 0$, $(i = H, V)$ which are dimensionless and such that $\int_0^H \kappa_i \, {\rm d}Y = H$, when the x-dependence is neglected. Otherwise, the latter constraint is not satisfied. We normalize 'horizontal' and 'vertical' scales, X and Y, by L and H, respectively:

$$x = X/L; \qquad y = Y/H, \tag{2}$$

and scale all velocities by the injection velocity q, time by L/q, and the fluid pressure by $Lq\mu_{\rm o}/k_{\rm H}$. If S denotes a 'water' saturation, the dimensionless balances become

$$\begin{aligned}
&\epsilon\left[\phi\frac{\partial S}{\partial t} + \frac{\partial u_{\rm w}}{\partial x}\right] + \frac{\partial v_{\rm w}}{\partial y} = 0,\\
&\epsilon\frac{\partial}{\partial x}(u_{\rm w} + u_{\rm o}) + \frac{\partial}{\partial y}(v_{\rm w} + v_{\rm o}) = 0,\\
&u_i = -\kappa_{\rm H}(x, y)\lambda_i\frac{\partial p_i}{\partial x}; \quad i = {\rm o}, w,\\
&\frac{\epsilon}{\delta}v_i = -\kappa_{\rm V}(x, y)\lambda_i\left[\frac{\partial p_i}{\partial y} + \frac{\epsilon}{\delta}\frac{\rho_i k_{\rm V} g}{q\mu_{\rm o}}\right]; \quad i = {\rm o}, w.
\end{aligned} \tag{3}$$

Here we defined $\delta \equiv k_{\rm V}/k_{\rm H}$, we have taken the y coordinate to increase upwards, and we have used u_i and v_i to denote the 'horizontal' and 'vertical' components, respectively, of the dimensionless velocity of fluid i. In the absence of capillarity and gravity, u_i and v_i can be expressed in terms of the total velocities $u \equiv u_{\rm w} + u_{\rm o}$ and $v \equiv v_{\rm w} + v_{\rm o}$, with the use of the fractional flow function $f_{\rm w}(S)$,

$$u_{\rm w} = uf_{\rm w}(S); \qquad v_{\rm w} = vf_{\rm w}(S). \tag{4}$$

In the presence of gravity (4) is also rate-dependent, and we shall treat this case in a subsequent section. In our notation, $\lambda_{\rm T}$ is the total mobility, $\lambda_{\rm T} \equiv \lambda_{\rm w} + \lambda_{\rm o}$, where $\lambda_{\rm w} \equiv \mu_{\rm o}/\mu_{\rm w}k_{\rm rw}$ and $\lambda_{\rm o} \equiv k_{\rm ro}$, thus $f_{\rm w} \equiv \lambda_{\rm w}/\lambda_{\rm T}$. We, then, obtain

$$\begin{aligned}
&\epsilon\left[\phi\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(uf_{\rm w})\right] + \frac{\partial}{\partial y}(vf_{\rm w}) = 0,\\
&\epsilon\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\
&u = -\kappa_{\rm H}(x, y)\lambda_{\rm T}\frac{\partial p}{\partial x},\\
&\frac{\epsilon}{\delta}v = -\kappa_{\rm V}(x, y)\lambda_{\rm T}\frac{\partial p}{\partial y}.
\end{aligned} \tag{5}$$

The above suggests the further substitution $v = \epsilon w$, which yields

$$\begin{aligned}
&\phi\frac{\partial S}{\partial t} + u\frac{\partial f_{\mathrm{w}}}{\partial x} + w\frac{\partial f_{\mathrm{w}}}{\partial y} = 0,\\
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} = 0,\\
&u = -\lambda_{\mathrm{T}}\kappa_{\mathrm{H}}(x,\, y)\frac{\partial p}{\partial x},\\
&\frac{1}{R_{\mathrm{L}}^2}w = -\lambda_{\mathrm{T}}\kappa_{\mathrm{V}}(x,\, y)\frac{\partial p}{\partial y}.
\end{aligned} \tag{6}$$

It follows that the relevant dimensionless group is R_{L}, defined as $R_{\mathrm{L}} \equiv \sqrt{\delta}/\epsilon = L/H\sqrt{k_{\mathrm{V}}/k_{\mathrm{H}}}$, in addition to the standard dimensionless functions f_{w}, λ_{T} and the permeability variables. The dimensionless parameter R_{L} is precisely the parameter used in justifying the use of VE (Zapata and Lake, 1981; Lake, 1989). To obtain the appropriate VE model, we expand in a regular asymptotic expansion

$$\begin{aligned}
&u = u_0 + \frac{1}{R_{\mathrm{L}}^2}u_1 + \cdots,\\
&w = w_0 + \frac{1}{R_{\mathrm{L}}^2}w_1 + \cdots,\\
&p = p_0 + \frac{1}{R_{\mathrm{L}}^2}p_1 + \cdots,\\
&S = S_0 + \frac{1}{R_{\mathrm{L}}^2}S_1 + \cdots.
\end{aligned} \tag{7}$$

and take the limit $R_{\mathrm{L}}^2 \gg 1$. Substitution into (6) yields to leading order,

$$\begin{aligned}
&\phi\frac{\partial S_0}{\partial t} + u_0\frac{\partial f_{\mathrm{w}}}{\partial x} + w_0\frac{\partial f_{\mathrm{w}}}{\partial y} = 0,\\
&\frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial y} = 0,\\
&u_0 = -\lambda_{\mathrm{T}}\kappa_{\mathrm{H}}(x,\, y)\frac{\partial p_0}{\partial x},\\
&\frac{\partial p_0}{\partial y} = 0.
\end{aligned} \tag{8}$$

Equation (8) dictates $p_0 = p_0(x,\, t)$, i.e., the pressure is independent of the transverse coordinate, which is the fundamental assumption of VE, rigorously derived here in the limit $R_{\mathrm{L}}^2 \gg 1$. Numerical evidence suggests that VE is satisfied quite well when $R_L \geqslant 10$, although good results with values as low as 2 have been reported (Lake, 1991). This is consistent with our asymptotic expansion, where the next order enters at $O(1/R_L^2)$, hence the error made by retaining only the first term should be of order 1%. The rigorous identification of the parameter R_L as the relevant variable for the validity of VE represents the first result of this paper.

Unless otherwise noted, we shall omit for convenience subscripts 0 and H. The next step is to eliminate $\partial p/\partial x$ by integrating (8c) along the y-coordinate. Since, by definition, $\int_0^1 u \, dy = 1$, we obtain after some algebra

$$u = \frac{\lambda_T(S)\kappa(x, y)}{\int_0^1 \lambda_T(S)\kappa(x, y) \, dy}, \tag{9}$$

where we have assumed a constant injection rate (although the extension to a variable rate is also straightforward). Subsequently, we eliminate w by integrating (8b)

$$w = -\int_0^y \frac{\partial u}{\partial x} \, dy, \tag{10}$$

where no-flow boundary conditions at $y = 0, 1$ were used. Substitution into (8a) yields the final result

$$\phi \frac{\partial S}{\partial t} + \frac{\lambda_T \kappa}{\int_0^1 \lambda_T \kappa \, dy} \frac{\partial f_w}{\partial x} - \frac{\partial}{\partial x} \left[\frac{\int_0^y \lambda_T \kappa \, dy}{\int_0^1 \lambda_T \kappa \, dy} \right] \frac{\partial f_w}{\partial y} = 0. \tag{11}$$

This equation represents the second result of this paper. The equation is a first-order, hyperbolic PDE, but in two spatial dimensions (x, y), and it contains only the saturation S as the dependent variable. More importantly, (11) contains a viscous mixing (cross-flow) term (third term in the LHS), which arises from VE. This term is fundamental to VE and it is absent from the corresponding case of non-communicating layers, where $v = 0$. In this sense, the VE approximation and particularly its subsequent extension to include transverse dispersion or capillarity, bears a relation to the typical boundary layer approximation for flow over a flat plate. When κ and S are independent of y, it naturally reduces to the traditional 1-D Buckley–Leverett (B–L) equation

$$\phi \frac{\partial S}{\partial t} + \frac{\partial f_w}{\partial x} = 0. \tag{12}$$

In this sense, Equation (11) is a 2-D generalization of the B–L equation under the conditions of VE. We must point out, however, that because of the absence of capillarity, Equation (11) does not satisfy all boundary conditions, thus it is not expected to hold near all boundaries. The capillary correction is discussed later in the paper.

Before we proceed, let us recall the conditions for the validity of the above. It was derived in the limit $R_{\mathrm{L}} \gg 1$ in the absence of gravity and capillarity. The first condition can be obeyed in systems where δ is $O(1)$ or smaller, but $\epsilon \ll 1$ (which also includes isotropic, but narrow and long systems), or where δ is large and ϵ is $O(1)$. The latter condition requires that the gravity terms in (3d) are small, or that

$$N_{\mathrm{G}} \ll 1 \tag{13}$$

where the gravity number was defined, $N_{\mathrm{G}} \equiv H k_{\mathrm{H}} g(\rho_{\mathrm{w}} - \rho_{\mathrm{o}})/L\mu_{\mathrm{o}} q$ (note the difference in notation with Fayers and Muggeridge, 1990). This allows for the two Equations (8d) and (9) to remain valid. However, in order for gravity and capillarity to be also absent from (4) and (11), one needs the stronger constraints (see also below)

$$N_{\mathrm{G}} \ll \frac{1}{R_{\mathrm{L}}^2} \quad \text{and} \quad N_{\mathrm{CT}} \ll \epsilon, \tag{14}$$

where the transverse capillary number was defined $N_{\mathrm{CT}} \equiv \sqrt{k_{\mathrm{V}}}/H N_{\mathrm{ca}}$ (see Yokoyama and Lake, 1981). As pointed out in the introduction, under these conditions, Equation (11) is a VE approximation that emphasizes viscous cross-flow and heterogeneity. It is expected, therefore, to control viscous fingering in such systems.

The validity of VE rests on the hypothesis that in (6d) the group $w/\lambda_{\mathrm{T}}\kappa_{\mathrm{V}}$ is $O(1)$, namely that it is not too large. We expect, therefore, that VE would be violated when λ_{T} becomes very small (as could be the case for a very large viscosity contrast, but not when a phase is at its residual, as only the total mobility is involved), when w becomes very large (which, as seen from (9) and (10), is possible near very sharp fronts), or when the vertical permeability vanishes somewhere. In all such cases, a boundary layer correction must be introduced. These complications, although quite important, will not be discussed further in this paper (see Yang and Yortsos, 1995).

3. Layered Reservoirs

While the full solution of (11) is possible (Yang and Yortsos, 1995), sometimes it can be more practical to consider a discrete treatment. For example, this would be the case of a layered reservoir, where we may take

$$\begin{aligned} \kappa(x, y) &\simeq \kappa_i(x); \\ S(x, y) &\simeq S_i(x); \end{aligned} \quad \frac{i-1}{N} < y < \frac{i}{N}; \quad i = 1, N. \tag{15}$$

Here, N is the number of equal thickness layers of the system. For simplicity, we shall denote $\psi = \lambda_{\mathrm{T}}\kappa / \int_0^1 \lambda_{\mathrm{T}}\kappa \, \mathrm{d}y$. We may then integrate (11) over y from $(i-1)/N$ to i/N, to obtain the system

$$\phi_i \frac{\partial S_i}{\partial t} + \psi_i \frac{\partial f_i}{\partial x} - (f_i - f_{i-1}) \frac{\partial}{\partial x} \sum_1^i \psi_j = 0; \quad i = 1, N. \tag{16}$$

Here, we have approximated y-integrals by sums, have denoted $f_i \equiv f_{\mathrm{w}}(S_i)$, $\psi_i \equiv \psi(S_i;\ \kappa_i)$ and defined $f_{\mathrm{o}} \equiv 0$. We may recast (16) in terms of a hyperbolic system

$$\frac{\partial \mathbf{S}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{S}}{\partial x} = \mathbf{B} \frac{\partial \kappa}{\partial x}, \tag{17}$$

where $\mathbf{S} = [S_1,\ S_2, \ldots, S_N]^{\mathrm{T}}$ and $\kappa = [\kappa_1,\ \kappa_2, \ldots, \kappa_N]^{\mathrm{T}}$ are $(N \times 1)$ vectors. $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ $(i,\ j = 1,\ N)$ are $N \times N$ square matrices with the following coefficients. Matrix $\mathbf{A}$ consists of two terms $\mathbf{A} = \mathbf{D} - \mathbf{E}$, where

$$\begin{aligned} d_{ii} &= \frac{\lambda_{\mathrm{T}}(S_i)\kappa_i}{\phi_i \Lambda} \left(\frac{\partial f_{\mathrm{w}}}{\partial S} \right)_i, \\ d_{ij} &= 0,\ i \neq j, \end{aligned} \tag{18}$$

and

$$\begin{aligned} e_{ij} &= \frac{(f_{\mathrm{w}}(S_i) - f_{\mathrm{w}}(S_{i-1}))}{\phi_i \Lambda^2} \kappa_j \left(\frac{\partial \lambda_{\mathrm{T}}}{\partial S} \right)_j \\ &\quad \begin{cases} \frac{1}{N} \sum_i^N \lambda_{\mathrm{T}}(S_{\mathrm{m}})\kappa_{\mathrm{m}}; & j \leqslant i - 1, \\ -\frac{1}{N} \sum_1^{i-1} \lambda_{\mathrm{T}}(S_{\mathrm{m}})\kappa_{\mathrm{m}}; & j \geqslant i, \end{cases} \end{aligned} \tag{19}$$

where, $1 \leqslant i \leqslant N$, and $e_{1j} = 0$. The inhomogeneous matrix $\mathbf{B}$ has coefficients

$$\begin{aligned} b_{ij} &= \frac{(f_{\mathrm{w}}(S_i) - f_{\mathrm{w}}(S_{i-1}))}{\phi_i \Lambda^2} \lambda_{\mathrm{T}}(S_j) \\ &\quad \begin{cases} \frac{1}{N} \sum_i^N \lambda_{\mathrm{T}}(S_{\mathrm{m}})\kappa_{\mathrm{m}}; & j \leqslant i - 1, \\ -\frac{1}{N} \sum_1^{i-1} \lambda_{\mathrm{T}}(S_{\mathrm{m}})\kappa_{\mathrm{m}}; & j \geqslant i. \end{cases} \end{aligned} \tag{20}$$

In the above we have denoted $\Lambda \equiv 1/N \Sigma_1^N \lambda_{\mathrm{T}}(S_{\mathrm{m}})\kappa_{\mathrm{m}}$.

This general formulation in discrete form is free of empirical arguments and represents a rigorous result, apparently also obtained here for the first time.

We observe the following:

(1) Matrix **A** is neither diagonal nor symmetric. This is because of the coupling between adjacent layers of different properties. The coupling is due to the variation of the mobility with saturation.
(2) When the layer permeabilities also depend on position ($\partial\kappa/\partial x \neq 0$) a source (sink) term arises on the RHS of (17). Thus, in this case, heterogeneity acts in the form of a reaction term. This feature is also due to the coupling between the layers and it is absent in the single layer case.
(3) When all layer properties are the same, the system reduces to the B–L equation, as expected.
(4) For a two-layer system, we may further simplify ($\Lambda = \lambda_T(S_1)\kappa_1 + \lambda_T(S_2)\kappa_2$) to obtain

$$d_{11} = \frac{\lambda_T(S_1)\kappa_1}{\phi_1\Lambda}\left(\frac{\partial f_w}{\partial S}\right)_1, \tag{21}$$

$$d_{22} = \frac{\lambda_T(S_2)\kappa_2}{\phi_2\Lambda}\left(\frac{\partial f_w}{\partial S}\right)_2, \tag{22}$$

$$d_{12} = d_{21} = 0, \tag{23}$$

$$e_{11} = e_{12} = 0, \tag{24}$$

$$e_{21} = \frac{1}{2}\left(\frac{f_w(S_2) - f_w(S_1)}{\phi_2\Lambda^2}\right)\kappa_1\kappa_2\left(\frac{\partial\lambda_T}{\partial S}\right)_1\lambda_T(S_2), \tag{25}$$

$$e_{22} = -\frac{1}{2}\left(\frac{f_w(S_2) - f_w(S_1)}{\phi_2\Lambda^2}\right)\kappa_1\kappa_2\left(\frac{\partial\lambda_T}{\partial S}\right)_2\lambda_T(S_1), \tag{26}$$

$$b_{11} = 0, \qquad b_{12} = 0, \tag{27}$$

$$b_{21} = \frac{1}{2}\left(\frac{f_w(S_2) - f_w(S_1)}{\phi_2\Lambda^2}\right)\lambda_T(S_1)\lambda_T(S_2)\kappa_2, \tag{28}$$

$$b_{22} = -\frac{1}{2}\left(\frac{f_w(S_2) - f_w(S_1)}{\phi_2\Lambda^2}\right)\lambda_T(S_1)\lambda_T(S_2)\kappa_1. \tag{29}$$

The above contains the formulation of Zapata and Lake (1981) and Pande and Orr (1989), but it is here also augmented by permeability heterogeneity along the x-direction. The latter result is also new.

4. Effects of Capillarity and Gravity

When the injection rates are low enough for capillary and gravity effects to be of some importance, but not very low for the phases to be segregated, Equations (4) must be reformulated. We obtain

$$\begin{aligned} u_w &= uf_w(S) + \frac{N_{CT}}{R_L}\kappa_H\lambda_o f_w\frac{\partial\Pi_c}{\partial x}, \\ v_w &= vf_w(S) + \kappa_V\lambda_o f_w\left\{N_{CT}\frac{\partial\Pi_c}{\partial y} - \frac{\delta}{\epsilon}N_G\right\}, \end{aligned} \tag{30}$$

where Π_c is the dimensionless capillary pressure, that can be expressed in terms of a J-function representation (e.g. $\Pi_c = J(S)/\sqrt{\kappa_H}$). Equations (5), (11) are appropriately modified. For instance, (5a) becomes

$$\epsilon\left[\phi\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}\left(uf_w+\frac{N_{CT}}{R_L}\kappa_H\lambda_o f_w\frac{\partial\Pi_c}{\partial x}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[vf_w+\kappa_V\lambda_o f_w\left\{N_{CT}\frac{\partial\Pi_c}{\partial y}-\frac{\delta}{\epsilon}N_G\right\}\right]=0. \tag{31}$$

We may proceed in exactly the same way as before, by substituting $v = \epsilon w$, identifying the large parameter as $R_L^2 = \delta/\epsilon^2$ and expanding appropriately. In order for the flow not to be segregated, condition (13) must still hold, $N_G \ll 1$. The longitudinal capillary term can be neglected, except near sharp fronts, much like in the classical Buckley–Leverett problem. However, transverse capillarity and gravity should be retained, if the following conditions are met, $N_{CT} \sim O(\epsilon)$ and $N_G \sim O(1/R_L^2)$. Then, the following equation is obtained

$$\phi\frac{\partial S}{\partial t}+\frac{\lambda_T\kappa}{\int_0^1\lambda_T\kappa\,\mathrm{d}y}\frac{\partial f_w}{\partial x}-\frac{\partial}{\partial x}\left[\frac{\int_0^y\lambda_T\kappa\,\mathrm{d}y}{\int_0^1\lambda_T\kappa\,\mathrm{d}y}\right]\frac{\partial f_w}{\partial y}$$
$$=-\frac{\partial}{\partial y}\left[\kappa_V\lambda_o f_w\left\{\frac{N_{CT}}{\epsilon}\frac{\partial\Pi_c}{\partial y}-R_L^2N_G\right\}\right]. \tag{32}$$

The relative importance of gravity over capillarity depends on the dimensionless ratio

$$\frac{\epsilon R_L^2 N_G}{N_{CT}}=\frac{g\Delta\rho H\sqrt{k_V}}{\gamma} \tag{33}$$

which is rate-independent. For thin beds of low vertical permeability, capillarity dominates. Then, if the typical assumption is made about Π_c as a single function of S, the RHS above represents capillary spreading (one should keep in mind, however, that permeability heterogeneity is likely to also imply capillary heterogeneity as well, see Yortsos and Chang, 1990, and Chaouche *et al.*, 1993. This case is of interest but will not be considered in this study). As shown below in the case of miscible displacement, capillary spreading can be equivalently represented in terms of a macro-dispersivity. Capillary effects in VE were considered by Yokoyama and Lake (1981).

When gravity dominates over capillarity in (32), then

$$\phi\frac{\partial S}{\partial t}+\frac{\lambda_T\kappa}{\int_0^1\lambda_T\kappa\,\mathrm{d}y}\frac{\partial f_w}{\partial x}-\frac{\partial}{\partial x}\left[\frac{\int_0^y\lambda_T\kappa\,\mathrm{d}y}{\int_0^1\lambda_T\kappa\,\mathrm{d}y}\right]\frac{\partial f_w}{\partial y}=+\Theta_G\frac{\partial}{\partial y}[\kappa_V\lambda_o f_w], \tag{34}$$

where we defined

$$\Theta_{\mathrm{G}} = \frac{Lk_{\mathrm{V}}g}{H\mu_{\mathrm{o}}q}(\rho_{\mathrm{w}} - \rho_{\mathrm{o}}).$$

In the discrete (layer) formulation, the contribution of the gravity term can be shown to act as a source/sink. Then, Equation (17) must be modified as follows

$$\frac{\partial \mathbf{S}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{S}}{\partial x} = \mathbf{B}\frac{\partial \kappa}{\partial x} + \mathbf{G}, \tag{35}$$

where the ith element of the $\mathbf{G}$ vector is

$$g_i = N\Theta_{\mathrm{G}}[\kappa_{\mathrm{V}}\lambda_{\mathrm{o}}(S_i)f_{\mathrm{w}}(S_i) - \kappa_{\mathrm{V}}\lambda_{\mathrm{o}}(S_{i-1})f_{\mathrm{w}}(S_{i-1})]. \tag{36}$$

When the flow rates are quite low, such that $N_{\mathrm{G}} \sim O(1)$, the phases are likely to be segregated. This case is described in a later section.

5. Miscible Displacement

Consider, next, a first-contact miscible process in the limit of negligible gravity, $N_{\mathrm{G}} \ll 1$. The mathematical description consists of Equations (5b)–(5d), where the total mobility is now a normalized inverse viscosity, $\lambda_{\mathrm{T}}(C) = \mu^*/\mu(C)$, and where the dimensionless concentration C is the dependent variable. The latter satisfies an advection-dispersion equation, which reads in dimensionless notation

$$\epsilon\left(\phi\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x}\right) + v\frac{\partial C}{\partial y} = \frac{\epsilon}{qL}\frac{\partial}{\partial x}\left[\phi D_{11}\frac{\partial C}{\partial x} + \phi\frac{D_{12}}{\epsilon}\frac{\partial C}{\partial y}\right]$$
$$+\frac{1}{qL}\frac{\partial}{\partial y}\left[\phi D_{21}\frac{\partial C}{\partial x} + \phi\frac{D_{22}}{\epsilon}\frac{\partial C}{\partial y}\right], \tag{37}$$

where the dispersion tensor has the following form (see Zimmerman and Homsy, 1991)

$$\mathbf{D} = \frac{q}{\sqrt{u^2 + v^2}}\begin{bmatrix} \alpha_{\mathrm{L}}u^2 + \alpha_{\mathrm{T}}v^2 & uv(\alpha_{\mathrm{L}} - \alpha_{\mathrm{T}}) \\ uv(\alpha_{\mathrm{L}} - \alpha_{\mathrm{T}}) & \alpha_{\mathrm{L}}v^2 + \alpha_{\mathrm{T}}u^2 \end{bmatrix}. \tag{38}$$

Here we have taken the standard approach to describe dispersion, based on a passive tracer in a random medium (although this can lead to several complications, see Yortsos and Zeybek, 1988), and we have further assumed that mechanical dispersion is dominant over molecular diffusion. Equation (37) is the analogue of (5a) for miscible displacement. In the above, we have denoted by α_{L} and α_{T} the longitudinal and transverse dispersivities, respectively. An order of magnitude analysis shows that in (37), the last term on the RHS dominates over the other dispersion terms. Thus longitudinal and off-diagonal dispersion can be neglected (except near sharp fronts).

To obtain asymptotic results we follow the same procedure as above to get the final result

$$\phi\frac{\partial C}{\partial t} + \frac{\lambda\kappa}{\int_0^1 \lambda\kappa\,\mathrm{d}y}\frac{\partial C}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\int_0^y \lambda\kappa\,\mathrm{d}y}{\int_0^1 \lambda\kappa\,\mathrm{d}y}\right]\frac{\partial C}{\partial y}$$

$$= N_{\mathrm{TC}}\frac{\partial}{\partial y}\left[\phi\left(\frac{u^2 + \frac{\alpha_{\mathrm{L}}\epsilon^2 w^2}{\alpha_{\mathrm{T}}}}{\sqrt{u^2+\epsilon^2 w^2}}\right)\frac{\partial C}{\partial y}\right] \tag{39}$$

where, subscript T was omitted for simplicity. In (39) we have assumed that $N_{\mathrm{TD}} \equiv \alpha_{\mathrm{T}}/\epsilon$ H remains finite in the limit of small ϵ. When transverse dispersion is neglected, the previous equation (11) is recovered, if the identification is made $S \leftrightarrow C$, $\lambda_{\mathrm{T}} \leftrightarrow \lambda$, and $f_{\mathrm{w}} \leftrightarrow C$. Comments on dispersive effects are given in the next section.

In the absence of dispersion, the equivalent of (17) reads

$$\frac{\partial \mathbf{C}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{C}}{\partial x} = \mathbf{B}\frac{\partial \kappa}{\partial x}, \tag{40}$$

where $\mathbf{C} = [C_1, C_2, \ldots, C_N]^{\mathrm{T}}$. The representation becomes simpler if we take the approximate mobility dependence $\lambda(C) = e^{rC}$, where r measures the mobility ratio, $r = \ell n M$. Then, for constant $\phi_i = \phi$, which can subsequently be absorbed in the dimensionless time, the coefficient matrices reduce to the following

$$d_{ii} \equiv d_i = \frac{\kappa_i e^{rC_i}}{\frac{1}{N}\sum_1^N \kappa_j e^{rC_j}},$$

$$e_{ij} = r(C_i - C_{i-1})d_j \begin{cases} \frac{1}{N}\sum_i^N d_{\mathrm{m}}, & j \leqslant i-1, \\ -\frac{1}{N}\sum_1^{i-1} d_{\mathrm{m}}, & i \leqslant j, \end{cases} \tag{41}$$

where $1 \leqslant i \leqslant N$, $e_{1j} = 0$, and

$$b_{ij} = \frac{e_{ij}}{r\kappa_j}. \tag{42}$$

One notes the interesting property that the sum of all the elements of each row of E vanishes, $\Sigma_{j=1}^N e_{ij} = 0$.

When the solute is not passive, an exact solution to the system is not available. Some interesting remarks can be made in the two limits where $r \to +\infty$ (unstable displacement, $M \gg 1$) or when $r \to -\infty$ (stable displacement, $M \ll 1$). In these two cases it can be shown readily that the matrix $\mathbf{C}$ becomes diagonal and, furthermore, that the only element with non-zero velocity is the layer with the highest ($r \to \infty$) or the lowest concentration ($r \to -\infty$), respectively. To show this, consider (41a) and rearrange as follows

$$d_i = \frac{\kappa_i}{\frac{1}{N}\Sigma_1^N \kappa_j e^{r(C_j - C_i)}}. \tag{43}$$

It is then straightforward that, in the limit $r \to \infty$, then $d_i \to 0$ for all i, except for $i = \max$, where $C_{\max}$ is the maximum concentration, hence $d_{\max} \to N$. The opposite applies for the case $r \to \infty$, in which $d_i \to 0$ for all i, except for $i = \min$, where $C_{\min}$ is the minimum concentration, hence $d_{\min} \to N$. In summary, in the case of very unstable displacement ($M \gg 1$), the highest concentration travels the fastest, in accord with viscous fingering notions. While, for very stable displacement ($M \ll 1$), it is the lowest concentration that travels the fastest, also as expected. Besides this simple result, however, Equation (40) contains a much richer structure (see Yortsos, 1992 and Yang and Yortsos, 1995).

6. Tracer Dispersion

In the passive solute case, where the viscosity is constant ($r = 0$), the off-diagonal terms e_{ij} vanish and we obtain the linear system

$$\frac{\partial C_i}{\partial t} + \kappa_i \frac{\partial C_i}{\partial x} = -(C_i - C_{i-1})\frac{\partial}{\partial x}\left(\sum_i^N \kappa_{\mathrm{m}}\right), \tag{44}$$

where, in the above κ_i is to be interpreted as normalized with the average permeability of a cross section. When $\partial \kappa_i / \partial x = 0$, this has the solution

$$C_i = H(t\kappa_i - x), \tag{45}$$

where $H(z)$ is the step function. In the continuum limit

$$\bar{C} = \lim_{N \to \infty} \frac{1}{N} \sum_1^N C_i \tag{46}$$

we further get

$$\bar{C} = \int_{\kappa_{\min}}^{\kappa_{\max}} f(\kappa) H(\kappa t - x)\, \mathrm{d}\kappa, \tag{47}$$

where $f(\kappa)$ is the *pdf* of κ. If we define the cumulative distribution $F(\kappa) = \int_{\kappa_{\min}}^{\kappa} f(\kappa)\, \mathrm{d}\kappa$, then we may rearrange to get

$$\bar{C} = H(t\kappa_{\max} - x) - F\left(\frac{x}{t}\right). \tag{48}$$

For a more general result in the case $\partial\kappa_i/\partial x = 0$, we consider transverse dispersion. We shall make use of the continuum formulation (39) and write (for $\phi = \text{const.}$ and neglecting the velocity dependence)

$$\frac{\partial C}{\partial t} + \kappa(y)\frac{\partial C}{\partial x} = N_{\mathrm{TD}}\frac{\partial^2 C}{\partial y^2}. \tag{49}$$

This equation represents Taylor–Aris dispersion as applied to porous media flows (Lake and Hirasaki, 1981). However, it must be stressed that in arriving at (49) we have neglected the velocity dependence shown on the RHS of (39) (effectively taking $u = 1$, $w = 0$). Strictly speaking, the more complicated dependence shown in (39) must be considered. Under this restrictive condition, and by using a coordinate moving with the average speed $\xi = x - t$, one gets (Taylor, 1953)

$$(\kappa(y) - 1)\frac{\partial C}{\partial \xi} \cong N_{\mathrm{TD}}\frac{\partial^2 C}{\partial y^2}, \tag{50}$$

which can be integrated to

$$C = \frac{1}{N_{\mathrm{TD}}}\frac{\partial \bar{C}}{\partial \xi}\int_0^y \int_0^{y'} (\kappa(y'') - 1)\,\mathrm{d}y''\,\mathrm{d}y'. \tag{51}$$

In the moving frame of reference the mass flux over a cross section is

$$\begin{aligned}&\int_0^1 (\kappa(y) - 1)C\,\mathrm{d}y\\ &\quad = \frac{1}{N_{\mathrm{TD}}}\int_0^1 (\kappa(y) - 1)\int_0^y \int_0^{y'} (\kappa(y'') - 1)\,\mathrm{d}y''\,\mathrm{d}y'\,\mathrm{d}y \cdot \frac{\partial \bar{C}}{\partial \xi},\end{aligned} \tag{52}$$

thus, yielding the macro-dispersion approximation

$$\frac{\partial \bar{C}}{\partial t} + \frac{\partial \bar{C}}{\partial x} = D_{\mathrm{m}}\frac{\partial^2 \bar{C}}{\partial x^2}, \tag{53}$$

where

$$D_{\mathrm{m}} \equiv \frac{1}{N_{\mathrm{TD}}}\int_0^1 (\kappa(y) - 1)\int_0^y \int_0^{y'} (\kappa(y'') - 1)\,\mathrm{d}y''\,\mathrm{d}y'\,\mathrm{d}y. \tag{54}$$

In dimensional notation, the macrodispersity is expressed as

$$\alpha_m = \frac{1}{H\alpha_{\mathrm{T}}(k_{\mathrm{H}})^2}\int_0^H (K_{\mathrm{H}}(y) - k_{\mathrm{H}})$$

$$\int_0^y \int_0^{y'} (K_\mathrm{H}(y'') - k_\mathrm{H})\, \mathrm{d}y''\, \mathrm{d}y'\, \mathrm{d}y. \tag{55}$$

A result similar to the above was first derived by Lake and Hirasaki (1981) by different means. A different result is expected when the full structure of the dispersion in (39) is considered. This is outside the present scope.

7. Gravity Segregated Flow

Consider, next, the case where the gravity terms in (3) are strong, $N_\mathrm{G} \sim O(1)$. Here, the approach is somewhat different. Gravity effects must be also considered in the distribution of pressure, not only in the equation for the mass balance. Consider, first, immiscible displacement with negligible capillarity. The general expression for the vertical flow velocity was derived before

$$\frac{\epsilon}{\delta} v = -\kappa_\mathrm{V} \left[\lambda_\mathrm{T} \frac{\partial p}{\partial y} + N_\mathrm{G}(\lambda_\mathrm{w}\bar{\rho}_\mathrm{w} + \lambda_\mathrm{o}\bar{\rho}_\mathrm{o}) \right], \tag{56}$$

where $\bar{\rho}_i \equiv \rho_i/(\rho_\mathrm{w} - \rho_\mathrm{o})$. In view of the fact that $v = \epsilon w$, the VE condition becomes

$$\lambda_\mathrm{T} \frac{\partial p}{\partial y} + N_\mathrm{G}(\lambda_\mathrm{w}\bar{\rho}_\mathrm{w} + \lambda_\mathrm{o}\bar{\rho}_\mathrm{o}) = 0 \tag{57}$$

thus gravity terms must be considered in the distribution of pressure. However, the fluids now become segregated and the previous analysis is not necessary. To show that flow segregation occurs we consider (30b) in the absence of capillary effects

$$v_\mathrm{w} = \epsilon[w f_\mathrm{w}(S) - \kappa_\mathrm{V}\lambda_\mathrm{o} f_\mathrm{w} N_\mathrm{G} R_\mathrm{L}^2]. \tag{58}$$

All terms above must be $O(1)$ or less. However, since we assumed $N_\mathrm{G} \sim O(1)$, the last term on the RHS would diverge at large R_L, resulting into a contradiction unless $\kappa_\mathrm{V}\lambda_\mathrm{o} f_\mathrm{w}$ vanishes, namely unless

$$\lambda_\mathrm{o} f_\mathrm{w} \to 0 \tag{59}$$

The solution of Equation (59) is either $\lambda_\mathrm{o} = 0$ or $f_\mathrm{w} = 0$, showing that there cannot be simultaneous flow of the two fluids, or equivalently that the flow is segregated. Under the further assumption that under this condition any other configuration would be gravitationally unstable, we further conjecture the full segregation solution

$$S = \begin{cases} S_\mathrm{wr}; & h < y < 1, \\ 1 - S_\mathrm{or}; & 0 < y < h, \end{cases} \tag{60}$$

where the location of the 'macroscopic interface' $h \equiv h(x, t)$ is to be determined as a function of position and time (Figure 3). This is the classical case of gravity tonguing which as shown above, holds under the conditions $R_{\rm L} \gg 1$, $N_{\rm G} \sim O(1)$. The first is a geometric condition, while the second also involves flow rates. To our knowledge, this is the first time that flow segregation was proved analytically.

To obtain the evolution of h, one needs to consider an integral balance. For this, equation (5a) is integrated over y between 0 and 1 to yield

$$\phi\frac{\partial}{\partial t}\int_0^1 S\,{\rm d}y + \int_0^1 \frac{\partial}{\partial x}(uf_{\rm w})\,{\rm d}y = 0. \tag{61}$$

Next, we use (57) to solve with respect to p. We obtain

$$\frac{\partial p}{\partial x} = -N_{\rm G}\frac{\partial}{\partial x}\left[\int_1^y \frac{(\lambda_{\rm w}\bar{\rho}_{\rm w} + \lambda_{\rm o}\bar{\rho}_{\rm o})\,{\rm d}y}{\lambda_{\rm T}}\right] + \frac{\partial \Pi}{\partial x}, \tag{62}$$

where $\Pi \equiv \Pi(x, t)$ is the pressure at $y = 1$ and depends on x and t only. Our ultimate goal is to obtain an expression for u. By subsequent substitution of (62) into (3b) and (61), we finally get

$$\phi\frac{\partial}{\partial t}\int_0^1 S\,{\rm d}y + \int_0^1 \frac{\partial}{\partial x}\left\{\kappa_{\rm H}\lambda_{\rm T}f_{\rm w}\frac{\partial}{\partial x}\left(N_{\rm G}\int_1^y\left(\frac{\lambda_{\rm w}\bar{\rho}_{\rm w}+\lambda_{\rm o}\bar{\rho}_{\rm o}}{\lambda_{\rm T}}\right){\rm d}y' - \Pi\right)\right\}$$
$$ {\rm d}y = 0. \tag{63}$$

Next, the total mass balance is considered. In a straightforward manner it can be shown that the following equation results

$$N_{\rm G}\int_0^1 \frac{\partial}{\partial x}\left\{\kappa_{\rm H}\lambda_{\rm T}\frac{\partial}{\partial x}\left(N_{\rm G}\int_1^y\left(\frac{\lambda_{\rm w}\bar{\rho}_{\rm w}+\lambda_{\rm o}\bar{\rho}_{\rm o}}{\lambda_{\rm T}}\right){\rm d}y' - \Pi\right)\right\}{\rm d}y = 0. \tag{64}$$

The two equations must be solved in conjunction with the distribution (60). We illustrate this application below.

8. The Dupuit and Dietz Approximations

Under the full segregation assumption, consider first the case in which the displaced fluid is 'air' (switch for a moment to subscript a) so that we may take $\mu_{\rm a} \ll \mu_{\rm w}$ and $\Pi = $ const. This conveniently eliminates the last terms in (63) and (64). Expression (60) can be used to evaluate all the integrals. For example, we have

$$\frac{\lambda_{\rm w}\bar{\rho}_{\rm w} + \lambda_{\rm a}\bar{\rho}_{\rm a}}{\lambda_{\rm T}} = \begin{cases} \bar{\rho}_{\rm a}; & h < y < 1, \\ \bar{\rho}_{\rm w}; & 0 < y < h. \end{cases} \tag{65}$$

Hence,

$$\int_1^y [] \, dy' = \begin{cases} (y-1)\bar{\rho}_a; & h < y < 1, \\ (h-1)\bar{\rho}_a + (y-h)\bar{\rho}_w; & 0 < y < h, \end{cases} \tag{66}$$

thus,

$$\frac{\partial}{\partial x}\int_1^y [] \, dy' = \begin{cases} 0 & h < y < 1, \\ (\bar{\rho}_a - \bar{\rho}_w)\frac{\partial h}{\partial x}; & 0 < y < h. \end{cases} \tag{67}$$

Proceeding similarly with the evaluation of the other integrals the final result is obtained

$$\phi\frac{\partial h}{\partial t} = N_G'\frac{\partial}{\partial x}\left[\int_0^h \kappa_H(x, y)\, dy \cdot \frac{\partial h}{\partial x}\right], \tag{68}$$

where $N_G' = N_G/M$ and M is the ratio of the mobilities of the displacing to the displaced phase, (here $M = (\mu_w k_{ra}(S_{wr})/(\mu_a k_{rw}(1 - S_{ar})))$. This is the standard, non-linear diffusion equation used in the water infiltration literature, where water infiltrates an unsaturated porous medium. It usually arises under the so-called Dupuit approximation where $\kappa_H = 1$ (see Bear, 1972). Here, it was derived explicitly and rigorously.

In the more general case, where the displaced fluid is viscous, a similar approach applies. The evaluation of the various integrals is much simplified, if generalized functions (like step and delta functions and their derivatives) are used. For instance, we can take

$$\lambda_T = H(y-h) + MH(h-y), \tag{69}$$

so that the integral in (62) is expressed in the compact form $\bar{\rho}_o(y-1)H(y-h) + (\bar{\rho}_o(h-1) + \bar{\rho}_w(y-h))H(h-y)$, etc. The properties of the generalized functions needed are $H'(z) = \delta(z)$ and $z\delta'(z) = -\delta(z)$, where $\delta(z)$ is the delta function of z. Without going into the considerable details, we shall only present the final results. The total mass balance yields

$$N_G\frac{\partial}{\partial x}\left[\int_0^h \kappa_H \, dy \cdot \frac{\partial h}{\partial x}\right] - \frac{\partial}{\partial x}\left[\left(\int_0^h \kappa_H \, dy + \frac{1}{M}\int_h^1 \kappa_H \, dy\right) \cdot \frac{\partial \Pi}{\partial x}\right] = 0, \tag{70}$$

while the ‘water’ mass balance becomes

$$\phi\frac{\partial h}{\partial t} + MN_G\frac{\partial}{\partial x}\left[\int_0^h \kappa_H \, dy \cdot \frac{\partial h}{\partial x}\right] - M\frac{\partial}{\partial x}\left[\int_0^h \kappa_H \, dy \cdot \frac{\partial \Pi}{\partial x}\right] = 0. \tag{71}$$

Here, $M = (\mu_o k_{rw}(1-S_{or})/(\mu_w k_{ro}(S_{wr}))$. We can integrate (70) once with respect to x to get

$$N_G \int_0^h \kappa_H \, dy \cdot \frac{\partial h}{\partial x} - \left(\int_0^h \kappa_H \, dy + \frac{1}{M} \int_h^1 \kappa_H \, dy \right) \frac{\partial \Pi}{\partial x} = C, \tag{72}$$

where $C = 1$, without loss, and then eliminate $\partial \Pi / \partial x$ between (71) and (72) to obtain the final equation

$$\phi \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left[\frac{\int_0^h \kappa_H \, dy}{\int_0^h \kappa_H \, dy + \frac{1}{M} \int_h^1 \kappa_H \, dy} \right]$$

$$= N_G \frac{\partial}{\partial x} \left[\frac{\int_0^h \kappa_H \, dy \int_h^1 \kappa_H \, dy}{\int_0^h \kappa_H \, dy + \frac{1}{M} \int_h^1 \kappa_H \, dy} \frac{\partial h}{\partial x} \right] \tag{73}$$

This is the generalization of the well known parallel flow approximation, derived by Dietz (1954) for the case of constant permeability. The process can be approximated as an 1-D displacement with equivalent 'saturation' $\bar{S} = S_{wr} + h(1 - S_{or} - S_{wr})$, equivalent relative permeabilities given by $\bar{k}_{rw} = k_{rw}(1 - S_{or}) \int_0^h \kappa_H \, dy$ and $\bar{k}_{ro} = k_{ro}(S_{or}) \int_h^1 \kappa_H \, dy$, and a straight line 'capillary pressure' with dispersion coefficient N_G (Lake, 1989). The pseudofunctions become straight-lines when the permeability is constant (Dietz, 1953). In any other case, they are non-linear functions of $\bar{S}$, and, in fact, they may also vary with position x. Clearly, because of the integral form, the order of the layers affects drastically the shape of the pseudofunctions (Lake *et al.*, 1990).

9. Capillarity–Gravity Segregation

When the dimensionless ratio in (33) is not large, capillarity is also of importance. The equivalent of (58) is now

$$v_w = \epsilon \left[w f_w(S) + \kappa_V \lambda_o f_w \left(\frac{N_{CT}}{\epsilon} \frac{\partial \Pi_C}{\partial y} - N_G R_L^2 \right) \right] \tag{74}$$

Following the same arguments as before, when N_{CT} and N_G are not small, we are led to the capillary equilibrium condition

$$N_{CT} \frac{\partial \Pi_c}{\partial y} = \frac{\delta}{\epsilon} N_G, \tag{75}$$

which can be integrated to yield

$$\Pi_c = \Pi_{co} + y \frac{\delta N_G}{\epsilon N_{CT}}, \tag{76}$$

where Π_{co} is the capillary pressure at $y = 0$. If Π_c is assumed to be a single function of saturation, the above determines the vertical distribution of saturation, given its value $S_o(x, t)$ at $y = 0$.

Again, an integral approach is needed. However, the problem here is quite simpler. Indeed, it can be readily shown that the 'water' pressure is hydrostatic, such that

$$p_w = -N_G \bar{\rho}_w (y - 1) + \Pi(x, t), \tag{77}$$

hence, the total flow rate u is described by expression (9) (assuming negligible capillarity along the x-direction). Thus, we can use directly Equation (61) to get

$$\phi \frac{\partial}{\partial t} \int_0^1 S \, \mathrm{d}y + \frac{\partial}{\partial x} \left\{ \frac{\int_0^1 \kappa_H \lambda_w(S) \, \mathrm{d}y}{\int_0^1 \kappa_H \lambda_T(S) \, \mathrm{d}y} \right\} = 0. \tag{78}$$

Equations (76) and (78) completely specify the problem. For example, all integrals in (78) can be explicitly calculated from the solution of (76), in terms of $S_0(x, t)$, the evolution of which can be obtained from (78). The result would be an equivalent to the Buckley–Leverett equation, this time in terms of $S_0(x, t)$. Whether, however, appropriate pseudofunctions can be defined for this problem is unclear.

Finally, if capillarity predominates in Equation (76), the saturation profiles along the vertical direction follow the capillary heterogeneity. Specifically, if k_V does not vary greatly with y, then the saturation profile is flat, $S = S_0(x, t)$, and Equation (78) becomes the Buckley–Leverett equation in the absence of capillarity (12).

10. Summary

In this paper, using a formal approach, the various manifestations of Vertical Equilibrium were derived. Key to the analysis was the identification of the parameter R_L as the proper asymptotic variable and the development of a formal asymptotic method in terms of $1/R_L^2$. The analysis confirms previously known numerical results and, for the first time, it rigorously establishes their validity in the limit of large R_L^2. Because the condition is geometric-structural it applies independently of flow and process parameters, hence it can be used regardless of the particular displacement process. Due to the ensuing reduction in the dimensionality of the problem, the process description is facilitated significantly. To our knowledge, this is one of the few cases in multiphase flow in porous media where such a reduction is possible.

The formal approach presented has many advantages, as it allows for a plethora of special cases to be readily derived. An analysis along these lines is also possible for any EOR process (Yortsos, 1992). In all cases, heterogeneity is the key variable and it is only the relative interplay of viscous to other forces that dictates the various approximations. Viscous, gravity and capillary effects were considered in the case

TABLE I.

Viscous	Gravity	Capillarity	Conditions	Equations	Comments
Strong	Negligible	Negligible	$N_G \ll \frac{1}{R_L^2}$, $N_{CT} \ll \epsilon$	(11) and (17)	Viscous Fingering
Strong	Moderate	Moderate	$N_G \sim \frac{1}{R_L^2}$, $N_{CT} \sim \epsilon$	(32) and (35)	Viscous Fingering with Dispersion
Moderate	Strong	Negligible	$N_G \sim 1$, $N_{CT} \ll \frac{\delta}{\epsilon}$	(68) and (73)	Gravity Tonguing (Dietz)
Moderate	Strong	Moderate	$N_G \sim 1$, $N_{CT} \sim \frac{\delta}{\epsilon}$	(78)	Gravity–Capillary Equilibrium
Moderate	Strong	Strong	$N_G \sim 1$, $N_{CT} \gg \frac{\delta}{\epsilon}$	(12)	Capillary Equilibrium (Buckley–Leverett)

of immiscible displacement. The classification of the various regimes depends on the relative importance of these forces, as described by the dimensionless parameters. A summary is given in Table I. A similar table can be constructed for miscible displacement, where the role of capillarity is played by transverse dispersion. We should point out that the above can be extended to include a second 'horizontal' dimension by appropriate modifications (Yortsos, 1992).

Many of the results obtained here are new, in the sense that they extend previous approximate analyses. Thus, Equations (11) and (17) are an extension of Zapata and Lake (1981), Equations (32) and (34) extend the work of Yokoyama and Lake (1981), Equations (40) and (53) are extensions of Lake and Hirasaki (1981), and Equations (68), (73) and (78) extend the Dupuit and Dietz approximations. In addition to their formal aspects, our results also offer insight on effects of viscous cross flow, as in (11), (18)–(20), and they suggest directly the relevant pseudofunctions for each case. Finally, it is the hope that a more detailed analysis of equations (11) and (40) would lead to improved approximations and to rigorously establishing the validity of the various empirical viscous fingering models (such as Koval, 1963, Todd and Longstaff, 1979 and Fayers, 1984).

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