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Convergence to Equilibrium for Delay-Diffusion Equations with Small Delay

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It is shown that for scalar dissipative delay-diffusion equations $u_t - \Delta u = f(u(t), u(t-\tau))$ with a small delay, all solutions are asymptotic to the set of equilibria as t tends to infinity.

KEY WORDS: Parabolic equations with time delay; asymptotic behavior.

1. INTRODUCTION

It is well-known and not hard to prove (see Refs. 4, 6, and 13) that for scalar reaction-diffusion equations,

$$u_t - \Delta u = f(u) \qquad (x \in \Omega \subset \mathbb{R}^n) \tag{1}$$

subject to homogeneous boundary conditions, all globally defined bounded solutions must approach the set of equilibria as t tends to infinity. This is a consequence of the fact that (1) is a gradient system, thanks to the Lyapunov function

$$V(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u)$$

where F is a primitive of f. It is also well-known that if we introduce a time delay into the right-hand side so that (1) becomes

$$u_t - \Delta u = f(u(t), u(t-\tau)) \tag{2}$$

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solutions will typically oscillate in t as $t \to \infty$: see Ref. 12 for an up-to-date survey of results on the underlaying ordinary delay differential equation

$$u_t = f(u(t), u(t-\tau)) \tag{3}$$

[which in the case of Neumann boundary values governs the evolution of the invariant subsystem obtained from (2) by considering functions constant in space], and see Refs. 2, 3, 14, 15, and 18 for existence and stability results on both spatially homogeneous and inhomogeneous oscillations for (2).

A natural question that then arises is whether such oscillations persist if the delay is decreased towards zero. The aim of this article is to show that this does not happen: if Eq. (2) is dissipative with respect to the $L^\infty(\Omega)$-norm and the delay is sufficiently small, then all trajectories get attracted by the set of equilibria as t tends to infinity (see Theorems 1 and 2, below). The result is of some interest even in the simpler case of Eq. (3), since introducing a small time delay into a scalar ODE makes a one-dimensional dynamical system infinite-dimensional.

2. PRELIMINARIES AND STATEMENT OF THE RESULT

Let Ω be an open bounded regular domain in $\mathbb{R}^n$, and let $\tau > 0$ be a positive parameter (the delay). We study the scalar delayed initial-boundary-value problem

$$u_t - \Delta u = f(u(t), u(t-\tau)) \qquad \text{in} \qquad \Omega \times \mathbb{R}^+ \tag{2a}$$

$$u = u_0 \qquad \text{in} \qquad \Omega \times [-\tau, 0] \tag{2b}$$

$$u = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \qquad \partial\Omega \times \mathbb{R}^+ \tag{2c}$$

where the nonlinearity f: $\mathbb{R}^2 \to \mathbb{R}$ is assumed to be locally lipschitz and to satisfy the one-sided growth estimates

$$\begin{aligned} f(u, v) &\leqslant (u+1)\,\gamma(v) \qquad &\text{for} \quad u \geqslant 0 \\ f(u, v) &\geqslant -(|u|+1)\,\gamma(v) \qquad &\text{for} \quad u \leqslant 0 \end{aligned} \tag{4}$$

for some continuous γ. Note that we do not impose any growth condition on the delayed part of f, or on undelayed terms tending to $\mp\infty$ as $u \to \pm\infty$. In particular, the equations we study include Hutchinson's equation

$$u_t - \Delta u = \alpha u(t)[1 - u(t-\tau)] \qquad (\alpha > 0) \tag{5}$$

studied in Refs. 2, 3, 11, 14, 15, and 18, its modification

$$u_t - \Delta u = \alpha u(t)[1 - \beta u(t-\tau) + \gamma u(t-\tau)^2 - \delta u(t-\tau)^3] \qquad (\alpha, \beta, \gamma, \delta > 0) \tag{6}$$

studied in one space dimension in Ref. 14, and the equations

$$u_t - \Delta u = -u(t-\tau)^3 \tag{7}$$

$$u_t - \Delta u = -u(t)\, u(t-\tau)^2 \tag{8}$$

$$u_t - \Delta u = -u(t)^2\, u(t-\tau), \qquad u \geqslant 0 \tag{9}$$

investigated in a three-dimensional domain in Refs. 8 and 9. By a solution of (2) on $(0, T)$ we mean a function $u\colon \Omega \times [-\tau, T) \to \mathbb{R}$ lying in

$$C^0([-\tau, T-\varepsilon]; L^2(\Omega)) \cap \mathscr{L}^\infty([-\tau, T-\varepsilon]; L^\infty(\Omega))$$

for all $\varepsilon \in (0, T)$ and satisfying

$$u(t) = T_\Delta(t)\, u_0(0) + \int_0^t T_\Delta(t-s)\, f(u(s), u(s-\tau))\, ds, \quad \forall t \in (0, T) \tag{10a}$$

$$u|_{[-\tau, 0]} = u_0 \tag{10b}$$

where $\mathscr{L}^\infty([a, b]; Y)$ denotes the space of bounded (not just essentially bounded) functions from $[a, b]$ into Y, and $T_\Delta(t)$ is the analytic semigroup generated by the Laplacian on $L^2(\Omega)$ subject to the choice of domain

$$\mathscr{D}(\Delta) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$$

in the Dirichlet case and

$$\mathscr{D}(\Delta) = \left\{ u \in W^{2,2}(\Omega) \colon \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} = 0 \right\}$$

in the Neumann case. Here the requirement that $u(t)$ be bounded in $L^\infty(\Omega)$ ensures that the integrand $f(s)$ in (10a) lies in $L^2(0, T-\varepsilon; L^2(\Omega))$ so that the integral makes sense; moreover, by the regularity theory for analytic semigroups (see Ref. 16), any such "mild" solution lies in

$$W^{1,2}_{\text{loc}}(0, T; L^2(\Omega)) \cap L^2_{\text{loc}}(0, T; \mathscr{D}(\Delta))$$

and is thus a "strong" solution. In particular, the boundary values are attained in the $W^{2,2}(\Omega)$ sense for almost all $t \in (0, T)$, since $u(t)$ lies in $\mathscr{D}(\Delta)$ for almost all t.

To shorten the notation, let us denote the Banach space $C^0([-\tau, 0]; L^2(\Omega))$ by X and its dense subspace $C^0([-\tau, 0]; L^2(\Omega)) \cap \mathscr{L}^\infty([-\tau, 0];$

$L^\infty(\Omega)$) by X_∞. Treating Eq. (2) stepwise as a nonautonomous undelayed parabolic PDE, it is more or less standard (see Section 3) to show the following.

Proposition (Global Existence and Uniqueness). *For all $u_0 \in X_\infty$ there exists a unique solution u of (2), defined globally in t. In particular, via*

$$(\Phi(t)u_0)(s) := u(t+s)$$

Eq. (2) defines a continuous semiflow (or "dynamical system") on the dense subspace X_∞ of X.

Remark. If we wanted to obtain a continuous semiflow on the Banach space $C^0([-\tau, 0]; Y)$ (Y an appropriate L^p or Sobolev space) itself rather than just a dense subspace—a strategy usually adopted in the literature (see Refs. 1, 8, 14, 15, 17, 18)—we would either have to impose severe growth conditions on f, including delayed and negative feedback terms, or to work in Sobolev spaces of higher and higher order so as to weaken these growth restrictions; otherwise we would not even obtain local existence results.[3] In particular, we would not be able to incorporate the above examples (for which existence has been proved in the literature in various spaces, depending on the dimension of domains and the order of nonlinearities) into a unified context.

As usual, an equilibrium solution of (2) is defined as a solution which does not depend on t; the equilibrium states are thus the functions $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfying the elliptic boundary value problem

$$-\Delta u = f(u, u) \qquad \text{in} \quad \Omega$$

$$u = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial\Omega$$

in the weak sense [so that they in particular lie in $W^{2,2}(\Omega)$], and their totality is denoted E for the remainder of this paper. We can now state our main result.

Theorem 1. *Given $K > 0$ there exists $\tau_0 = \tau_0(f, K)$ independent of Ω such that for $\tau < \tau_0$ all trajectories u of (2) with $\overline{\lim}_{t\to\infty} \|u(t)\|_\infty < K$ satisfy*

$$dist_{L^2(\Omega)}(u(t), E) \to 0$$

as t tends to infinity.

[3] For example, the local existence results for abstract retarded parabolic equations in Ref. 1 or 17 applied to Eq. (2) with initial data in $C^0([-\tau]; L^2(\Omega))$ require, among other hypotheses, a global linear growth bound $|f(u, v)| \leq C(|u| + |v| + 1)$.

The proof is given in Section 4, and its main ingredients are decay estimates on u_t obtained by using the Lyapunov function for the undelayed counterpart of (2) (Lemmata 1 and 2).

Under further assumptions of f, the result can be improved: provided that f satisfies a one-sided linear growth condition independent of the delayed terms together with a negative feedback condition at infinity,

$$\begin{aligned} f(u, v) &\leqslant (u+1)\gamma && \text{for} \quad u \geqslant 0 \\ f(u, v) &\geqslant -(|u|+1)\gamma && \text{for} \quad u \leqslant 0 \end{aligned} \qquad (\gamma \text{ independent of } v) \tag{11}$$

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} f(\lambda u, \lambda v) \to \pm\infty \qquad \text{for} \quad u, v \gtrless 0 \tag{12}$$

it is a nontrivial result of Luckhaus [11] that for small τ, all solutions of (2) satisfy $\overline{\lim}_{t \to \infty} \|u(t)\|_{L^2(\Omega)} < K$ for some K independent of τ and u. In Ref. 11 this is proved for nonnegative weak subsolutions, but the proof can easily be adapted to the present setting by considering u_+, $-u_-$ instead of u and using the fact that both are nonnegative weak subsolutions. Since, moreover, (11) implies an L^1–L^∞ estimate,

$$\|u(t)\|_\infty \leqslant c(\gamma, \Omega, \varepsilon)(\|u(t-\varepsilon)\|_1 + 1)$$

for all solutions of (2) (see, e.g., Ref. 6, exercise 3.5.4), we obtain the following.

Theorem 2. *Assume in addition that f satisfies (11), (12). Then there exists $\tau_0 = \tau_0(f, \Omega)$ such that for $\tau \leqslant \tau_0$, all trajectories u of (2) satisfy*

$$\text{dist}_{L^2(\Omega)}(u(t), E) \to \infty$$

as t tends to infinity.

In particular, this implies global stability of zero for Eqs. (7), (8), and (9), since in these cases $E = \{0\}$. Also, by applying the above results to spatially homogeneous solutions of the Neumann problem, we obtain a corresponding statement for ordinary delay differential equations.

Corollary 1. *Let f be locally lipschitz and satisfy (4), and assume that the dynamical system generated by (3) on $C := C^0([-\tau, 0]; \mathbb{R})$ is dissipative, uniformly with respect to τ for small τ. Then there exists $\tau_0 = \tau_0(f)$ such that for $\tau < \tau_0$ all solutions of (3) converge in C to the set of equilibria as $t \to \infty$. Moreover, if the zero set of $f(u, u)$ is discrete, each solution stabilizes to a single equilibrium.*

It is left to the interested reader to extract a simpler proof from Section 4 exploiting the fact that the "pseudo-Lyapunov"-function V reduces on spatially homogeneous solutions to a multiple of a primitive of $f(u, u)$.

Let us now return to the diffusive case and end this section with a rather less trivial example than (7)–(9), where our results lead to a complete picture of the global dynamics.

Example. Hutchinson's equation in an interval

$$\begin{aligned} &u_t - u_{xx} = u(t)[1 - u(t-\tau)] \qquad \text{for} \quad x \in (0, \pi), \quad t > 0 \\ &u_x(0, t) = u_x(\pi, t) = 0 \qquad \text{for} \quad t > 0 \end{aligned} \tag{13}$$

subject to nonnegative initial data $u_0 \geqslant 0$.

Without diffusion, this equation was proposed in 1948 by Hutchinson [7] as a continuous model (derived from the familiar discrete logistic equation) for population fluctuations and is one of the best-studied ODDEs in the literature (see Ref. 5, Section 11.4, Ref. 12, and the references therein); its diffusive version was studied by Green and Stech [3], Yoshida [18], Luckhaus [11], and the author [2], and recently by Memory [14, 15]. It is easy to see that the solutions of (13) stay nonegative for all t (in keeping with their interpretation as density distributions), that therefore Theorem 2 applies, and that $u \equiv 0$, $u \equiv 1$ are the only equilibria: if u is an equilibrium, then

$$0 \leqslant \int_0^\pi u_{xx}^2 - \int_0^\pi u_x^2 = -2 \int_0^\pi u u_x^2$$

which implies $uu_x \equiv 0$, hence $u_x \equiv 0$. Consequently, by Theorem 2 each solution converges to either $u \equiv 0$ or $u \equiv 1$ if the delay is small. Let us examine whether there can exist nontrivial orbits converging to zero. Unfortunately, linearizing (13) about zero proves unsuccessful here because it so happens that zero possesses an infinite-dimensional stable, an infinite-dimensional unstable, and an infinite-dimensional center manifold [with respect to the flow generated according to Ref. 18 by (13) on $C^0([-\tau, 0]; W_N^{2,2}(0, \pi))$], and it seems difficult to predict where the intersection of these manifolds with the constraint set $\{u \geqslant 0\}$ lies. So let us argue differently. Suppose $u(\cdot, 0) \not\equiv 0$, $u(\cdot, t) \to 0$ in $L^2(0, \pi)$. By orbit precompactness in $W^{2,2}(0, \pi)$ [see Ref. 14 or show directly via energy estimates that $u(\cdot, t)$ stays bounded in $W^{3,2}(0, \pi)$], $u(\cdot, t) \to 0$ in $W^{2,2}(0, \pi)$ and in $L^\infty(0, \pi)$, so there exist $T > 0$, $\varepsilon < 1$ such that

$$u(x, t) < \varepsilon \qquad \text{for} \quad x \in (0, \pi), \qquad t \geqslant T \tag{14}$$

On the other hand, since $\|u(\cdot, t)\|_{\infty} \leqslant K$ for all $t \geqslant -\tau$ and $\|u(\cdot, 0)\|_{L^1(\Omega)} > 0$, we have $\inf_{x \in (0,\pi)} u(x, t) > 0$ for all $t > 0$. [This Harnack-type estimate follows from an elementary comparison argument here: u is estimated from below by the solution of $u_t - u_{xx} = (1-K)u$ on the whole of $\mathbb{R}$ with the same initial data as u, and the latter is seen to have positive infimum on $x \in (0, \pi)$ by writing it explicitly as an integral against the heat kernel.] Now letting $\delta := \inf_{x \in (0,\pi)} u(x, T)$ we define the comparison function $\bar{u}(x, t) := \delta e^{(1-\varepsilon)(t-T)}$ and compute, using (14),

$$\bar{u}_t - \bar{u}_{xx} - (1-\varepsilon)\bar{u} = 0 \leqslant u_t - u_{xx} - (1-\varepsilon)u \qquad (t \geqslant T)$$

Hence $u(t) \geqslant \bar{u}(t)$ for $t \geqslant T$, which contradicts (14) for large enough t. Therefore no nontrivial solution can converge to zero as $t \to \infty$, and our Theorem 2 gives the following result (which improves the local stability result in Ref. 15, p. 140).

Corollary 2. *If the delay τ is small, then all solutions $u(\cdot, t)$ of (13) with initial data $u_0 \geqslant 0$, $u_0(\cdot, 0) \not\equiv 0$ stabilize in $W^{2,2}(0, \pi)$ to the nontrivial stationary state $u \equiv 1$ as t tends to infinity.*

For large delays, the above stability results break down for various reasons. First, bounded solutions might contain oscillating orbits in their ω-limit sets [i.e., $\omega(u_0) \not\subset E$]; see, in particular, Refs. 14 and 18 for interesting existence and stability results for periodic solutions to Eqs. (5), (6), and (13), or Ref. 3, where Eq. (13) subject to Dirichlet boundary conditions is studied numerically and a stable spatially inhomogeneous periodic orbit is observed. Second, Theorem 2 breaks down since dissipativity itself may fail for large delays, even under the negative feedback condition (12): trajectories might escape exponentially to infinity with respect to the $L^1(\Omega)$-norm [i.e., $\omega(u_0) = \emptyset$] by moving their "mass" aropund along periodic paths in space, as happens for example for Hutchinson's equation (5) in more than one space dimension or in an interval with periodic boundary conditions (see Ref. 2).

3. PROOF OF THE PROPOSITION

To prove existence, we treat Eq. (2) stepwise as a nonautonomous undelayed parabolic PDE on the time intervals $[(j-1)\tau, j\tau]$ ($j \in \mathbb{N}$) by regarding the delayed values as fixed. However, despite the vast literature on existence for such equations, we have been unable to locate a result directly applicable to (2) since rather mild regularity assumptions on the nonautonomous terms are required here. For example, results involving Hölder continuity assumptions with respect to t as in Ref. 6 cannot be applied since our initial data, considered as functions from $[-\tau, 0]$ into

$L^2(\Omega)$, are only continuous. To work in $C^{0,\alpha}([\tau, 0]; L^2(\Omega))$ instead of $C^0([-\tau, 0]; L^2(\Omega))$ is no way out of the dilemma since solutions of parabolic equations do not in general attain their initial values Hölder-continuously, thus time translates of typical solutions with Hölder continuous initial data are not even locally Hölder in t. Thus our strategy is to mimick the results of Henry (Ref. 6, Theorem 3.3.3 and Corollary 3.3.5) but with his assumption of Hölder continuity in t replaced by p-integrability. But first, let us prove uniqueness.

3.1. Proof of Uniqueness

Let u be any solution of (2). For $s \in (0, t_1)$ $(t_1 < T)$, $u(s)$ is uniformly bounded in $L^\infty(\Omega)$; thus by continuity of f, so is $f(u(s), u(s-\tau)) =: f(s)$, in particular, $f(s) \in L^2(0, t_1; L^2(\Omega))$. But since the semigroup is analytic and $L^2(\Omega)$ is Hilbert, the variation-of-constants formula has maximal regularity in $L^2(0, t_1; L^2(\Omega))$ (see Ref. 16 for an elegant three-line proof using the Fourier transform), hence

$$\int_0^t T_\Delta(t-s)\, f(s)\, ds \quad \in \quad W^{1,2}(0, t_1; L^2(\Omega)) \cap L^2(\mathscr{D}(\Delta))$$

Thus since a difference $w := u_1 - u_2$ of two solutions consists only of two terms of the above form [the singular terms $T_\Delta(t-s)u_0$ cancel], it lies in $W^{1,2}(0, t_1; L^2(\Omega)) \cap L^2(\mathscr{D}(\Delta))$, and thus we can "test the equation with w," i.e., compute

$$\begin{aligned}
\tfrac{1}{2}\int_\Omega \|w(t)\|_2^2 - \tfrac{1}{2}\int_\Omega \|w(0)\|_2^2 &= \int_0^t \int_\Omega w_t w \\
&= \int_0^t \int_\Omega (\Delta w + [f(\cdot, u_1(\cdot)) - f(\cdot, u_2(\cdot))])w \\
&\leqslant -\int_0^t \|\nabla w\|_2^2 + s(t_1) \int_0^t \|w\|_2^2 \\
&\leqslant s(t_1) \int_0^t \|w\|_2^2
\end{aligned}$$

where

$$s(t_1) := \sup \left\{ \frac{|f(u_1', v') - f(u_2', v')|}{|u_1' - u_2'|} : |u_1'|, |u_2'|, |v'| \leqslant \max_i \sup_{s \in [-\tau, t_1]} \|u_i(s)\|_\infty \right\}$$

Since $w \in C^0([0, t_1]; L^2(\Omega))$ and $w(0) = 0$, Gronwall's inequality now implies $\|w(s)\| \equiv 0$ on $[0, t_1]$. This proves uniqueness.

3.2. Proof of Existence

3.2.1. Local Existence on $(0, T)$ *for Some Small* $T > 0$:

Let $v(t) := u(t-\tau)$ for $t \in [0, \tau]$, and let $\bar{f} := f \cdot \zeta$ where ζ is a cutoff function with values between 0 and 1, with compact support in $\mathbb{R}^2$, and satisfying $\zeta \equiv 1$ on $[-R, R]^2$ (R to be specified later). Then $\bar{f}$ is globally lipschitz on $\mathbb{R}^2$ with some lipschitz constant $L(R)$ and obeys the same growth conditions as f:

$$\begin{aligned} &\bar{f}(u, v) \leqslant (u+1)C \qquad \text{for} \quad u \geqslant 0 \\ &\bar{f}(u, v) \geqslant -(|u|+1)C \qquad \text{for} \quad u \leqslant 0 \\ &(C = \sup\{\gamma(v') : |v'| \leqslant \sup_{s \in [0,\tau]} \|v(s)\|_\infty\}) \end{aligned} \tag{15}$$

Since $\bar{f}$ is lipschitz we readily obtain for $u, u_1, u_2 \in L^2(\Omega_T)$ [where $\Omega_T := \Omega \times (0, T))$]:

$$\|\bar{f}(u_1, v) - \bar{f}(u_2, v)\|_{L^2(\Omega_T)} \leqslant L(R) \|u_1 - u_2\|_{L^2(\Omega_T)} \tag{16}$$

$$\|\bar{f}(u, v)\|^2_{L^2(\Omega_T)} \leqslant L(R)^2 (\|u\|^2_{L^2(\Omega_T)} + \|v\|^2_{L^2(\Omega_T)}) \tag{17}$$

Hence the operator

$$(Gu)(t) := T_\Delta(t)\, u_0(0) + \int_0^t T_\Delta(t-s)\, \bar{f}(u(s), v(s))\, ds$$

is well defined, maps $L^2(\Omega_T)$ into itself, and also turns out to be a contraction for sufficiently small T: in fact, by the Schwarz inequality,

$$\|(Gu_2)(t) - (Gu_1)(t)\|_2 \leqslant \sqrt{t} \sup_{s \in [0,T]} |||T_\Delta(s)||| \; \|\bar{f}(u_1, v) - \bar{f}(u_2, v)\|_{L^2(\Omega_T)}$$

thus, by integrating over t and by (16),

$$\|Gu_2 - Gu_1\|_{L^2(\Omega_T)} \leqslant \frac{T}{\sqrt{2}} \sup_{s \in [0,T]} |||T_\Delta(s)||| \; L(R) \|u_1 - u_2\|_{L^2(\Omega_T)}$$

Hence, for T small enough, G possesses a unique fixed point in $L^2(\Omega_T)$, which satisfies

$$u(t) = T_\Delta(t-s)\, u_0(0) + \int_0^t T_\Delta(t-s)\, \bar{f}(u(s), v(s))\, ds \qquad \forall t \in (0, T) \tag{18}$$

Since the integrand $\bar{f}(s)$ lies in $L^2(0, T; L^2(\Omega))$, the integral in the above equation varies continuously with t, and thus $u \in C^0([-\tau, 0]; L^2(\Omega))$. Moreover, by the growth estimates (15) on $\bar{f}$ and the comparison principle

$$\|u(t)\|_\infty \leqslant (\|u_0(0)\|_\infty + 1)e^{Ct} - 1 \tag{19}$$

[where we have used the solutions constant in space of $u_t - \Delta u = \pm C(|u|+1)$ with initial data $\pm \|u_0(0)\|_\infty$ as comparison functions], so that $u \in \mathscr{L}^\infty([-\tau, 0]; L^\infty(\Omega))$. Consequently u is a local solution of (2) with nonlinearity $\bar{f}$.

3.2.2. Existence on [0, τ]

Suppose $t_1 \leqslant \tau$ is the maximal time of existence. Then as in Ref. 6 (Corollary 3.3.5), either

(A) $$\frac{\|f(u(t), v(t))\|_Y}{1 + \|u(t)\|_Y} \to \infty \qquad (t \to t_1) \qquad \text{or}$$

(B) $$u(t) \to u^* \quad \text{in } Y \quad (t \to t_1) \quad \text{for some } u^* \in Y$$

[where the failure of (A) implies (B) since then $\{u(t)\colon t \to t_1\}$ is bounded in the fractional power spaces Y^β $(0 \leqslant \beta < 1)$ associated with the Laplacian on $Y = L^2(\Omega)$, which in turn implies $\{u(t)\colon t \to t_1\}$ Cauchy in Y]. Here (A) is excluded by (19), thus $t_1 = \tau$ and $u(t) \to u^*$ $(t \to t_1)$ in $L^2(\Omega)$. Moreover, (19) implies that $u^* \in L^\infty(\Omega)$ and $\|u(t)\|_\infty$, $\|u^*\|_\infty$ are bounded by some constant independent of t, thus u is a solution of (2) with right-hand side $\bar{f}$ on $[0, \tau]$. Now use the fact that the bound on $\|u(t)\|_{L^\infty(\Omega)}$ in (19) depends only on $\|v\|_\infty$, and not on R [it is exactly at this point where the one-sided linear growth conditions (4) on the undelayed part of f are needed]; thus choosing

$$R > (\|v\|_\infty + 1)e^{C\tau} - 1$$

implies $\bar{f} = f$ for all values attained by u, v on $[-\tau, 0] \times \Omega$ so that u is a solution of (2) with right-hand side f on $[0, \tau]$.

3.2.3. Global Existence

Repeating the above procedure iteratively on the time intervals $[j\tau, (j+1)\tau]$ $(j \in \mathbb{N})$ now gives a global solution of (2) for all $t > 0$. This completes the proof of the proposition.

4. PROOF OF CONVERGENCE TO EQUILIBRIUM

The proof of Theorem 1 is split into several lemmata. Our first goal is to obtain appropriate decay estimates on u_t. To this end, consider the Lyapunov function for the undelayed counterpart of (2), which was already mentioned in the Introduction:

$$V(u) := \tfrac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u)$$

where $f(u)$ is a primitive of $f(u, u)$ with respect to u. V is defined on $W^{1,2}(\Omega) \cap L^\infty(\Omega)$, and thus, in particular, along trajectories of (2). In the case $\tau = 0$, one would have

$$\frac{d}{dt} V(u(t)) = -\int_\Omega u_t^2$$

and thus V would strictly decrease along trajectories except at equilibria. In the presence of a delay $\tau > 0$, this is no longer true, but we have the following.

Lemma 1. *Let u be a solution of (2) satisfying* $\|u(t)\|_\infty < K$ *for* $t \geqslant T$, *and define*

$$B(K, f) := \sup_{u, v, v' \in [-K, K]} \frac{|f(u, v) - f(u, v')|}{|v - v'|}$$

Then for $t \geqslant T + \tau$, *we have*

$$\frac{d}{dt} V(u(t)) \leqslant \left(\frac{B\tau}{2} - 1\right) \int_\Omega u_t^2(t) + \frac{B}{2} \int_{t-\tau}^t \int_\Omega u_t^2 \tag{20}$$

Proof. Using the differential equation for u, the lipschitz continuity of f as manifested in the definition of B, Young's inequality, the fundamental theorem of calculus, and the Schwarz inequality, we compute

$$\begin{aligned}
\frac{d}{dt} V(u(t)) &= -\int_\Omega u_t^2 + \int_\Omega (f(u(t), u(t-\tau)) - f(u(t), u(t)))u_t \\
&\leqslant -\int_\Omega u_t^2 + B \int_\Omega |u(t-\tau) - u(t)|\,|u_t| \\
&\leqslant \left(\frac{B\tau}{2} - 1\right) \int_\Omega u_t^2 + \frac{B}{2\tau} \int_\Omega |u(t-\tau) - u(t)|^2 \\
&\leqslant \left(\frac{B\tau}{2} - 1\right) \int_\Omega u_t^2(t) + \frac{B}{2} \int_{t-\tau}^t \int_\Omega u_t^2
\end{aligned}$$

The above decay estimate is good enough to deduce the following.

Lemma 2. *Given* $K > 0$, *there exists* $\tau_0(K, f)$ *independent of* Ω *such that for* $\tau < \tau_0$, *all trajectories u of (2) with* $\overline{\lim}_{t \to \infty} \|u(t)\|_\infty < K$ *satisfy*

$$\int_{t-\tau}^t \int_\Omega u_t^2 \to 0 \qquad (t \to \infty) \tag{21}$$

Proof. For sufficiently large t, estimate (20) holds for all $s \geqslant t$, thus

$$\begin{aligned} V(u(t+T)) - V(u(t)) &\leqslant \left(\frac{B\tau}{2} - 1\right) \int_t^{t+T} \int_\Omega u_t^2 + \frac{B}{2} \int_t^{t+T} \int_{s-\tau}^{s} \int_\Omega u_t^2 \\ &\leqslant \left(\frac{B\tau}{2} - 1\right) \int_t^{t+T} \int_\Omega u_t^2 + \frac{B\tau}{2} \int_{t-\tau}^{t+T} \int_\Omega u_t^2 \\ &= (B\tau - 1) \int_t^{t+T} \int_\Omega u_t^2 + \frac{B\tau}{2} \int_{t-\tau}^{t} \int_\Omega u_t^2 \end{aligned}$$

so that (letting $T \to \infty$)

$$[\overline{\lim_{T \to \infty}}\ V(u(t+T))] - V(u(t)) + (1 - B\tau) \int_t^\infty \int_\Omega u_t^2 \leqslant \frac{B\tau}{2} \int_{t-\tau}^{t} \int_\Omega u_t^2 \qquad (22)$$

But by the boundedness of $\|u(t)\|_\infty$, $\overline{\lim}_{T \to \infty} V(u(t+T)) > -\infty$, thus for $\tau < (1/B)$

$$\int_t^\infty \int_\Omega u_t^2 < \infty \qquad (23)$$

and the statement of the lemma holds with $\tau_0 = (1/B)$.

Remark. The appearance of the constant B is rather natural here, as it measures how substantially the nonlinearity f depends on the delayed values of u (recall its definition from Lemma 2); in particular, we regain estimate (21) for arbitrary τ if f depends on undelayed terms only.

Another Remark. Using (23) and letting $t \to \infty$ in (22) gives $\overline{\lim}\, V(u(t)) - \underline{\lim}\, V(u(t)) \leqslant 0$, that is, $V(u(t))$ stabilizes to some real number as $t \to \infty$. This fact is not, however, made use of below.

Lemma 4 suggests that the orbits under consideration must in some sense become stationary as $t \to \infty$. But since the equation of evolution is parabolic we in addition expect their limit points to be regular enough to satisfy the differential equation of stationary states. In the language of dynamical systems, this preservation of regularity at $t = \infty$ is called "orbit precompactness" and its verification is indeed straightforward here.

Lemma 3 (Orbit Precompactness). *Let τ be arbitrary. If $\overline{\lim}_{t \to \infty} \|u(t)\|_\infty < \infty$, then the orbit $\{\Phi(t) u_0\}_{t \geqslant 0}$ is precompact in X.*

Proof. Due to the compact imbedding (see, e.g., Ref. 10),

$$W^{1,2}((0, T) \times \Omega) \subsetneq C^0([0, T]; L^2(\Omega))$$

it suffices to show that u_t, ∇u remain bounded in $L^2((t-\tau, t)\times\Omega)$ as $t\to\infty$. To obtain the desired estimates, we proceed in exactly the same way as one would in the case of an undelayed parabolic equation. First, testing (2a) with u, applying Young's inequality, and integrating over t gives

$$\int_{t-\tau}^{t}\int_{\Omega}|\nabla u|^2\leqslant\tfrac{1}{2}\int_{\Omega}u^2(t-\tau)-\tfrac{1}{2}\int_{\Omega}u^2(t)+\tfrac{1}{2}\int_{t-\tau}^{t}\int_{\Omega}(u^2+f^2)$$

so that

$$\overline{\lim_{t\to\infty}}\int_{t-\tau}^{t}\int_{\Omega}|\nabla u|^2<\infty \tag{24}$$

Next, texting (2a) with u_t and applying Young's inequality yields a growth estimate for $\int_{\Omega}|\nabla u(t)|^2$:

$$\frac{1}{2}\int_{\Omega}u_t^2+\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla u|^2\leqslant\frac{1}{2}\int_{\Omega}f^2 \tag{25}$$

Combined with (24), this gives

$$\overline{\lim_{t\to\infty}}\int_{\Omega}|\nabla u(t)|^2<\infty \tag{26}$$

Finally, we integrate (25) over t and obtain by (26)

$$\overline{\lim_{t\to\infty}}\int_{t-\tau}^{t}\int_{\Omega}u_t^2<\infty \tag{27}$$

This completes the proof.

Remark. Obviously, orbits are also precompact in "higher" norms than the norm of X: For example, using the fact that $f\in W^{1,\infty}_{\text{loc}}(\mathbb{R}^2,\mathbb{R})$, we could differentiate (2a) for $t>\tau$ weakly with respect to t (thereby preserving the boundary condition), and analogous energy estimates as above yield boundedness of $\nabla u_t(t)$ in $L^2(\Omega)$ and of u_{tt} (thus also of Δu_t, D^2u_t) in $L^2(t-\tau, t; L^2(\Omega))$.

Proof of Theorem 1. Let u be any solution of (1), and assume $\overline{\lim}_{t\to\infty}\|u(t)\|_{\infty}<K$ and $\tau<\tau_0(K)$ as in Lemma 2. According to Lemma 3, the associated orbit $\{\Phi(t)u_0\}_{t\geqslant 0}$ is precompact and thus possesses a nonempty ω-limit set

$$\omega(u_0):=\begin{array}{l}\text{closure with respect to the topology of } X\\ \text{of } \bigcap_{T>0}\bigcup_{t>T}\Phi(t)u_0\end{array}$$

by which it is attracted as $t \to \infty$, i.e.,

$$\operatorname{dist}_X(\Phi(t)u_0, \omega(u_0)) \to 0 \qquad (t \to \infty) \tag{28}$$

Therefore, we need to show only that the set $\omega(u_0)$ consists of equilibria. We do not have a Lyapunov function so that the La Salle–Hale invariance principle in its standard form cannot be applied, but the decay estimate in Lemma 4 will do just as well. Take $v_0 \in \omega(u_0)$ and pick $t_i \to \infty$ such that $\Phi(t_i)u_0 \to v_0$ in X. Since $(d/ds)[\Phi(t_i)u_0](s) \to 0$ in $L^2((-\tau, 0) \times \Omega)$ by Lemma 4, v_0 lies in $W^{1,2}(-\tau, 0; L^2(\Omega))$, $\Phi(t_i)u_0 \to v_0$ in $W^{1,2}(-\tau, 0; L^2(\Omega))$, and

$$\frac{d}{ds} v_0(s) \equiv 0 \qquad (s \in (-\tau, 0))$$

Now is $\omega(u_0)$ positively invariant under Φ? This requires a little care, since the flow is not continuous with respect to the $L^\infty(\Omega)$-norm (even in the case of the plain heat equation $u_t - \Delta u = 0$, L^∞ initial data are not in general attained continuously in L^∞), thus ω-limit sets may not lie in the domain of definition X_∞ of the flow. However, the orbits under consideration here satisfy $\overline{\lim}_{t\to\infty} \|u(t)\|_\infty < \infty$, so their ω-limit sets do lie in X_∞ (since L^2-limits of a sequence of functions which is bounded in L^∞ must also lie in L^∞), and then they must trivially be positively invariant under Φ since Φ is continuous with respect to the topology of X. Thus the above arguments for v_0 also apply to $\Phi(t)v_0$, and we see that

$$\frac{d}{ds} v(s) \equiv 0 \qquad (s \in \mathbb{R}^+) \tag{29}$$

where $v(s)$ is the solution of (2) with initial data v_0. In other words, the constant state attained by $v(s)$ is an equilibrium state. Finally, combining (28) and (29) and retranslating them into a statement about $u(t)$ gives

$$\sup_{s \in [t-\tau, t]} \operatorname{dist}_{L^2(\Omega)}(u(s), E) \to 0 \qquad (t \to \infty)$$

as claimed. In fact, the remark following the proof of Lemma 3 gives slightly more:

$$\operatorname{dist}_{W^{1,2}(\Omega)}(u(s), E) \to 0 \qquad (t \to \infty)$$

The proof of Theorem 1 is now complete.

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