Transport in Porous Media 7: 283–304, 1992

Self-Similar Solution for Deep-Penetrating Hydraulic Fracture Propagation

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(Received: 11 April 1990)

Abstract. The propagation of a vertical hydraulic fracture of a constant height driven by a viscous fluid injected into a crack under constant pressure, is considered. The fracture is assumed to be rectangular, symmetric with respect to the well, and highly elongated in the horizontal direction (the Perkins and Kern model). The fracturing fluid viscosity is assumed to be different from the stratum saturating fluid viscosity, and the stratum fluid displacement by a fracturing fluid in a porous medium is assumed to be piston-like. The compressibility of the fracturing fluid is neglected. The stratum fluid motion is governed by the equation of transient seepage flow through a porous medium.

A self-similar solution to the problem is constructed under the assumption of the quasi-steady character of the fracturing fluid flow in a crack and in a stratum and of a locally one-dimensional character of fluid-loss through the crack surfaces. Crack propagation under a constant injection pressure is characterized by a variation of the crack size l in time t according to the law $l(t) = l_0(1 + At)^{1/4}$, where the constant A is the eigenvalue of the problem. In this case, the crack volume is $V \sim l$, the seepage volume of fracturing fluid $V_f \sim l^3$, and the flow rate of a fluid injected into a crack is $Q_0 \sim l^{-1}$.

Key words. Hydraulic fracturing, transient seepage flow, self-similar solution.

1. Statement of the Problem

The principles of hydraulic fracturing theory were laid down in [26, 3, 4]. These papers have studied the mechanism and basic laws of rock destruction during hydraulic fracturing and proposed the models for a penny-shaped horizontal crack and for a vertical hydraulic fracturing of a large height, which have been widely applied [7, 10, 12, 27, 28]. Both exact and approximate solutions to some associated elastic-hydrodynamic problems of hydraulic fracturing were later obtained for these models [1, 2, 11, 14, 19, 21, 22, 24].

An alternative model of a vertical hydraulic fracture was proposed in [16, 18] and further developed in [6, 8, 13, 15, 17, 20, 23] for calculating the processes of massive or deep-penetrating hydraulic fracturing. It is this model which is applied below [9].

Suppose that in the hydraulic fracturing of a stratum of thickness $2h$, a vertical crack is formed of constant height $2H$ and large length $2l \gg 2H \leqslant 2h$. The crack is symmetric with respect to the well (Figure 1). Then the pressure in a crack p at any time may be assumed to be constant at an arbitrary vertical cross-section of a crack $x = \text{const}$, i.e. $p = p(x, t)$. By virtue of the elongated shape of the crack, one

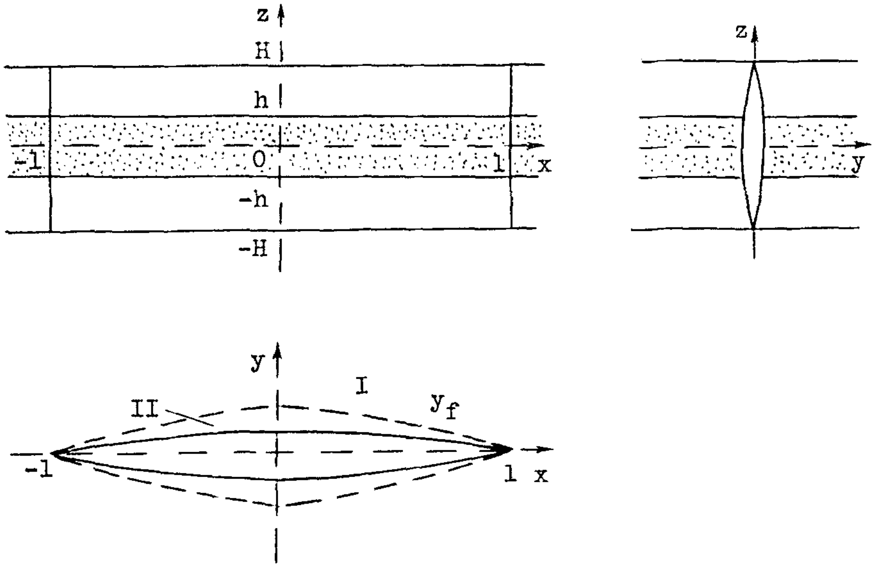

Fig. 1. Perkins and Kern fracture.

can assume the plane cross-section hypothesis according to which at any cross-section $x = \text{const}$, the crack width $2w$ is determined by the solution of a plane problem of elasticity theory and is proportional to the difference between the local value of pressure in the crack $p(x, t)$ and a confining horizontal stress $\sigma = \text{const}$. In this case, at the crack tips $x = \pm l(t)$, one can specify the condition of a smooth contact of surfaces: $p = \sigma$ (the analogue of C. A. Khristianovich's condition [26, 3, 4]). The formation permeability k is assumed to be small enough, the fracturing fluid viscosity μ_f to be different from the formation fluid viscosity μ_0, the displacement of one fluid by another to be piston-like, and the zone of fracturing fluid penetration into a stratum to be thin and adjacent to the crack surfaces intersecting the pay zone. Then the flow inside the penetration zone can be considered to be quasi-stationary, one-dimensional, and directed perpendicular to the crack surface. The flow outside the penetration zone is supposed to be planar and obeying the transient seepage flow equation.

Within the framework of above assumptions, the process of hydraulic fracturing is governed by the following system of equations:

$$\frac{\partial \langle w \rangle}{\partial t} + \frac{\partial \langle wv \rangle}{\partial x} = -2hq_L, \qquad \langle f \rangle = \int_{-H}^{H} f(z)\, \mathrm{d}z, \tag{1.1}$$

$$v(x, z, t) = -\frac{w^2}{3\mu_f} \frac{\partial p}{\partial x}, \tag{1.2}$$

$$w(x, z, t) = \frac{2(1-v^2)}{E} \sqrt{H^2 - z^2}\, [p(x, t) - \sigma], \tag{1.3}$$

$$|x| \leqslant l(t), \qquad |z| \leqslant H,$$

$$u = -\frac{k}{\mu_f}\frac{\partial p_{II}}{\partial y}, \qquad \frac{\partial u}{\partial y} = 0, \qquad w \leqslant |y| \leqslant y_f, \tag{1.4}$$

$$u = -\frac{k}{\mu_0}\nabla p_I, \qquad \frac{\partial p_I}{\partial t} = \kappa\left(\frac{\partial^2 p_I}{\partial x^2} + \frac{\partial^2 p_I}{\partial y^2}\right), \tag{1.5}$$

$$|x| \leqslant l(t), \qquad |y| \geqslant y_f; \qquad |x| \geqslant l(t), \qquad |y| \geqslant 0,$$

$$\frac{\partial}{\partial t} y_f(x, t) = \frac{1}{m} u(x, y_f, t). \tag{1.6}$$

Here v and u are velocities of fluid flow in a crack and through a porous medium in a stratum, $q_L(x, t)$ is the seepage velocity of flow through crack surfaces, E and ν are the Young modulus and the Poisson ratio of the formation and surrounding rock, $y_f(x, t)$ is the depth of the fracturing fluid penetration into a stratum, p_I and p_{II} are pressures in a stratum inside the region of formation fluid flow through porous medium (I) and inside the penetration zone (II), κ is the formation piezoconductivity, and m is porosity.

Equation (1.1) is the equation of mass conservation in a crack, (1.2) is the Boussinesq formula for a laminar viscous fluid flow in a narrow channel, (1.3) is the solution of a plane elasticity problem on a pressurized crack for the homogeneous loading of its surfaces, (1.4) and (1.5) are Darcy's laws of flow through a porous medium and the continuity equations for a fracturing fluid flow inside the penetration zone and for the formation fluid flow outside the latter, and (1.6) is the kinematic equation of a penetration zone boundary.

In order that the system (1.1)–(1.6) is closed, we have the condition of smooth contact of crack surfaces

$$p(\pm l(t), t) = \sigma \tag{1.7}$$

and conditions of continuity of pressures and flow rates at crack surfaces $y = w(x, t)$ and at the penetration front $y = y_f(x, t)$

$$p(x, t) = p_{II}(x, w, t), \qquad q_L(x, t) = u(x, w, t), \tag{1.8}$$

$$p_{II}(x, y_f, t) = p_I(x, y_f, t), \qquad u(x, y_f, t) = -\frac{k}{\mu_0}\frac{\partial p_I}{\partial y}\bigg|_{y=y_f}. \tag{1.9}$$

Besides, one should specify the initial conditions – the pore pressure p^0, the initial crack length l_0, and the initial depth of fracturing fluid penetration into a stratum $y_f^0(x)$ – as well as the regime of fluid injection into a crack. Thus, we have

$$p_I(x, y, 0) = p^0, \qquad l(0) = l_0, \qquad y_f(x, 0) = y_f^0(x), \qquad p(0, t) = p_0, \tag{1.10}$$

where p_0 is the well-bore pressure ($p_0 > \sigma$).

By virtue of the above assumptions, $w \ll l_0$ and $y_f \ll l_0$. Therefore, we shall hereafter use the following natural simplifications of the problem. In calculating the flow in the penetration zone II, we shall extrapolate the conditions at the crack surfaces $|y| = w$ into the plane $y = 0$, from which we shall further measure the penetration zone depth $y_f(x, t)$. In the same manner, in solving the outer problem in region I, the penetration zone boundary $y = y_f$ may be displaced by the plane $y = 0$.

Substituting (1.2) and (1.3) into (1.1), one obtains the equation for the pressure distribution in a crack

$$a\frac{\partial}{\partial t}(p-\sigma) - \frac{\partial^2}{\partial x^2}(p-\sigma)^4 + \beta q_L = 0,$$

$$|x| \leqslant l(t), \qquad a = \frac{4\mu_f E^2}{(1-\nu^2)^2 H^2}, \qquad b = \frac{8h\mu_f E^3}{\pi(1-\nu^2)^3 H^4}, \tag{1.11}$$

that contains an unknown fluid-loss velocity $q_L(x, t)$.

It follows from Equations (1.4), that in the penetration layer, the pressure $p_{\mathrm{II}}(x, y, t)$ is a linear function of the coordinate y. As a result, the flow velocity in a layer $u(x, t)$ does not depend on y and coincides with $q_L(x, t)$ by virtue of (1.8). Denoting by $p_f(x, t)$ the pressure at the penetration zone boundary $y = y_f$, one gets

$$\frac{\partial}{\partial t} y_f(x, t) = \frac{1}{m} q_L(x, t), \tag{1.12}$$

$$q_L(x, t) = \frac{k}{\mu_f} \frac{p(x, t) - p_f(x, t)}{y_f(x, t)}, \tag{1.13}$$

$$p_{\mathrm{I}}(x, 0, t) = p_f(x, t), \qquad \left.\frac{\partial p_{\mathrm{I}}}{\partial y}\right|_{y=0} = -\frac{\mu_0}{k} q_L(x, t). \tag{1.14}$$

Thus, we have four equations (1.11)–(1.14) for determining the unknown functions $p(x, t)$, $q_L(x, t)$, $y_f(x, t)$, $p_f(x, t)$. Function $p_{\mathrm{I}}(x, y, t)$ in these equations must satisfy the 'piezoconductivity' equation (1.5), and condition (1.7) serves for finding the crack length $l(t)$. These relations, together with conditions (1.10), compose a complete system of equations for solving the problem stated.

2. Transition to Dimensionless Variables

Now we shall introduce new variables according to formulae

$$\tau = \frac{t}{t_0}, \qquad X = \frac{x}{l(t)}, \qquad Y = \frac{y}{y_0 f(\tau)}, \qquad L(\tau) = \frac{l(t)}{l_0},$$

$$P(X, \tau) = \frac{p}{\sigma}, \qquad P_{\mathrm{I}}(X, Y, \tau) = \frac{p_{\mathrm{I}}}{\sigma}, \qquad P_f(X, \tau) = \frac{p_f}{\sigma}, \qquad Q_L(X, \tau) = \frac{q_L}{q_{L0}\varphi(\tau)}, \tag{2.1}$$

$$Y_f(X, \tau) = \frac{y_f}{y_0 f(\tau)}, \qquad P_0 = \frac{p_0}{\sigma}, \qquad P^0 = \frac{p^0}{\sigma}, \qquad \mu = \frac{\mu_f}{\mu_0},$$

where t_0, y_0, q_L are scaling factors, and $f(\tau)$ and $\varphi(\tau)$ are unknown functions to be determined.

Using the following expressions for derivatives

$$\frac{\partial}{\partial t}=\frac{1}{t_0}\left(\frac{\partial}{\partial \tau}-\frac{\dot{L}}{L}X\frac{\partial}{\partial X}-\frac{\dot{f}}{f}Y\frac{\partial}{\partial Y}\right),$$

$$\frac{\partial^n}{\partial x^n}=(l_0 L)^{-n}\frac{\partial^n}{\partial X^n},\qquad \frac{\partial^n}{\partial y^n}=(y_0 f)^{-n}\frac{\partial^n}{\partial Y^n},$$

where the dot means differentiation with respect to dimensionless time τ, one gets, after transformations, the system of equations

$$-\frac{\varepsilon}{\varphi(\tau)}\left(\frac{\partial}{\partial \tau}-\frac{\dot{L}}{L}X\frac{\partial}{\partial X}\right)(P-1)+\frac{L^{-2}(\tau)}{\varphi(\tau)}\frac{\partial^2}{\partial X^2}(P-1)^4=Q_L,$$

$$4\delta\left(\frac{\partial}{\partial \tau}-\frac{\dot{L}}{L}X\frac{\partial}{\partial X}\right)[f(\tau)Y_f]=\varphi(\tau)Q_L,$$

$$\varphi(\tau)Q_L=\frac{1}{f(\tau)}\frac{P-P_f}{Y_f}, \tag{2.2}$$

$$P_{\mathrm{I}}(X,0,\tau)=P_f(X,\tau),\qquad \frac{\mu}{f(\tau)}\frac{\partial P_{\mathrm{I}}}{\partial Y}\bigg|_{Y=0}=-\varphi(\tau)Q_L,$$

$$4f^2(\tau)\left(\frac{\partial P_{\mathrm{I}}}{\partial \tau}-\frac{\dot{L}}{L}X\frac{\partial P_{\mathrm{I}}}{\partial X}-\frac{\dot{f}}{f}Y\frac{\partial P_{\mathrm{I}}}{\partial Y}\right)=\frac{\partial^2 P_{\mathrm{I}}}{\partial Y^2}+\left[\frac{y_0 f(\tau)}{l_0 L(\tau)}\right]^2\frac{\partial^2 P_{\mathrm{I}}}{\partial X^2},$$

where

$$q_{L0}=\frac{\pi(1-\nu^2)^3H^4\sigma^4}{8hl_0^2\mu_f E^3},\qquad y_0=\frac{k\sigma}{\mu_f q_{L_0}},\qquad t_0=\frac{y_0^2}{4\kappa},$$

$$\varepsilon=\frac{\mu_f}{\sigma t_0}\left[\frac{2El_0}{(1-\nu^2)H\sigma}\right]^2,\qquad \delta=\frac{m\kappa\mu_0}{k\sigma}\mu. \tag{2.3}$$

The condition of a smooth contact of crack surfaces (1.7) takes the form

$$P(1,\tau)=1 \tag{2.4}$$

and from (1.10), one gets

$$P_{\mathrm{I}}(X,Y,0)=P^0,\qquad L(0)=1,\qquad Y_f(X,0)=Y_f^0(X),\qquad P(0,\tau)=P_0, \tag{2.5}$$

where $Y_f^0=y_f^0/[y_0 f(0)]$.

3. Quasi-Steady Crack Propagation Regime

System (2.2) includes a small parameter $(y_0/l_0)^2$ that characterizes the ratio of the thickness of a zone of fracturing fluid penetration into a stratum to the crack length. Letting this parameter be zero, one obtains the system of equations

$$-\varepsilon\left(L^2\frac{\partial}{\partial\tau}-L\dot{L}X\frac{\partial}{\partial X}\right)(P-1)+\frac{\partial^2}{\partial X^2}(P-1)^4=L^2\varphi(\tau)Q_L, \tag{3.1a}$$

$$4\delta\left(\frac{\partial}{\partial\tau}-\frac{\dot{L}}{L}X\frac{\partial}{\partial X}\right)[f(\tau)Y_f]=\varphi(\tau)Q_L, \tag{3.1b}$$

$$P-P_f=\varphi(\tau)f(\tau)Y_fQ_L, \tag{3.1c}$$

$$P_I(X,0,\tau)=P_f(X,\tau),\qquad \mu\frac{\partial P_I}{\partial Y}\bigg|_{y=0}=-\varphi(\tau)f(\tau)Q_L, \tag{3.1d}$$

$$4f^2(\tau)\left(\frac{\partial P_I}{\partial\tau}-\frac{\dot{L}}{L}X\frac{\partial P_I}{\partial X}-\frac{\dot{f}}{f}Y\frac{\partial P_I}{\partial Y}\right)=\frac{\partial^2 P_I}{\partial Y^2}. \tag{3.1e}$$

We shall seek its stationary solutions $P(X)$, $Q_L(X)$, $P_f(X)$, $Y_f(X)$, $P_I(X, Y)$. If functions $f(\tau)$ and $\varphi(\tau)$ are chosen in the form of

$$f(\tau)=L^2(\tau),\qquad \varphi(\tau)=L^{-2}(\tau) \tag{3.2}$$

and the function $L(\tau)$ is chosen as the solution to the equation

$$4\frac{f}{\varphi}\frac{\dot{L}}{L}\equiv 4f^2\frac{\dot{L}}{L}\equiv 4L^3\dot{L}=\alpha=\text{const}, \tag{3.3}$$

then Equations (3.1b)–(3.1e) will not be time-dependent for $\partial/\partial\tau=0$.

In this case, Equation (3.1a) takes the form

$$\frac{\varepsilon\alpha}{4L^2}X\frac{\partial}{\partial X}(P-1)+\frac{\partial^2}{\partial X^2}(P-1)^4=Q_L.$$

The time-dependence conserves in this equation due to the $\varepsilon\alpha/(4L^2)$ coefficient in the first term that takes into account the variation of a crack volume with changing pressure in a crack. In cases where the fluid-loss velocity Q_L is large, the contribution of this term into the redistribution of fluid flow in a crack becomes small ($\varepsilon\alpha \ll 1$) and can be neglected. In other words, the time of pressure redistribution in a crack becomes small when compared to the characteristic hydraulic fracturing time. Therefore, for determining the pressure in a crack, one can make use of the stationary flow continuity equation ($\varepsilon\alpha = 0$).

Equation (3.3) is explicitly integrationed and with due account of the initial condition $L(0) = 1$ one finds

$$L(\tau)=(1+\alpha\tau)^{1/4}. \tag{3.4}$$

Parameter α along with $P(X)$, $Q_L(X)$, $P_f(X)$, $Y_f(X)$, and $P_I(X, Y)$ functions, are to be determined from the system of equations

$$\frac{d^2}{dX^2}(P-1)^4=Q_L, \tag{3.5}$$

$$\delta\alpha\left(2Y_f - X\frac{\mathrm{d}Y_f}{\mathrm{d}X}\right) = Q_L, \tag{3.6}$$

$$P_f = P - Y_f Q_L, \tag{3.7}$$

$$P_{\mathrm{I}}(X, 0) = P_f, \qquad \mu\frac{\partial P_{\mathrm{I}}}{\partial Y}\bigg|_{Y=0} = -Q_L, \tag{3.8}$$

$$-\alpha\left(X\frac{\partial P_{\mathrm{I}}}{\partial X} + 2Y\frac{\partial P_{\mathrm{I}}}{\partial Y}\right) = \frac{\partial^2 P_{\mathrm{I}}}{\partial Y^2}, \tag{3.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}X}(P-1)^4 = 0 \quad (|X| = 1). \tag{3.10}$$

Here, the additional relation (3.10) expresses the nonpenetration condition at the crack tips – the equality of the crack propagation rate to the rate of its filling with a fracturing fluid averaged over the vertical cross-section of a crack. It is this condition which is used to determining the parameter α.

Functions $P(X)$ and $P_{\mathrm{I}}(X, Y)$, according to (2.4) and (2.5), must satisfy the conditions

$$P(1) = 1, \qquad P(0) = P_0 \tag{3.11}$$

and the condition that the flow is unperturbed at infinity. The statement of the latter condition in the given case does not occur to be trivial, however.

Indeed, after lowering the order of a piezoconductivity equation in variable X, that implies the transition to the scheme of locally one-dimensional seepage flow, its solution should be sought in a half-band $|X| \leqslant 1$, $Y \geqslant 0$ rather than in a half-plane $Y \geqslant 0$. In this case, it is sufficient to state only one boundary condition in variable X. In the initial problem, however, there were two conditions of this type for a complete piezoconductivity equation; namely, the conditions that $P_{\mathrm{I}}(X, Y)$ is an even function of X and that the flow is unperturbed at $|X| \to \infty$. The question of which of these conditions should be conserved in a given case is solved as follows. Equation (3.9), after replacement $\Theta = -\ln|X|$, is reduced to the equation of heat conductivity with convection

$$\alpha\left(\frac{\partial P_{\mathrm{I}}}{\partial \Theta} - 2Y\frac{\partial P_{\mathrm{I}}}{\partial Y}\right) = \frac{\partial^2 P_{\mathrm{I}}}{\partial Y^2} \quad (0 \leqslant \Theta < \infty),$$

where Θ is a false time.

This equation requires an 'initial' condition at $\Theta = 0$ and, hence, the boundary conditions for Equation (3.9) should be stated at the boundaries of a half-band $|X| = 1$. Since (3.9) is a first-order equation in X, it follows that the problem should be solved independently in regions $0 \leqslant X \leqslant 1$ and $-1 \leqslant X \leqslant 0$. By virtue of the fact that the initial problem is even in X, it is sufficient to find its solution in the half-band $0 \leqslant X \leqslant 1$, $Y \geqslant 0$ and then to continue the solution into the half-band $-1 \leqslant X \leqslant 0$, $Y \geqslant 0$ symmetrically about the Y axis.

As a result, one obtains the boundary condition

$$P_{\mathrm{I}}(X, Y) = P^0 \quad (|X| = 1,\ Y \geqslant 0;\ |X| < 1,\ Y = \infty). \tag{3.12}$$

The condition that the flow is unperturbed at half-band boundaries $|X| = 1$, has a simple physical meaning. Namely, by virtue of the one-dimensional character of the fluid loss, the flow perturbations are absent before the crack tips ($|X| > 1$). Therefore, the perturbations are also absent at the boundaries between the perturbed and unperturbed regions – at straight lines $Y \geqslant 0$, $X = \pm 1$. Thereby, the continuity of the solution of the problem is provided.

Condition (3.12) is closely related to the statement of a boundary condition for Equation (3.6) with respect to $Y_f(X)$. In order that the fracturing fluid flow be consistent with (3.12) in the penetration zone, the penetration depth $Y_f(X)$ should be zero at the crack tips, i.e.

$$Y_f(\pm 1) = 0. \tag{3.13}$$

Finally, the initial condition $Y_f(X, 0) = Y_f^0(X)$ cannot be satisfied, generally speaking, since function $Y_f^0(X)$ must coincide with the solution of problem (3.5)–(3.13) and, therefore, it cannot be specified in an arbitrary manner. In other words, the type of solution of the initial problem that is sought does exist at some special initial condition, namely, at $y_f^0(x) = y_0 f_0 Y_f(x/l_0) = y_0 Y_f(x/l_0)$, where the function $Y_f(X)$ is determined from (3.5)–(3.13).

4. Construction of a Self-Similar Solution

Problem (3.5)–(3.13) is nonlinear and, therefore, we shall construct its solution by the iteration method. For this purpose, we shall specify an initial approximation for function P_f and find the solution to Equation (3.9) satisfying the first condition of (3.8). Then we shall express Q_L from the second condition of (3.8). Further, we shall calculate α from condition (3.10), find Y_f from Equation (3.6) and, finally, correct the value of p_f using Equation (3.7). The iterations continue until we achieve the convergency of the iteration process.

So, we shall first consider Equation (3.9). It is convenient to seek its solution within a half-band $0 \leqslant X \leqslant 1$, $Y \geqslant 0$ in new variables

$$\xi = X^2, \qquad \eta = \sqrt{2\alpha}\, Y, \qquad \phi_{\mathrm{I}} = P_{\mathrm{I}} - P^0, \qquad \phi_f = P_f - P^0. \tag{4.1}$$

Then function $\phi_{\mathrm{I}}(\xi, \eta)$ satisfies the equation

$$-\left(\xi \frac{\partial \phi_{\mathrm{I}}}{\partial \xi} + \eta \frac{\partial \phi_{\mathrm{I}}}{\partial \eta}\right) = \frac{\partial^2 \phi_{\mathrm{I}}}{\partial \eta^2} \tag{4.2}$$

whose general solution can be found by the separation of variables method. Substituting into (4.2)

$$\phi_{\mathrm{I}}(\xi, \eta) = R(\xi) F(\eta) \tag{4.3}$$

one obtains two equations

$$\xi \frac{\mathrm{d}R}{\mathrm{d}\xi} = \lambda R, \qquad \frac{\mathrm{d}^2F}{\mathrm{d}\eta^2} + \eta \frac{\mathrm{d}F}{\mathrm{d}\eta} = -\lambda F, \quad \lambda = \text{const.} \tag{4.4}$$

We shall make use of their solutions for $\lambda = 0$

$$R_0 = c_0 = \text{const}, \qquad F_0 = \operatorname{erfc}(2^{-1/2}\eta) \tag{4.5}$$

and for $\lambda = n + 1$, $n \geqslant 0$

$$R_{n+1} = \xi^{n+1}, \qquad F_{n+1} = \exp(-2^{-1}\eta^2)H_n(-2^{1/2}\eta), \tag{4.6}$$

where $H_n(\zeta)$ are the Ermith polynomials orthogonal on $-\infty < \zeta < \infty$ with weight $\exp(-\zeta^2)$.

The system of functions $F_{n+1}(\eta)$, $n \geqslant 0$ is complete on $-\infty < \eta < \infty$. However, the system of functions $R_{n+1} = \xi^{n+1}$, associated with $F_{n+1}(\xi)$, is not complete in the interval $0 \leqslant \xi \leqslant 1$, because it does not contain any constant. This circumstance does not allow us to construct the solution by its expansion over the products of functions (4.6) $R_{n+1}(\xi)F_{n+1}(\eta)$ and requires the attraction of the $R_0F_0(\eta)$ solution of (4.5). As a result, one obtains the general solution from Equation (4.2) in the form

$$\phi_{\mathrm{I}}(\xi, \eta) = c_0 \operatorname{erfc}(2^{-1/2}\eta) + \exp\left(-\frac{\eta^2}{2}\right) \sum_{n=0}^{\infty} c_{n+1}\xi^{n+1}H_n(2^{-1/2}\eta) \tag{4.7}$$

which must be found in the half-band $0 \leqslant \xi \leqslant 1$, $\eta \geqslant 0$ with the boundary conditions following from (3.8), (3.12)

$$\phi_{\mathrm{I}}(\xi, 0) = \phi_f(\xi), \qquad \phi_{\mathrm{I}}(1, \eta) = 0. \tag{4.8}$$

The specificity of expansion (4.7) is in the fact that when the interval narrows down to a half-line, the property of the Ermith polynomial orthogonality on $-\infty < \zeta < \infty$ with weight $\exp(-\zeta^2)$ separately conserves for polynomials of even and odd numbers only. Besides, function $F_0(\eta)$ is not orthogonal to any function $F_{n+1}(\eta)$, $n \geqslant 0$. Nevertheless, this fact does not prevent the determination of $c_0, c_1, \ldots$ coefficients of expansion (4.7).

We shall begin by using the first condition of (4.8). For $\eta = 0$, one obtains from (4.7)

$$\phi_f = c_0 + \sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1},$$

$$a_{2n+1} = c_{2n+1}H_{2n}(0) = (-1)^n \frac{(2n)!}{n!} c_{2n+1}. \tag{4.9}$$

If the function $\Psi(\xi) = \phi_f(\xi) - c_0$ is known, then coefficients a_{2n+1} and, hence, c_{2n+1}, are found by expanding $\Psi(\xi)$ in the $0 \leqslant \xi \leqslant 1$ interval over the system of functions ξ^{2n+1}, $n = 0, 1, \ldots$, which is complete on this segment.

Coefficients a_{2n+1} have been sought, practically, in two stages. First, the collocation method was used for expanding the $\Psi(\xi)$ function over odd Chebyshev polynomials which are orthogonal in the $0 \leqslant \xi \leqslant 1$ interval with weight $(1-\xi^2)^{-1/2}$. In so doing, the collocation points were chosen equidistant in the $0 \leqslant \beta \leqslant \pi/2$ interval, where $\beta = \arcsin \xi$. Then, the coefficients at ξ^{2n+1} were recalculated in terms of the coefficients of expansion over Chebyshev polynomials.

Once the c_{2n+1} coefficients are found, the second condition of (4.8) can be used for determining c_{2n} coefficients at even powers of ξ. We have

$$\exp\left(-\frac{\eta^2}{2}\right)\sum_{n=1}^{\infty} c_{2n}H_{2n-1}(2^{-1/2}\eta) = G(\eta),$$

$$G(\eta) = -c_0 \operatorname{erfc}(2^{-1/2}\eta) - \exp\left(-\frac{\eta^2}{2}\right)\sum_{n=0}^{\infty} c_{2n+1}H_{2n}(2^{-1/2}\eta), \tag{4.10}$$

where $G(\eta)$ is an already-known function.

Using the property of orthogonality of Ermith polynomials $H_{2n-1}(2^{-1/2}\eta)$, $n = 1, 2, \ldots$ on the half-line $\eta \geqslant 0$ with weight $\exp(-\frac{1}{2}\eta^2)$, the c_{2n} coefficients can be explicitly expressed in terms of c_0 and c_{2n+1}. For this purpose, it is sufficient to multiply (4.10) by $H_{2i-1}(2^{-1/2}\eta)$ and integrate the result over η from 0 to ∞. Using relations

$$\int_0^{\infty} H_{2n-1}\left(\frac{\eta}{\sqrt{2}}\right)H_{2i-1}\left(\frac{\eta}{\sqrt{2}}\right)\exp\left(-\frac{\eta^2}{2}\right)\mathrm{d}\eta = \begin{cases}\sqrt{2\pi}\, 2^{2(i-1)}(2i-1)!, & i=n,\\ 0, & i \neq n,\end{cases}$$

$$\int_0^{\infty} \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)H_{2i-1}\left(\frac{\eta}{\sqrt{2}}\right)\mathrm{d}\eta = (-1)^{i+1}\frac{(2i-1)!}{\sqrt{2}\, i!},$$

$$\int_c^{\infty} H_{2n}\left(\frac{\eta}{\sqrt{2}}\right)H_{2i-1}\left(\frac{\eta}{\sqrt{2}}\right)\exp\left(-\frac{\eta^2}{2}\right)\mathrm{d}\eta = \frac{\sqrt{2}\,(-1)^{i+n-1}(2i-1)!(2n)!}{(2i-2n-1)(i-1)!n!},$$

one gets

$$c_{2i} = \frac{(-1)^i b_{2i}}{\sqrt{\pi}\, 2^{2i-1}(i-1)!}, \qquad \beta_{2i} = \frac{c_0}{i} + \sum_{n=0}^{\infty}\frac{a_{2n+1}}{i-n-1/2}. \tag{4.11}$$

Now the unknown functions Q_L, P, Y_f can be expressed in terms of $c_0, c_1, c_2, \ldots$ coefficients.

Substituting (4.7) into the second condition of (3.8), and by taking (4.1) and (4.11) into account, one obtains

$$Q_L(\xi) = \mu\sqrt{\frac{\alpha}{\pi}}\left[2c_0 + \sum_{n=1}^{\infty}\frac{(2n-1)!!b_{2n}}{(2n-2)!!}\xi^{2n}\right]. \tag{4.12}$$

Upon integration and taking into account conditions (3.11), the substitution of (4.12) into (3.5) yields

$$P(\xi) = 1 + \left\{\left[(P_0-1)^4 - c_0\mu\sqrt{\frac{\alpha}{\pi}}\xi^{1/2}\right](1-\xi^2) - \right.$$
$$\left. -\mu\sqrt{\frac{\alpha}{\pi}}\xi^{1/2}\sum_{n=1}^{\infty}\frac{(2n-1)!!b_{2n}(1-\xi^{2n+1/2})}{(2n-2)!!(4n+1)(4n+2)}\right\}^{1/4}. \tag{4.13}$$

Finally, using Equation (3.6) and condition (3.13), one finds

$$Y_f(\xi) = \frac{\mu}{\delta\sqrt{\pi\alpha}}\left[c_0(1-\xi) + \frac{\xi}{2}\sum_{n=1}^{\infty}\frac{(2n-1)!!b_{2n}(1-\xi^{2n-1})}{(2n-2)!!(2n-1)}\right]. \tag{4.14}$$

The unknown constant α, that appears in expressions (4.12)–(4.14) is found from condition (3.10) which, upon transition to variable ξ, takes the form

$$\mathrm{d}(P-1)^4/\mathrm{d}\xi = 0 \qquad (\xi = 1). \tag{4.15}$$

Substituting expression (4.13) into (4.15), one obtains

$$\alpha = \frac{\pi}{\mu^2}(P_0 - 1)^8\left[c_0 + \sum_{n=1}^{\infty}\frac{(2n-1)!!b_{2n}}{(2n-2)!!(4n+2)}\right]^{-2}. \tag{4.6}$$

Now, the coefficient c_0 can be found explicitly if one uses Equation (3.7) for $\xi = 0$. We have

$$c_0 = \frac{\pi\delta}{4\mu^2}\left[\sqrt{1 + \frac{8\mu^2(P_0 - P^0)}{\pi\delta}} - 1\right]. \tag{4.17}$$

Thus, all parameters of the hydraulic fracturing – the crack propagation law $L(\tau)$, the fluid pressure in a crack P, the intensity of fluid-loss through crack surface Q_L, the depth of fracturing fluid penetration into a stratum Y_f, and the pressure at the penetration front P_f – have been expressed in terms of the coefficients c_{2n+1} of the expansion of pressure in a stratum (4.7) for odd powers of ξ, which are efficiently found by the iteration method (see Section 5). In this case, there is no necessity in calculating the pressure in a stratum $P_{\mathrm{I}}(\xi, \eta)$ for the determination of principal hydraulic fracturing parameters. Thereby, the dimension of an initial problem is lowered.

5. Calculation Method

The self-similar solution has been numerically constructed according to the following procedure. First, some initial approximation for pressure at the front of the fracturing fluid penetration into a stratum was specified: $P_f = P_f^{(0)}(\xi)$, $0 \leqslant \xi \leqslant 1$. Usually, function $P_0 + c_0(1-\xi)^{1/4}$ was chosen as $P_f^{(0)}(\xi)$, satisfying Equation (3.7) for $\xi = 0$ and the condition of absence of pressure perturbations in a stratum before the crack tip: $\xi = 1$. Then, the function $\Psi_f(\xi) = P_f(\xi) - P_f(0)$ was calculated and expanded over odd Chebyshev polynomials in the $0 \leqslant \xi \leqslant 1$ interval:

$$\Psi_f(\xi) = \sum_{j=1}^{\infty} d_{2j-1}T_{2j-1}(\xi), \tag{5.1}$$

$$d_j = \frac{2}{\pi}\int_0^{\pi}\Psi_f(\cos\Theta)\cos_j\Theta\,\mathrm{d}\Theta = \frac{1}{\pi}\sum_{k=1}^{N}[\Psi_f(\cos\Theta_{k-1})\cos_j\Theta_{k-1} +$$
$$+\Psi_f(\cos\Theta_k)\cos_j\Theta_k]\,\Delta\Theta, \quad \Delta\Theta = \pi/N,\ \Theta_k = k\,\Delta\Theta.$$

Coefficients d_j rapidly decrease as number j grows. Therefore, in calculations, the series (5.1) was truncated at some term, the number M of which was determined from the condition $d_M \leqslant 10^{-7}$. As usual, nine terms or less were retained in this procedure.

Then, the a_{2n-1} coefficients of expansion (4.9) $\phi_f(\xi)$ over odd powers of ξ were recalculated in terms of coefficients d_j. For this purpose, the recurrent relation

$$a_{2n-1} = \frac{2^{2(n-1)}}{(2n-1)!} \sum_{i=0}^{\infty} \frac{(-1)^i (2n+i-2)!(2n+2i-1)}{i!} d_{2(n+i)-1} \tag{5.2}$$

was utilized.

Further, coefficients b_{2i} were calculated by Equations (4.17) and (4.11), after which Equations (4.12)–(4.14) and (4.16) were used for finding the values of the functions $Q_L(\xi)$, $P(\xi)$ and $Y_f(\xi)$ at collocation points $\xi_k = \sin(\Theta_k - \pi/2)$, $\Theta_k = \pi k/N$, $k = 0, 1, \ldots, N$. These values were substituted into formula (3.7) for determining a new pressure distribution $P_f^{(1)}(\xi)$ at the penetration front, and the iteration was repeated. The iterations were stopped when the sequence $P_f^{(0)}$, $P_f^{(1)}, \ldots$ converged to the required accuracy. The number of iterations was mainly determined by the specified accuracy and by the value of the dimensionless parameter δ/μ. For example, when the maximum deviation ε between the $P_f^{(i)}(\xi_k)$ and $P_f^{(i-1)}(\xi_k)$ values of the order of 10^{-4} and $\delta/\mu = 0.1$, the number of iterations was about 30. As δ/μ increased (i.e. as the fracturing fluid-loss from a crack became more intensive), the number of iterations grew. To accelerate the convergency of a sequence of functions $P_f^{(i)}$ the relaxation method was applied.

6. Asymptotics

Some features of the self-similar solution can be elucidated analytically prior to its numerical construction. This information occurs to be useful for a qualitative analysis of basic laws of the hydraulic fracturing as well as for practical application of the numerical method.

First, we shall find the main term of the similarity solution expansion. This term corresponds to the condition that b_{2n} coefficients in formulae (4.12)–(4.14), (4.16) are zeroes. Using expression (4.17) for c_0, one obtains

$$Q_L(\xi) = 2(P_0 - 1)^4,$$

$$P(\xi) = 1 + (P_0 - 1)(1 - \xi^{1/2})^{1/2},$$

$$Y_f(\xi) = \frac{c_0^2 \mu^2 (1-\xi)}{\pi\delta (P_0 - 1)^4} = \frac{\pi\delta(1-\xi)}{16\mu^2 (P_0-1)^4}\left[\sqrt{1 + \frac{8\mu^2(P_0 - P^0)}{\pi\delta}} - 1\right]. \tag{6.1}$$

Substituting (6.1) into (3.7), one can find the pressure at the front of the fracturing fluid penetration into a stratum:

$$P_f(\xi) = 1 + (P_0 - 1)(1 - \xi^{1/2})^{1/2} - \frac{2c_0^2\mu^2(1-\xi)}{\pi\delta}$$
$$= 1 + (P_0 - 1)(1 - \xi^{1/2})^{1/2} - \frac{\pi\delta(1-\xi)}{8\mu^2} \times \left[\sqrt{1 + \frac{8\mu^2(P_0 - P^0)}{\pi\delta}} - 1\right]. \tag{6.2}$$

This pressure distribution does not however, satisfy the condition that the pressure is unperturbed before the crack tip: $P_f(1) = P^0$. Indeed, in this case, $P_f(1) = 1$, i.e. the pressure P_f at the penetration zone boundary tends to the fluid pressure in a crack $P(1) = 1$, as one approaches the crack tip. As a result, the pore pressure perturbation zone arises in the vicinity of a crack tip, at the $\xi = 1$ cross-section. Its characteristic size is evaluated by the $\eta_1 \simeq \sqrt{\pi\alpha/2}$ value in dimensionless variables, and in dimension ones it decreases with time as an inverse square of the crack length. This deficiency of a solution follows from truncating the next terms of the expansion.

Thus, the approximate solution (6.1), (6.2) coincides with the exact solution at the middle of a crack ($\xi = 0$), where the well is situated, and deviates from the exact solution when approaching the crack tip ($\xi = 1$). The same behaviour tendency also conserves for improved approximate solutions obtained with retaining the next terms of the expansion. This is due to the fact that the similarity solution expansion constructed does not take into account explicitly the features of its behaviour near the crack tip.

In order to find the exact solution asymptotics in the crack tip vicinity, it is sufficient to make use of Equations (3.5)–(3.7). From (3.7), one has

$$Q_L = \frac{P - P_f}{Y_f} \approx \frac{1 - P^0}{Y_f} \quad (\xi \to 1). \tag{6.3}$$

The substitution of this relation into (3.6) yields the equation with respect to Y_f, which is integrated in the explicit form as follows

$$Y_f = \frac{1}{2}\sqrt{\frac{2(1-P^0)}{\delta\alpha}(1-\xi^2)} \quad (\xi \to 1). \tag{6.4}$$

Then, the asymptotics of $Q_L(\xi)$ and $P(\xi)$ behaviour in the crack-tip vicinity can be found from (6.3) and (3.5). As a result, one obtains the relations

$$Q_L(\xi) \sim \Delta^{-1/2}, \quad P(\xi) - 1 \sim \Delta^{3/8}, \quad Y_f(\xi) \sim \Delta^{1/2}, \quad \Delta = 1 - \xi^{/12} = 1 - X \to 0 \tag{6.5}$$

Comparing them with Equations (6.1), one can see that for the exact solution, the reduced pressure $P(\xi) - 1$ and the penetration zone depth $Y_f(\xi)$ near the crack tip tend to zero lower than is predicted by first-approximation formulae (6.1). Besides, the fluid-loss velocity distribution $Q_L(\xi)$ near the crack tip has a root singularity, as it takes place in the stationary problem of fluid flow from a motionless crack of infinite conductivity into a porous medium [5]. This coincidence would hardly be forecasted a priori, however, since in the given case, the fluid-loss into a porous medium have been determined with due account of the finite hydraulic conductivity of a fracture. So, in the problem of flow injection into a formation through a penny-shaped crack or through the vertical Zheltov–Zhristianovich fracture [26, 28], the consideration of the dependence of a crack conductivity versus its opening by the Bousinesq formula results in the limitation of a flow near the crack tips: the velocity of flow through a porous medium decreases as a root of the distance to the crack tip [25].

7. Discussion of the Results

The purpose of the calculations was to study the influence of two principal parameters – the formation permeability k and the fracturing fluid viscosity μ_f – upon the dynamics of major characteristics of a hydraulic fracturing – the crack length l, its opening $2w$, volume V, the depth of fracturing fluid penetration into a formation y_f, and the total quantity of fluid penetrated into a formation, V_f. The remaining parameters were fixed at the values $2H = 2h = l_0 = 10$ m, $E = 10^4$ MPa; $\nu = 0.25$; $\sigma = 50$ MPa; $\kappa = 0.1$ m^2/s; $m = 0.1$; $\mu_0 = 5$ MPa s; $p^0 = 20$ MPa; $p_0 = 52.5$ MPa.

The initial crack length $l_0 = 10$ m was chosen in a rather arbitrary way, since it only slightly influences the ultimate process parameters for $l \gg l_0$.

The inlet pressure value p_0 was chosen from the following considerations. Usually, the inlet pressure variation range is limited from above because of the danger of rock fracture in the vertical direction. The $(p_0 - \sigma)/\sigma$ value of a relative increase of confining horizontal stress σ in the hydraulic fracturing varies, as a rule, within the limits of 0.001–0.05. Letting $(p_0 - \sigma)/\sigma = 0.05$, one obtains $p_0 = 1.05\,\sigma = 52.5$ MPa.

The accepted values of the confining stress $\sigma = 50$ MPa and of initial pore pressure $p^0 = 20$ MPa, correspond to the formation deposition depth of the order of 2 km.

Finally, note that the crack height $2H$ is not determined within the framework of the Perkins and Kern model, but it should be specified from some additional considerations. It was assumed in the calculations that it coincides with the formation thickness $2h = 10$ m. This situation frequently takes place when the top and bottom of a formation are in contact with plastic rocks (clays or salts) possessing a higher strength than the formation.

Four cases of the process have been considered corresponding to two values of fracturing fluid viscosity: $\mu_f = 50$ and 500 MPa s and to two values of a formation permeability: $k = 0.01$ and $0.1\ \mu$m^2. The corresponding values of dimensionless parameters μ and δ are given in Table I. Here are also presented the found values of parameter α that characterize the crack propagation rate. The dimensionless

Table I. Parameters of the hydraulic fracturing processes

Version	1	2	3	4
μ_f, Pa s	0.05	0.05	0.5	0.5
k, μm^2	0.01	0.1	0.01	0.1
μ	10	10	100	100
δ	10^3	10^2	10^4	10^3
$\alpha \times 10^{13}$	7.74	18.3	0.183	0.819
$\Pi_1 \times 10^2$	3.61	3.52	3.5	3.48
$\Pi^2 \times 10^{-4}$	0.412	1.87	1.87	4.24
$Y_f(0) \times 10^{-4}$	0.401	2.05	2.1	4.93
$\Omega_V \times 10^2$, m^2	5.32	5.2	5.16	5.12
$\Omega_f \times 10^4$	0.326	14.8	1.48	33.6
$\Omega_y \times 10^5$, m^{-1}	0.793	40.7	4.07	97.7
Ω_Q, $m^4 s^{-1}$	1.9	2.08	0.208	0.211
Ω_l, $m^4 s^{-1}$	$7.9.10^4$	$1.87.10^3$	$1.87.10^3$	83.5

self-similar distributions of pressure P in a crack, the velocities Q_L of flow through its surfaces, the depths Y_f of fracturing fluid penetration into a stratum, and the pressures P_f at the penetration front for these cases are shown in Figures 2 to 5. The notations of curves in all plots coincide with numbers of cases in Table I.

Consider some of the features of self-similar solutions found above. The pressure P in a crack (Figure 2) behaves nearly linearly in X in the middle part of a crack and rapidly decreases near its tips. It changes very weakly from one version to

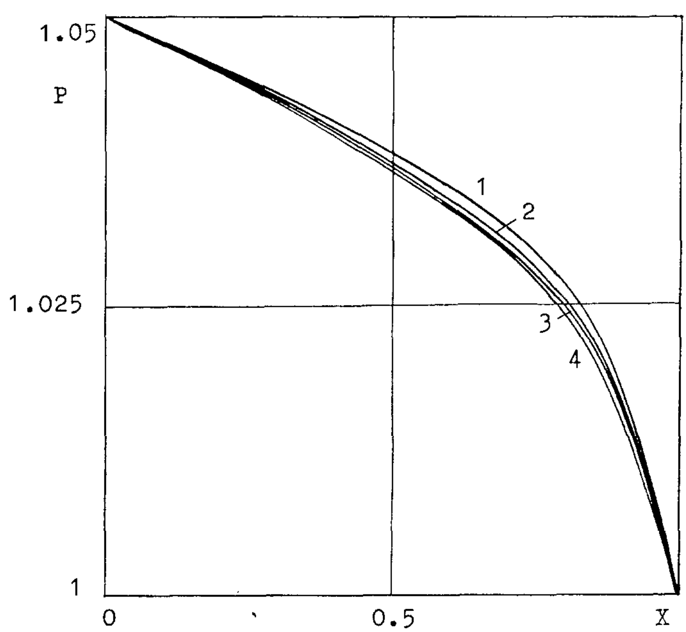

Fig. 2. Dimensionless pressure distribution in a fracture.

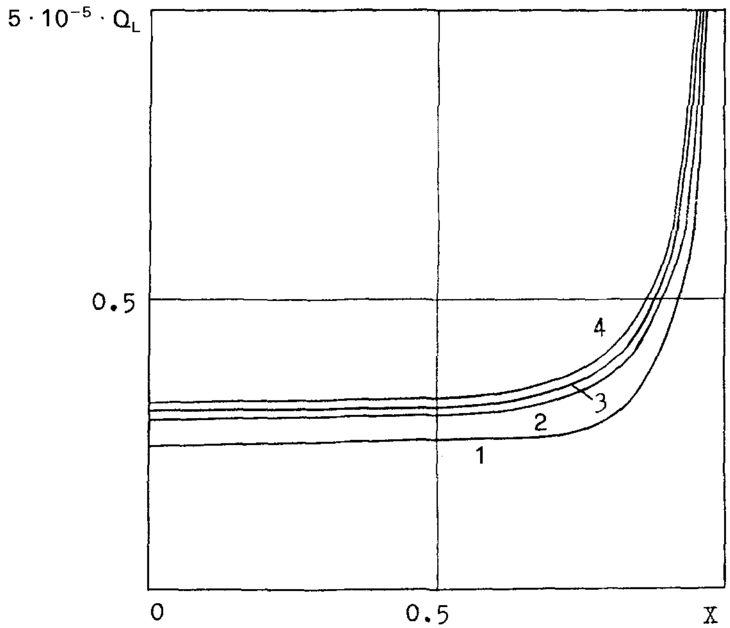

Fig. 3. Dimensionless fluid-loss velocity through the fracture surfaces.

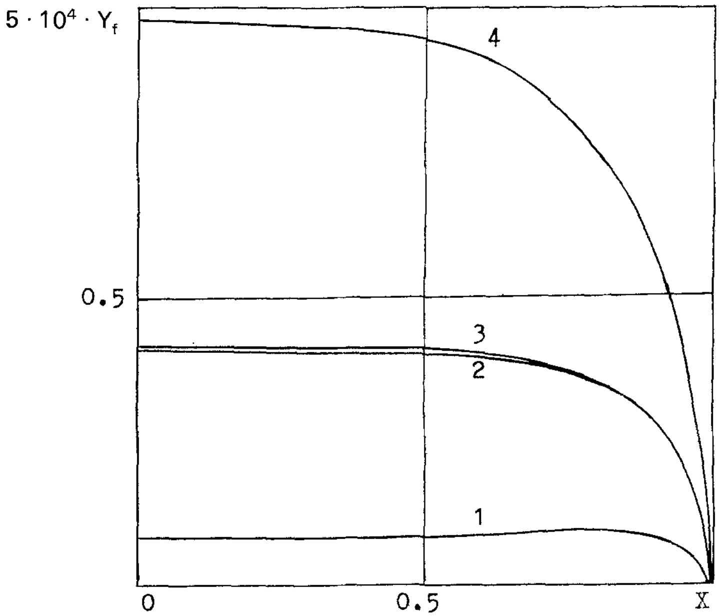

Fig. 4. Dimensionless depth of fracturing fluid penetration into a stratum.

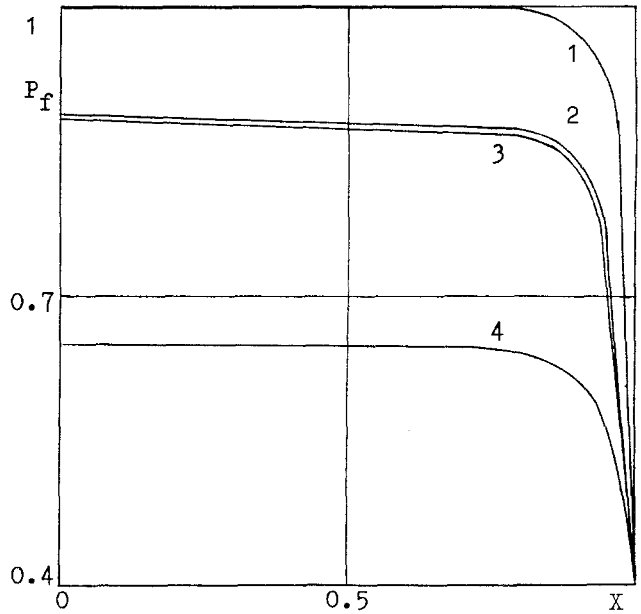

Fig. 5. Dimensionless pressure at the fracturing fluid penetration boundary.

another, and slightly decreases as the fracturing fluid viscosity and the formation permeability grow.

Function $Q_L(X)$, which determines the fluid-loss, slightly depends on the μ and δ parameters as well (Figure 3). In this case, the fluid-loss velocity is near constant in the middle of a crack and sharply increases towards its tips. Accordingly, it changes the depth of a fracturing fluid penetration into a stratum $Y_f(X)$ (Figure 4): it is nearly constant in the middle of a crack and sharply decreases down to zero near the crack tips. However, Y_f depends much more strongly on the μ and δ parameters, than Q_L does. The pressure distributions P_f at the penetration front have a similar behaviour (Figure 5); however, P_f tends more to their limiting value p^0/σ with $X \to 1$ in a steeper way than Y_f does.

Of interest is the fact that the Y_f and P_f functions practically coincide for the second and third versions, which differ in ten-fold, oppositely-directed variations of fracturing fluid viscosity and formation permeability for the constant k/μ_f ratio. Comparing these two versions with the first and fourth versions, one concludes that the increase of both fracturing fluid viscosity and formation permeability has the same consequence. Namely, when the cracks reach the same length, the penetration layer thickness increases.

The results obtained seems to be rather surprising. It seems that as the fracturing fluid viscosity grows, the fluid-loss should decrease. However, the growth of viscosity at a fixed inlet pressure results in the decrease of the flow rate of the fracturing fluid injection into the crack. In this case, the crack propagation rate lowers, and the hydraulic fracturing time grows. As a result, it occurs that though the injected fluid viscosity increases, the volume of leakage through the crack surface grows.

We shall express the principal hydraulic fracturing characteristics in terms of self-similar solution parameters. Using (2.1), (2.3), and (3.2)–(3.4), one obtains the following expressions for the crack volume V, the fluid-loss volume V_f, the maximum depth of fracturing fluid penetration into a stratum $y_{f0} = y_f(0, t)$, and the crack length

$$V = 2\int_{-l}^{l} \langle w \rangle \, dx = \frac{4\pi(1-\nu^2)\sigma H^2}{E} \Pi_1 l; \quad \Pi_1 = \int_0^1 (P-1)\, dX,$$

$$V_f = 4mh \int_{-l}^{l} y_f \, dx = \frac{64mkh^2}{\pi H^4}\left[\frac{E}{(1-\nu^2)\sigma}\right]^3 \Pi_2 l^3; \quad \Pi_2 = \int_0^1 Y_f \, dX,$$

$$y_{f0} = y_{0f} Y_f(0) = \frac{8kh}{\pi H^2}\left[\frac{E}{(1-\nu^2)\sigma}\right]^3 Y_f(0) l^2,$$

$$l = \left\{ l_0^4 + \alpha \left[\frac{(1-\nu^2)\sigma}{E}\right]^6 \left(\frac{\pi H^4}{4kh}\right)^2 \kappa t \right\}^{1/4} \tag{7.1}$$

The values of the Π_1, Π_2 and $Y_f(O)$ quantities in Equations (7.1), are given in Table I.

Thus, the crack length l grows as $t^{1/4}$ for large values of time. The proportionality factor includes the parameter αk^{-2}, that decreases as the permeability k and fracturing fluid viscosity μ_f grow (see Table I). As a result, the crack growth rate decelerates in this case.

Since the parameter Π_1 varies very slightly from one case to another, the crack volume V is determined, in fact, by its length only and does not explicitly depend upon k and μ_f: $V \sim l$.

The depth of fracturing-fluid penetration into a stratum and the fluid-loss volume V_f, to the opposite, rather strongly depend upon these parameters, and one should note that both these quantities grow with both the formation permeability k and fracturing fluid viscosity μ_f.

At the developed stage of the hydraulic fracturing ($l \gg l_0$), the volume of fluid penetrated into a stratum $V_f \sim l^3$ grows more rapidly than the crack volume does: $V \sim l$. Therefore, the flow rate Q_0 of a fluid injected into a stratum is determined, for $l \gg l_0$, by the fluid-loss dynamics. One has

$$Q_0 = \frac{d}{dt}(V + V_f) \approx \frac{dV_f}{dt}$$

$$= \frac{3\pi m\alpha \Pi_2 \kappa H^4}{k}\left[\frac{(1+\nu^2)\sigma}{E}\right]^3 l^{-1}. \tag{7.2}$$

It is this circumstance which has already been used above in constructing the self-similar solution: in the continuity equation of flow in a crack (3.1), the corresponding term that accounts for a crack-volume variation, was omitted. As a result, the flow rate Q_0, expressed in terms of self-similar solution parameters according to the formula

$$Q_0 = 4\langle wv\rangle\Big|_{x=0} = -\frac{\pi\sigma H^4}{\mu_f}\left[\frac{(1-\nu^2)\sigma}{E}\right]^3 \frac{\mathrm{d}(P-1)^4}{\mathrm{d}X}\Big|_{X=0} l^{-1}, \tag{7.3}$$

must coincide with that specified by asymptotic formula (7.2). Equating the right-hand sides of these formulae, one obtains the relation

$$-\frac{\mathrm{d}(P-1)^4}{\mathrm{d}X}\Big|_{X=0} = \frac{3m\kappa}{\sigma}\frac{\alpha\Pi_2\mu_f}{k}. \tag{7.4}$$

Using the data of Table I, one can see that the parameter $\alpha\Pi_2\mu_f/k$ and, hence, the $\mathrm{d}(P-1)^4/\mathrm{d}X$ value for $X = 0$, only slightly changes from one case to another. This implies that the flow rate Q_0 of a fluid injected into a crack, does depend on the formation permeability k and varies inversely proportional to the fracturing fluid viscosity μ_f.

Finally, note that Equation (7.2), rather than (7.3), is more convenient for calculating Q_0 because the latter equation includes a rather small quantity $\mathrm{d}(P-1)^4/\mathrm{d}X$ requires a thickening of the computation nodes in the vicinity of point $X = 0$ when the self-similar solution is found, whereas when using Equation (7.2), no special improvements of a self-similar solution are required.

Substituting specific values of parameters into Equations (7.1), (7.2), one reduces them to the form

$$V = \Omega_V l, \qquad V_f = \Omega_f l^3, \qquad y_{f0} = \Omega_y l^2, \qquad Q_0 = \Omega_Q/l \qquad l = (l_0^4 + \Omega_l t)^{1/4} \tag{7.5}$$

Coefficients Ω_V, Ω_f, Ω_y, Ω_Q, and Ω_l are given in Table I. Here, the quantities V, V_f, y_{f0}, Q_0, l, and t have dimensions m^3, m^3, m, m^3/s, m and s, respectively.

Some specific results of calculations by Equations (7.5) are presented below. Let us find the final characteristics of the hydraulic fracturing when forming cracks of length $l = 50$ and 100 m. The values of V, V_f, y_f, Q_o and t are given in Table II. Remember that Q_o is the flow rate of fluid injected at time t, corresponding to the cessation of fracturing. The same table presents the values of time-averaged flow rate $Q_0^* = V_f/t > Q_0$. The fact that Q_0 and Q_0^* are close suggests that the fracturing fluid injection stage with high flow rates is rather short.

The width $2w_0$ in the middle of a crack is the same for all cases, since it depends on the inlet pressure only, and equals 4.7 mm.

Table II shows that, for low fracturing fluid viscosities and low permeabilities (the 1st version), the crack volume is comparable with a fluid-loss volume; for large viscosities and permeabilities (the 4th version), $V \ll V_f$. The increase of permeability for $\mu_f = \mathrm{const}$ gives rise to a drastic increase of fluid-loss and, hence, to the growth of fracturing time for nearly the same flow rate of injected fluid (see versions 1 and 2 or 3 and 4).

Table II. Parameters of the hydraulic fracturing processes

l, m	50				100			
	1	2	3	4	1	2	3	4
V, m^3	2.66	2.6	2.58	2.56	5.32	5.2	5.16	5.12
V_f, m^3	4.08	185	18.5	420	32.6	1480	148	3360
y_f, cm	2	102	10.2	244	7.9	407	40.7	977
Q_0, m^3/min	2.38	2.5	0.25	0.253	1.19	1.25	0.125	0.126
t, min	1.32	55.7	55.7	1256	21.1	819	890	20100
Ω_0^*, m^3/min	3.09	3.32	0.332	0.335	1.54	1.66	0.166	0.167

Finally, the increase of viscosity μ_f for constant permeability k (see versions 1 and 3 or 2 and 4) results in the inversely proportional decrease of the flow rate Q_0, and also in the growth of fluid-loss which is essentially less, however, than that observed when increasing the permeability.

As follows from Equations (7.5), if the size (the volume) of a crack doubles, the hydraulic fracturing time grows as much as 16 times, the depth of fracturing fluid penetration into a stratum increases four times and the volume of the fluid penetrated into a stratum increases eight times.

Specifying one of the parameters appeared in these formulae, all the remaining parameters can be calculated. For example, with the injected fluid volume V_f known, one can find the crack length l and the hydraulic fracturing time t.

On the other hand, Equations (7.5) can be used for improving the model parameters by measuring any two quantities which are included in (7.5). Obviously, the injected fluid volume V_f and the flow rate Q_0 can naturally be considered as the measured values in the given case. Eliminating the crack length l from corresponding Equations (7.5), one obtains

$$V_f Q_0^3 = \Omega_f \Omega_Q^3 = \text{const.} \tag{7.6}$$

It follows from this relation that function the $V_f Q_0^3$, measured in the hydraulic fracturing, should tend with time to some constant value, and relation (7.6) can be considered to be the equation for one of the model parameters included in the coefficients

$$\Omega_f = \frac{64mkh^2}{\pi H^4}\left[\frac{E}{(1-\nu^2)\sigma}\right]^3 \Pi_2, \quad \Omega_Q = \frac{3\pi m\alpha \Pi_2 \kappa H^4}{k}\left[\frac{(1-\nu^2)\sigma}{E}\right]^3. \tag{7.7}$$

In particular, from (7.6) and (7.7), one can express the vertical crack size H in terms of the other parameters and the measured value $V_f Q_0^3$:

$$H = \frac{(V_f Q_0^3)^{1/8}}{(12\alpha\kappa)^{3/8}}\left[\frac{kE}{\pi h m^2 \Pi_2^2 (1-\nu^2)\sigma}\right]^{1/4}. \tag{7.8}$$

Here, the parameters α and Π_2 are found from the self-similar solution; they depend on dimensionless complexes μ and δ and not on H. The remaining quantities in the right-hand side of (7.8) must be known.

Thus, formula (7.8) allows us to find, in principle, a rarely defined, or, more frequently, merely unknown model parameter – the hydraulic fracture height $2H$ from the data of measurements of flow rate Q_0 and the injected fluid volume V_f.

Let us resumé the results obtained.

In forming vertical, highly elongated hydraulic fractures by the injection of viscous fluid under constant well-bore pressure, a large amount of fracturing fluid penetrates into a stratum, the fluid-loss volume per fracture length or volume unit being increased with fracture size as a square of the fracture length.

For the fixed final fracture length and for the same inlet pressure, the hydraulic fracturing time and the fluid-loss volume increase with the formation permeability and fracturing fluid viscosity. This implies that it is more beneficial, from technological and economical points of view, to form a hydraulic fracture by using low-viscosity fluids, and high-viscosity fluids should be used at the fracture fixation stage only. This conclusion, of course, cannot be spread to injection regimes with a constant flow rate in which the inlet pressure should increase with time and with the fracturing fluid viscosity.

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