SPACES OF PIECEWISE-CONTINUOUS ALMOST-PERIODIC FUNCTIONS AND ALMOST-PERIODIC SETS ON THE REAL LINE. II

A. M. Samoilenko and S. I. Trofimchuk

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The authors consider a series of spaces of piecewise-continuous almost periodic functions and study the properties of the elements of these spaces. The theory developed in the paper is then applied to investigate almost periodic linear pulse systems.

1. In this paper, which is a sequel to [1], we study spaces of piecewise-continuous almost periodic (p.c.a.p.) functions.

1.1. We consider one general construction (all the notation is taken from [1]). Let $T \in \mathfrak{A}$, and arrange the real numbers of T in a strictly increasing sequence $\{t_n\}$. The set $s(T) = \{t_n\}$ will be called the support of T. Thus, we have defined a map $s:\mathfrak{A} \to \mathfrak{A}$. The set $s(\mathfrak{A})$ is obviously invariant under the map θ_s, $\theta_s(T) = T + s$.

We introduce a Hausdorff metric χ in $s(\mathfrak{A})$; if $P,\ Q \in s(\mathfrak{A})$ and $F_a(P)$ is a closed a-neighborhood of the set P (in the usual topology of R), then

$$\chi(P, Q) = \inf\{a: F_a(P) \supset Q,\ F_a(Q) \supset P\}$$

(χ may take the value $+\infty$ for certain P, Q).

Let $\mathfrak{G}$ be a subset of $\mathfrak{A}$ such that $\mathfrak{G} \supset s(\mathfrak{A})$ and $\theta_s(\mathfrak{G}) = \mathfrak{G}\ \forall s \in R$. Let δ denote a metric on $\mathfrak{G}$ with the following properties:

a1) $\delta(\theta_s(T),\ \theta_s(Q)) = \delta(T, Q)\ \forall s$;

a2) $\chi(T, Q) \leqslant \delta(T, Q)$;

a3) $\delta(\theta_s(Q), Q) \leqslant |s|\ \forall s,\ Q$.

In addition, we will need a commutative binary operation in $\mathfrak{G}$ — the sum of sets $\sqcup$: $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, with the following properties:

b0) $(T \sqcup P) + a = (T + a) \sqcup (P + a)$;

b1) $s(T \sqcup P) = s(T) \cup s(P)$;

b2) $\delta(T_1 \sqcup T_2,\ P_1 \sqcup P_2) \leqslant \max(\delta(T_1, P_1),\ \delta(T_2, P_2))$.

The map θ_s: $\mathfrak{G} \times R \to \mathfrak{G}$ defines a continuous dynamical system in $(\mathfrak{G}, \delta)$ (because, by conditions a1, a3, we have

$$\delta(\theta_s(Q), \theta_r(T)) \leqslant |s - r| + \delta(Q, T)).$$

Examples of spaces $(\mathfrak{G}, \delta, \sqcup, \theta_s)$ are the space $(\mathfrak{A}, \rho, \sqcup, \theta_s)$ studied in [1] (recall that in that paper $\sqcup$ denoted the free union of sets [2]) and the space $(s(\mathfrak{A}),\ \chi,\ \cup, \theta_s$ ($\cup$ denotes union of sets).

As done in [1], we can define in $(\mathfrak{G}, \delta, \theta_s)$ the notions of a.p. sets in Bohr's sense (briefly: δ-a.p. sets in Bohr's sense) and a.p. sets in Bochner's sense (δ-a.p. sets in Bochner's sense). An important distinction between the present, general situation and that considered in [1] is that these notions need not coincide, since $(\mathfrak{G}, \delta)$ is not complete.

Example. Consider the following set in the space $(s(\mathfrak{A}), \chi,\ \cup, \theta_s)$:

$$T = \bigcup_{k=0}^{+\infty} (2^k \mathbb{Z} + 2^{-2k}); \qquad 2^k \mathbb{Z} + 2^{-k} = \{2^k n + 2^{-k},\ n \in \mathbb{Z}\}.$$

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Taking into account that $\chi(T + 2^k m, T) = 2^{-(k+1)}$ for arbitrary k, m, that the set $\{2^k m\}$ is relatively dense in R for fixed $k, m \in \mathbb{Z}$, and that $\lim_{k\to+\infty} 2^{-(k+1)} = 0$, we see that T is a χ-a.p. set in Bohr's sense. On the other hand, it is easy to verify that, starting from the sequence $\{T + 2^{2^k}\}_{k=0}^{+\infty}$, it is not always possible to extract a subsequence that converges in $(s(\mathfrak{A}), \chi)$, and therefore T cannot be χ-a.p. in Bochner's sense.

Nevertheless, a look at the proof of Theorem 1 in [1] will show that δ-almost periodicity in Bochner's sense always implies δ-almost periodicity in Bohr's sense [$(\mathfrak{G}, \delta)$ does not have to be complete for this to be true].

1.2. Fix some space $(\mathfrak{G}, \delta, \dashv, \theta_s)$. Let $x(t)$: $R \to R$ be a p.c. function with discontinuities of the first kind (possibly zero) over a set $T \in \mathfrak{G}$. To fix ideas, let us assume that x(t) is left continuous.

Definition 1. The pair $X = (x(t), T)$ will be called δ-a.p.p.c. in Bohr's sense if

Γ_1) for every $\varepsilon > 0$, there exists a set Ω_ε of numbers τ, relatively dense in R, such that

$$|x(t+\tau) - x(t)| < \varepsilon \quad \forall t \in R \setminus F_\varepsilon(s(T))$$

[the elements of Ω_ε will be called ε-a. periods of x(t)];

Γ_2) T is a δ-a.p. set in Bohr's sense;

Γ_3) $\forall a > 0$ the function $x(t)$: $R \setminus F_a(S(T)) \to R$ is uniformly continuous.

The set of all δ-a.p.p.c. functions in Bohr's sense will be denoted by $AP(\mathfrak{G}, \delta)$.

We claim that, given a pair X, we can choose the set Ω_ε in such a way that any $\tau \in \Omega_\varepsilon$ will also be an ε-almost period of the δ-a.p. set T in Bohr's sense.

LEMMA 1. Let T be a δ-a.p. set in Bohr's sense. Then $\forall \eta > 0\ \exists L(\eta) > 0$ such that for every positive $\gamma < \eta$ and every interval J of length $L(\eta)$, there exists an integer m such that $m\gamma \in J$ and

$$\delta(T, T + m\gamma) < \eta. \tag{1}$$

Proof. Indeed, by the definition of a δ-a.p. set in Bohr's sense, it follows that $\forall \eta > 0\ \exists L_1(\eta/2) > 0$ such that in any interval (a, b) of length $L_1(\eta/2)$ there exists ω such that $\delta(T, T + \omega) < \eta/2$. Let $L(\eta) = L_1(\eta/2) + \eta$, and, in any interval (α, β) of length $L(\eta)$, pick out a subinterval $(\alpha + \eta/2, \beta - \eta/2)$ of length $L_1(\eta/2)$ with the above ω, such that $\Delta = [\omega - \eta/2, \omega + \eta/2] \subset (\alpha, \beta)$. Obviously, for positive $\gamma < \eta$ it follows from the Dirichlet principle that there exists m for which $m\gamma \in \Delta \subset (\alpha, \beta)$. At the same time, $\delta(T, T + m\gamma) \leqslant \delta(T, T + \omega) + |\omega - m\gamma| < \eta$.

LEMMA 2. Let $X = (x(t), T)$ be a δ-a.p.p.c. function in Bohr's sense. Then $\forall \eta > 0\ \exists L(\eta) > 0$, $\delta(\eta) > 0$ such that, for every positive $\gamma < \delta(\eta)$ and every interval J of length $L(\eta)$, there is an integer m such that $m\gamma \in J$ and

$$|x(t + m\gamma) - x(t)| < \eta \quad \forall t \in R \setminus F_\eta(s(T)). \tag{2}$$

Proof. By condition Γ_1, $\forall \eta > 0\ \exists L_1(\eta/2) > 0$ such that in every interval of length $L_1(\eta/2)$ there exists τ such that

$$|x(t+\tau) - x(t)| < \eta/2 \quad \forall t \in R \setminus F_{\eta/2}(s(T)).$$

The function x(t) is uniformly continuous on $R \setminus F_{\eta/2}(s(T))$. Let $\delta(\eta) < \eta/2$ be a positive number such that for any $t', t'' \in R \setminus F_{\eta/2}(s(T))$, $|t' - t''| < \delta(\eta)$ (in that case t', t'' must belong to the same interval in $R \setminus F_{\eta/2}(s(T))$), we have $|x(t') - x(t'')| < \eta/2$. Let $L(\eta) = L_1(\eta/2) + \delta(\eta)$. Take any interval (α, β) of length $L(\eta)$. The subinterval $(\alpha + \delta/2, \beta - \delta/2)$ of length $L_1(\eta)$ contains τ such that $|x(t+\tau) - x(t)| < \eta/2\ \forall t \in R \setminus F_{\eta/2}(s(T))$. For a positive $\gamma < \delta(\eta)$, there must be an integer m such that $m\gamma \in [\tau - \delta/2, \tau + \delta/2] \subset (\alpha, \beta)$. Moreover, if $t \in R \setminus F_\eta(s(T))$ and we define $t' = t + m\gamma - \tau$, then $|t' - t| = |m\gamma - \tau| < \delta/2 < \eta/4$, and therefore $t' \in R \setminus F_{\eta/2}(s(T))$ and t', t are in the same interval in $R \setminus F_{\eta/2}(s(T))$. Consequently, if $t \in R \setminus F_\eta(s(T))$, then $|x(t + m\gamma) - x(t)| \leqslant |x(t) - x(t + m\gamma - \tau)| + |x(t + m\gamma - \tau) - x(t + m\gamma)| < \eta$.

LEMMA 3. Let the pair $X = (x(t), T)$ be δ-a.p.p.c. Then $\forall \eta > 0\ \exists L(\eta) > 0$, $\delta(\eta) > 0$ such that for every positive $\gamma < \delta(\eta)$ and very interval J of length $L(\eta)$ there exists an integer m such that $m\gamma \in J$ and both inequalities (1) and (2) hold.

Proof. The proof follows [3, 4] (Appendix, Sec. 6). Let $L_1(\eta/8)$, $L_2(\eta/8)$, $\delta(\eta/8)$ be the constants existing according to Lemmas 1 and 2 for $\eta/8 > 0$, $L = \max(L_1, L_2)$. For any interval of length $L(\delta)$ and every positive number $\gamma < \delta(\eta/8)$, there exist integers m, m' such that the interval contains points $m\gamma$, $m'\gamma$ with

$$\delta(T+m\gamma, T)<\eta/8, \quad |x(t+m'\gamma)-x(t)|<\eta/8 \quad \forall t\in R\setminus F_{\eta/8}(s(T)). \tag{3}$$

Since $|m\gamma - m'\gamma|$, the differences $m - m'$ may take only finitely many values n_i, $i = 1,\dots,p$. For each n_i there exists a pair (m_i, m_i') satisfying (3) – fix this pair once and for all. Define $\lambda=\max_i|m_i'\gamma|$, $l=L+2\lambda$. Let $J = (\alpha, \alpha + l)$ be any interval of length l; then, by the foregoing reasoning, the subinterval $J''=(\alpha+\lambda, \alpha+\lambda+L)$ will contain numbers $m\gamma$, $m'\gamma$ such that (3) is true and also $m\gamma-m'\gamma=n_i\gamma=(m_i-m_i')\gamma$. Let $q = m - m_i$; then obviously $q\gamma\in J$. Further, we have

$$\delta(T+q\gamma, T)=\delta(T+(m-m_i)\gamma, T)\leqslant\delta(T+m\gamma, T)+\delta(T+m\gamma,$$
$$T+m\gamma-m_i\gamma)=\delta(T+m\gamma, T)+\delta(T, T+m_i\gamma)<\eta/4.$$

We now show that if $t\in R\setminus F_\eta(s(T))$, then $t+q\gamma\in R\setminus F_{\eta/2}(s(T))$. Indeed, otherwise it would follow that, for some $s\in T$, $|t+q\gamma-s|\leqslant\eta/2$. Taking into account that $\delta(T + q\gamma, T) < \eta/4$, we see that for this s, by property a2 of the metric δ, there exists a number $r\in T$ such that $|s-(r+q\gamma)|<\eta/2$. Finally, we obtain $|t-r|=|t+q\gamma-s+(s-(r+q\gamma))|=|t+q\gamma-s|+|s-(r+q\gamma)|<\eta$ and therefore $t\in F_\eta(s(T))$ and $t\notin R\setminus F_\eta(s(T))$. Consequently, for $t\in R\setminus F_\eta(s(T))$, we have the following chain of inequalities:

$$|x(t+q\gamma)-x(t)|=|x(t+(m'-m_i')\gamma)-x(t)|=|x(t+m'\gamma-m_i'\gamma)-$$
$$-x(t+m'\gamma)|+|x(t+m'\gamma)-x(t)|<\eta/8+\eta/8<\eta.$$

COROLLARY. The pair $X = (x(t), T)$ is a δ-a.p.p.c. function in Bohr's sense if and only if: Γ) for every $\varepsilon > 0$ there exists a set Ω_ε of numbers τ, relatively dense in R, such that $\delta(T + \tau, T) < \varepsilon$ and

$$|x(t+\tau)-x(t)|<\varepsilon \quad \forall t\in R\setminus(F_\varepsilon(s(T))\cup F_\varepsilon(s(T-\tau))).$$

Indeed, suppose that condition Γ is fulfilled. Then T is obviously δ-a.p. Fix an arbitrary $\varepsilon > 0$. By condition Γ there exists a relatively dense set $\Omega_{\varepsilon/4}$ such that for $\tau\in\Omega_{\varepsilon/4}$

$$|x(t+\tau)-x(t)|<\varepsilon/4 \quad \forall t\in M=R\setminus(F_{\varepsilon/4}(s(T))\cup F_{\varepsilon/4}(s(T-\tau))),$$
$$\delta(T+\tau, T)<\varepsilon/4.$$

But then $\chi(T+\tau, T)\leqslant\varepsilon/4$ and therefore $F_{\varepsilon/2}(s(T-\tau))\subset F_\varepsilon(s(T))$. Consequently, $F_{\varepsilon/4}(s(T-\tau))\cup F_{\varepsilon/4}(s(T))\subset F_\varepsilon(s(T))$ and if $t\in R\setminus F_\varepsilon(s(T))$, then $t\in M$. Finally, we see that for $\tau\in\Omega_{\varepsilon/4}$

$$|x(t+\tau)-x(t)|<\varepsilon/4<\varepsilon \quad \forall t\in R\setminus F_\varepsilon(s(T)).$$

We now prove the validity of condition Γ_3 (weak uniform continuity). Fix $a > 0$. By the foregoing reasoning, for any positive $\varepsilon < a$ there exists a number $L(\varepsilon/4) > 0$ such that $\forall\alpha$ $\exists\tau\in(\alpha, \alpha+L(\varepsilon/4))$:

$$\delta(T+\tau, T)<\varepsilon/4,$$
$$|x(t+\tau)-x(t)|<\varepsilon/4 \quad \forall t\in R\setminus F_{\varepsilon/4}(s(T)).$$

Consider the p.c. function $x(t)$ over the interval $[-\varepsilon, L(\varepsilon/4)]$: for any $\varepsilon > 0$ one can find a positive number $\delta(\varepsilon) < \varepsilon/2$ such that $\forall t', t''\in[0, L(\varepsilon/4)]$, $|t'-t''|<\delta(\varepsilon)$ we have $|x(t')-x(t'')|<\varepsilon/2$ [for $(t', t'')\cap T=\varnothing$]. Now let the numbers $t_1, t_2\in R\setminus F_a(s(T))$ be such that $|t_1-t_2|<\delta(\varepsilon)$ [in that case t_1 and t_2 $(t_1\leqslant t_2)$ lie in the same interval of $R\setminus F_a(s(T))$].

If $t_i\in R\setminus F_a(s(T))\subset R\setminus F_\varepsilon(s(T))$, then by the inequality $\chi(T+\tau, T)\leqslant\delta(T+\tau, T)<\varepsilon/4$ we must have $t_i-\tau\in R\setminus F_{\varepsilon/4}(s(T))$, and therefore $|x(t_1)-x(t_2)|\leqslant|x(t_1)-x(t_1-\tau)|+|x(t_1-\tau)-x(t_2-\tau)|+|x(t_2-\tau)-x(t_2)|<\varepsilon$, $((t_1-\tau, t_2-\tau)\cap T=\varnothing$, $\tau\in[t_2-L(\varepsilon/4), t_2]$, $\varepsilon<2L(\varepsilon/4))$.

Conversely, suppose that conditions Γ_1-Γ_3 are fulfilled. By Lemma 3, we may assume that the set Ω_ε consists of ε-a. periods common to T and $x(t)$, and the truth of the inequality $|x(t+\tau)-x(t)|<\varepsilon$ for all $t\in R\setminus F_\varepsilon(s(T))$ implies its truth for all $t\in(R\setminus F_\varepsilon(s(T)))\cap(R\setminus F_\varepsilon(s(T+\tau)))$.

Definition 2. Given functions $X = (x(t), T)$, $Y = (y(t), P)$, we can define multiplication by a scalar $\alpha X = (\alpha x(t), T)$; addition $X + Y = (x(t) + y(t), T - P)$; multiplication $X\cdot Y = (x(t)y(t), T - P)$; if $x(t)\neq 0$ $\forall t\in R$, then we can also define the quotient $Y/X = (y(t)/x(t), T - P)$.

THEOREM 1. The space $AP(\mathfrak{G}, \delta)$ is closed with respect to addition and multiplication. If $X \in AP(\mathfrak{G}, \delta)$ is such that $x(t) \neq 0 \ \forall t$ and $\forall a > 0 \ \exists \mu(a) > 0$ such that $|x(t)| \geq \mu(a) \ \forall t \in R \setminus F_a(s(T))$, then $X^{-1} = (x^{-1}(t), T) \in AP(\mathfrak{G}, \delta)$.

LEMMA 4 [weak boundedness of a function $X \in AP(\mathfrak{G}, \delta)$]. If $X = (x(t), T) \in AP(\mathfrak{G}, \delta)$, then $\forall a > 0 \ \exists M(a) > 0$ such that $|x(t)| < M(a) \ \forall t \in R \setminus F_a(s(T))$.

Proof. By Lemma 3, $\forall a > 0 \ \exists L(a/4) > 0$ such that any interval of length L(a/4) > 0 will contain a number τ for which

$$|x(t+\tau) - x(t)| < a/4 \ \forall t \in R \setminus F_{a/4}(s(T)); \ \delta(T+\tau, T) \leq \frac{a}{4}. \tag{4}$$

The (p.c.) function x(t) is bounded on [0, L(a/4)]: x(t) < m(a/4). Let $t \in R \setminus F_a(s(T))$. The interval (t − L(a/4), t) contains a number τ such that (4) is true, and then $s = t - \tau \in (0, L(a/4))$. We claim that $s \in R \setminus F_{a/2}(s(T))$; otherwise, there would be some $r \in T$ for which $|s - r| \leq a/2$, and by (4) this r determines a number $p \in T$ such that $|r + \tau - p| < a/2$, and therefore

$$|t - p| = |t - \tau - r + r + \tau - p| < a/2 + a/2 = a, \quad t \in F_a(s(T)).$$

Finally, $|x(s) - x(t)| = |x(t-\tau) - x(t)| < a/4, \quad |x(t)| \leq |x(s) - x(t)| + |x(s)| < a/4 + m(a/4) = M(a)$.

LEMMA 5 [uniform almost periodicity of a finite number of functions in $AP(\mathfrak{G}, \delta)$]. Let $X_i = (x_i(t), T_i) \in AP(\mathfrak{G}, \delta)$. Then for every $\varepsilon > 0$, i = 1,...,m, there exists a set Ω_ε of numbers τ, relatively dense in R, such that

$$|x_i(t+\tau) - x_i(t)| < \varepsilon \quad \forall t \in R \setminus F_\varepsilon(s(T_i))$$

and $\delta(T_i + \tau, T_i) < \varepsilon \ \forall i = 1, \ldots, m$.

Lemma 5 may be derived from Lemma 3 in exactly the same way as Lemma 3 from Lemmas 1 and 2 (obviously, the proof need be carried out only for the case m = 2).

Proof of Theorem 1. A (Addition). By Lemma 5, $\forall \varepsilon > 0 \ \exists L(\varepsilon/2) > 0$ such that any interval of length $L(\varepsilon/2)$ will contain a τ for which $\delta(T+\tau, T) < \varepsilon/2$, $\delta(P+\tau, P) < \varepsilon/2$; $|x(t+\tau) - x(t)| < \varepsilon/2 \ \forall t \in R \setminus F_{\varepsilon/2}(s(T))$; $|y(t+\tau) - y(t)| < \varepsilon/2 \ \forall t \in R \setminus F_{\varepsilon/2}(s(P))$. But then $\delta((T-P)+\tau, T-P) = \delta((T+\tau) - (P+\tau), T-P) \leq \max(\delta(T+\tau, T), \delta(P+\tau, P)) < \varepsilon$, and also $|x(t+\tau) + y(t+\tau) - (x(t) + y(t))| < \varepsilon \ \forall t \in (R \setminus F_{\varepsilon/2}(s(T) \cup s(P))) \supset R \setminus F_\varepsilon(s(T-P))$.

B (Multiplication). Fix an arbitrary number $\varepsilon > 0$. Considered over the set $R \setminus F_{\varepsilon/2}(s(T))$ $[R \setminus F_{\varepsilon/2}(s(P))]$, the function |x(t)| [respectively, |y(t)|] is bounded by $M(\varepsilon/2)$ by Lemma 4. Putting $\eta = \min\left(\frac{\varepsilon}{4}, \frac{\varepsilon}{2} M^{-1}\right)$ and using Lemma 5, we see that there exists a set Ω_η, relatively dense in R, such that $\delta(T+\tau, T) < \eta$, $\delta(P+\tau, P) < \eta$,

$$|x(t+\tau) - x(t)| < \eta \ \forall t \in R \setminus F_\eta(s(T)) \supset R \setminus F_\varepsilon(s(T)),$$

$$|y(t+\tau) - y(t)| < \eta \ \forall t \in R \setminus F_\eta(s(P)) \supset R \setminus F_\varepsilon(s(P)).$$

Further, if $t \in R \setminus F_\varepsilon(s(T))$, then $t + \tau \in R \setminus F_{\varepsilon/2}(s(T))$ (otherwise there would exist $p \in T$ for which $|t + \tau - p| \leq \varepsilon/2$ and $r \in T$ for which $|p - (r+\tau)| < 2\eta \leq \varepsilon/2$, so that finally $|t - r| = |t + \tau - p + p - (r+\tau)| \leq \varepsilon$). But then, for $\tau \in \Omega_\eta$, we have $|x(t+\tau) y(t+\tau) - x(t) y(t)| \leq |x(t+\tau)| \, |y(t+\tau) - y(t)| + |y(t)| \, |x(t+\tau) - x(t)| \leq \varepsilon \ \forall t \in (R \setminus F_\varepsilon(s)(T))) \cap (R \setminus F_\varepsilon(s(P))) = R \setminus F_\varepsilon(s(T-P))$.

C (Existence of X^{-1}). Fix an arbitrary number $\varepsilon > 0$. By assumption, considered on the set $R \setminus F_{\varepsilon/2}(s(T))$, the function |x(t)| is bounded away from zero: $|x(t)| \geq \mu(\varepsilon/2)$. Let $\eta = \min(\varepsilon/4, \varepsilon\mu^2)$. Then, since X is δ-a.p.p.c. there exists a set Ω_η, relatively dense in R, such that $\forall \tau \in \Omega_\eta \quad |x(t+\tau) - x(t)| < \eta \quad \forall t \in R \setminus F_\eta(s(T))$, $\delta(T+\tau, T) < \eta$. If $t \in R \setminus F_\varepsilon(s(T))$, then $t + \tau \in R \setminus F_{\varepsilon/2}(s(T))$ (see part B of this proof) and therefore, if $\tau \in \Omega_\eta$, then

$$|x^{-1}(t+\tau) - x^{-1}(t)| = |x(t+\tau) - x(t)| \cdot [|x(t) x(t+\tau)|]^{-1} < \eta\mu^{-2}(\varepsilon/2) \leq \varepsilon \ \forall t \in R \setminus F_\varepsilon(s(T)).$$

This completes the proof of Theorem 1.

1.3. Let x(t): R → R be a piecewise-continuous function with discontinuities of the first kind (possibly zero) at the points of a set $T \in \mathfrak{G}$. To fix ideas, let us assume that x(t) is left continuous. Consider the space $\mathfrak{M}$ of all such pairs X = (x(t), T).

Using condition Γ, it is not difficult to introduce a uniform topology on $\mathfrak{M}$ and, along the lines of the above construction, to define the concept of an a.p. function in Bohr's sense, ultimately obtaining a space $AP(\mathfrak{G}, \delta)$ — this is one of the specific features of the class of p.c.a.p. functions with which we are concerned.

Let a be a positive rational number,

$$U_\infty = \mathfrak{M}\times\mathfrak{M};\ U_a = \{(X, Y) = [(x(t), T);\ (y(t), P)] \in \mathfrak{M}\times\mathfrak{M} : \delta(T, P) < a,$$
$$|x(t) - y(t)| < a\ \ \forall t \in R\setminus \dot{F}_a(s(T-P))\}.$$

The following properties are obvious:

c1) $\forall a\ \ \forall X \in \mathfrak{M}\ (X, X) \in U_a$;

c2) if $(X, Y) \in U_a$, then also $(Y, X) \in U_a$ (i.e., $U_a^{-1} = U_a$);

c3) if $(X, Y);\ (Y, Z) \in U_{a/2}$, then $(X, Z) \in U_a$;

c4) $U_a \cap U_b = U_{\min(a,b)}$.

Therefore [5], the family of sets U_a forms the base of a uniformity $\mathcal{U}$ for $\mathfrak{M}$, so that $\mathfrak{M}$ is endowed with a uniform topology. In $(\mathfrak{M}, \mathcal{U})$ a sequence $X_n = (x_n(t), T_n)$ converges to an element $X = (x(t), T)$ if and only if, for every $\varepsilon > 0$, there exists n_0 such that, for all $n \geqslant n_0$, $\delta(T, T_n) < \varepsilon$,

$$|x_n(t) - x(t)| < \varepsilon\ \ \forall t \in R\setminus F_\varepsilon(s(T_n - T)).$$

It is a relatively simple matter to prove that $(\mathfrak{M}, \mathcal{U})$ is a Hausdorff uniform space (but not complete). $\mathfrak{M}$ is invariant with respect to translation: if $X = (x(t), T) \in \mathfrak{M}$, then also $\theta_s(X) = (x(t+s), T-s) \in \mathfrak{M}$. Moreover, if U_a is an element of the base of the uniformity $\mathcal{U}$, then for any $s \in R$

$$(\theta_s(X),\ \theta_s(Y)) \in U_a \Leftrightarrow (X, Y) \in U_a \quad \forall X, Y. \tag{5}$$

(We also observe that the map $\theta_s : \mathfrak{M}\times R \to \mathfrak{M}$ is not jointly continuous in all its arguments.)

The definition of a δ-a.p.p.c. function in Bohr's sense, given in the corollary to Lemma 3, may now be rephrased in our new terms:

<u>Definition 3.</u> A pair $X = (x(t), T) \in \mathfrak{M}$ is a δ-a.p.p.c. function in Bohr's sense if, for every element V of the uniformity $\mathcal{U}$, there exists a set of numbers τ, relatively dense in R, such that $(\theta_\tau(X), X) \in V$.

<u>Definition 4.</u> A pair X = (x(t), T) is a δ-a.p.p.c. function in Bochner's sense if, from every sequence $\{h_k'\}$ of real numbers, one can extract a subsequence $\{h_n\} = \{h_{k_n}'\}$ such that in $(\mathfrak{M}, \mathcal{U})$, for some $Y \in \mathfrak{M}$,

$$\lim_{n\to\infty} \theta_{h_n}(X) = Y.$$

<u>THEOREM 2.</u> A function $A \in \mathfrak{M}$ that is δ-a.p.p.c. in Bochner's sense is also δ-a.p.p.c. in Bohr's sense.

The converse need not be true. An example of a δ-a.p.p.c. function in Bohr's sense that is not δ-a.p.p.c. in Bochner's sense is the pair X = (0, T), where $T \in (s(\mathfrak{A}), \chi)$ is the set described in the example at the beginning of this paper.

<u>Proof.</u> By the Aleksandrov–Urysohn theorem [5, p. 248 of Russian translation], the uniform space $(\mathfrak{M}, \mathcal{U})$ is metrizable as a Hausdorff space with countable uniformity base, and by (5) and the proof of the metrization lemma in [5] we may assume that the metric d generated by $\mathcal{U}$ has the following invariance property:

$$d(\theta_s(X), \theta_s(Y)) = d(X, Y)\quad \forall s \in R\quad \forall X, Y \in \mathfrak{M}. \tag{6}$$

In what follows, $AP_h(\mathfrak{G}, \delta)$ will denote the subspace of $(\mathfrak{M}, d)$ consisting of all δ-a.p.p.c. functions in Bochner's sense.

Let $A \in AP_h(\mathfrak{G}, \delta);\ \Pi(A) = \{\theta_s(A),\ s \in R\}$. Obviously, Π(A) is invariant with respect to translation θ_s and, therefore, we can define a map $\theta_s : \Pi(A)\times R \to \Pi(A)$. Let $\omega(s) = d(\theta_s(A), A)$. We will prove that $\lim_{s\to 0}\omega(s) = 0$. Indeed, if this were not true, there would exist a sequence $s_n' \to 0$, $n \to +\infty$, such that $\omega(s_n') \geqslant \varepsilon_0 > 0\ \forall n$. Since $A \in AP_h(\mathfrak{G}, \delta)$, we can extract from $\{s_n'\}$ a subsequence $\{s_k\}$ such that for some $Y \in \mathfrak{M}$ $\lim_{k\to+\infty} \theta_{s_k}(X) = Y$. It is easily verified that Y = X and therefore $\lim_{k\to+\infty}\omega(s_k) = 0$, contrary to the inequality $\omega(s_k) \geqslant \varepsilon_0\ \forall k$.

The map θ_s: $\Pi(A)\times R\to\Pi(A)$ is uniformly continuous, because

$$d(\theta_s(X),\ \theta_r(Y))=d(\theta_{s-r}(X),Y)\leqslant d(\theta_{s-r}(X),X)+d(X,Y)=$$
$$=d(\theta_{s-r}(A),A)+d(X,Y)=\omega(s-r)+d(X,Y).$$

Let H(A) denote the closure of Π(A) in $(\mathfrak{M}, d)$. It is easy to verify that H(A) is invariant with respect to translation θ_s and, therefore, we can define the map θ_s: H(A) × R → H(A), which is also uniformly continuous. (If $X, Y\in H(A)$, then there exist sequences $\{X_n\}, \{Y_n\}\subset$ Π(A) such that $X_n\to X$, $Y_n\to Y$ as $n\to\infty$; since

$$d(\theta_s(X_n),\ \theta_r(Y_n))\leqslant\omega(s-r)+d(X_n,\ Y_n)\ \forall n,$$

passage to the limit gives

$$d(\theta_s(X),\ \theta_r(Y))\leqslant\omega(s-r)+d(X,Y)).$$

We now consider the continuous dynamical system (H(A), θ_s). Now all the arguments in Sec. 3 of [1] [other than those requiring the space (H(A), d) to be complete] go through for this system; we may therefore consider Theorem 2 proven. Moreover, on the same grounds, we arrive at the following conclusion:

THEOREM 3. $A\in AP_h(\mathfrak{G},\delta)$ if and only if the set $H(A)=\mathrm{Closure}\{\theta_s(A),\ s\in R\}$ is compact in $(\mathfrak{M},\mathcal{U})$. Under those conditions $\forall B\in H(A)$ we have H(B) = H(A) and so

$$H(A)\subset AP_h(\mathfrak{G},\delta)\subset AP(\mathfrak{G},\delta).$$

2.1. Definition 5. Let APH(R) denote the subspace of $AP(\mathfrak{A},\rho)$ consisting of all the functions that satisfy conditions H_0, H_1, H_2 of [1]. Let $(\mathfrak{R},\mathcal{U})$ denote the subspace of $(\mathfrak{M},\mathcal{U})$ consisting of all the functions that satisfy condition H_2 of [1].

THEOREM 4. $APH(R)=AP(\mathfrak{A},\rho)\cap\mathfrak{R}=AP_h(\mathfrak{A},\rho)\cap\mathfrak{R}$.

Proof. It will suffice to show that $AP(\mathfrak{A},\rho)\cap\mathfrak{R}\subseteq AP_h(\mathfrak{A},\rho)\cap\mathfrak{R}$. Let $A=(a(t),L)\in APH(R)$, $\Pi(A)=\{\theta_s(A),\ s\in R\}$. The set Π(A) is invariant with respect to θ_s, and so the map θ_s: $\Pi(A)\times R\to$ Π(A) is well defined. Set $\omega_A(s)=\inf\{a:(\theta_s(A),A)\in U_a\}$, $\sigma_A(|s|)=\sup|a(t')-a(t'')|$, where the supremum extends over all t' < t" such that $|t'-t''|\leqslant|s|$, $[t',t'']\cap L=\varnothing$. By condition H_2, $\lim\limits_{s\to 0}\sigma_A(|s|)=0$ and since $\omega_A(s)\leqslant\max(|s|,\ \sigma_A(|s|))$, it follows that also $\lim\limits_{s\to 0}\omega_A(s)=0$. The map θ_s: $\Pi(A)\times R\to\Pi(A)$ is uniformly continuous. Indeed, fix an arbitrary number ε > 0 and take δ > 0 such that, if $|s|\leqslant\delta$, then $\sigma_A(|s|)+|s|<\varepsilon/2$. Then for any $|t-r|<\delta$, $X, Y\in\Pi(A):(X,Y)\in U_{\varepsilon/2}$, we have $(\theta_{t-r}(A), A)\in U_{\varepsilon/2}\Leftrightarrow(\theta_t(X),\theta_r(X))\in U_{\varepsilon/2}$; $(X,Y)\in U_{\varepsilon/2}\Leftrightarrow(\theta_r(X),\ \theta_r(Y))\in U_{\varepsilon/2}$, and finally $(\theta_t(X),\ \theta_r(Y))\in U_\varepsilon$.

Let H(A) denote the closure of Π(A) in $(\mathfrak{M},\mathcal{U})$. We claim that $H(A)\subset\mathfrak{R}$. If B = (b(t), $K)\in H(A)$, there exists a real sequence $\{s_n\}$ such that $\theta_{s_n}(A)=(a(t+s_n),\ L-s_n)\to B$ as $n\to\infty$. Let t' < t" be numbers such that $[t',t'']\cap K=\varnothing$. There exists $\varkappa>0$ for which $[t'-\varkappa,\ t''+\varkappa]\cap K=\emptyset$; for this $\varkappa$, choose n_0 so that $\forall n\geqslant n_0$ $(\theta_{s_n}(A),B)\in U_{\varkappa/4}$. But then, if $n\geqslant n_0$, necessarily $[t'-\varkappa/4,\ t''+\varkappa/4]\cap(L-s_n)=\varnothing$, and so

$$|a(t'+s_n)-a(t''+s_n)|\leqslant\sigma_A(|t'-t''|)\quad\forall n\geqslant n_0.$$

Noting that the sequence of functions $a(t+s_n)$ converges uniformly in the interval $[t'-\varkappa/4,$ $t''+\varkappa/4]$ to b(t) as $n\to\infty$, $n\geqslant n_0$, we obtain $|b(t')-b(t'')|\leqslant\sigma_A(|t'-t''|)$, that is, $B\in\mathfrak{R}$. We have also shown that if $\sigma_B(|s|)=\sup|b(t')-b(t'')|$, where the supremum extends over all t' < t" such that $[t',t'']\cap K=\varnothing$, $|t'-t''|\leqslant|s|$, then $\sigma_B(|s|)\leqslant\sigma_A(|s|)$.

Now H(A) is invariant under translation θ_s, so the map θ_s: H(A) × R → H(A) is well defined. We will now show that it is uniformly continuous. For any $B\in H(A)$, let $\omega_B(s)=$ $\inf\{a:(\theta_s(B),B)\in U_a\}$: obviously, $\omega_B(s)\leqslant\max(|s|,\ \sigma_B(|s|))\leqslant\max(|s|,\ \sigma_A(|s|))$. Now fix any number ε > 0 and choose δ > 0 so that, if $|s|\leqslant\delta$, we have $\max(|s|,\ \sigma_A(|s|))<\varepsilon/2$. Then for any $|t-r|<$ δ, $X, Y\in H(A)$; $(X,Y)\in U_{\varepsilon/2}$, we have $(\theta_t(X),\ \theta_r(Y))\in U_{\varepsilon/2}$ and $(\theta_r(X),\ \theta_r(Y))\in U_{\varepsilon/2}$, so that $(\theta_t(X),\ \theta_r(Y))\in U_\varepsilon$.

Thus, we can consider the continuous dynamical system $(H(A),\theta_s,\mathcal{U})$. We claim that the space $(H(A),\mathcal{U})$ is complete. If $Y_n=(y_n(t),T_n)\in H(A)$ is a Cauchy sequence, then $T_n\in\mathfrak{A}$ is also a Cauchy sequence, and, since $(\mathfrak{A},\rho)$ is complete, there exists a set $T\subset\mathfrak{A}$ such that $T_n\to T$, $n\to\infty$. Let $\rho_n=\rho(T_n,T)\to 0$, $n\to\infty$; we may assume without loss of generality that $\rho_{n+1}<$ ρ_n $\forall n$. Now, considering the set $G_m=R\setminus F_{2\rho_m}(s(T))\subset R\setminus(F_{\rho_m}(s(T))\cup F_{\rho_m}(s(T_m)))$ and letting $n\geqslant m$, we can define a sequence of functions $y_n(t)$; $y_n(t)\in C(G_m)$, which is a Cauchy sequence; hence there exists a function $y(t)\in C(G_m)$ such that $y_n(t)$ converges to y(t) uniformly on G_m as

$n \to \infty$. In addition, for any t_1, t_2 that lie in the same interval of G_m, we have $|y_n(t_1) - y_n(t_2)| \leqslant \sigma_A(|t_1 - t_2|)$; hence also $|y(t_1) - y(t_2)| \leqslant \sigma_A(|t_1 - t_2|)$. Since $G_m \subset G_{m+1}$; $\bigcup_m G_m = R \setminus T$, there exists a function $y(t) \rightrightarrows [y(t)\colon R \setminus T \to R]$ such that $y_n(t) \rightrightarrows y(t)$ on each of the sets G_m as $n \to \infty$ and moreover $y(t)$ satisfies condition H_2 of [1]. Extending the definition of $y(t)$ to the points of $s(T)$ so that it becomes left continuous, we obtain $Y = (y(t), T) \in \mathfrak{N}$. It is not hard to see now that $\lim_{n\to\infty} Y_n = Y$; this, since $H(A)$ is closed, implies that the space $(H(A), \mathcal{U})$ is indeed complete.

Theorem 4 now follows from Theorem 1 of [1] and the remark just before formula (6).

2.2. We now point out the most important properties of the space $APH(R) \subset AP(\mathfrak{A}, \rho)$.

A. APH(R) is closed with respect to addition and multiplication by scalars.

This follows from Theorem 1 and the fact that if $X, Y \in \mathfrak{N}$, then $X + Y \in \mathfrak{N}$, $\alpha X \in \mathfrak{N}$ $\forall \alpha \in R$.

B. By Lemma 4, a function $X \in APH(R)$ is weakly bounded. There exists an unbounded function $A \in APH(R)$ (compare Theorem 23.1 in [3]).

Indeed, let $T = \{n\} \cup \{n + c_n\}_{n=-\infty}^{+\infty}$, where $\{c_n\}$ is an a.p. sequence such that $0 < c_n < 1/4$ and $\inf_n c_n = 0$; let $a(t)\colon R \to R$ be a function such that

$$a(t) = \begin{cases} (t - n) + c_n^{-1}, & t \in (n, n + c_n]; \\ 0, & t \in (n + c_n,\ n + 1]. \end{cases}$$

Obviously, $(a(t), T) \in \mathfrak{N}$, the function $a(t)$ is unbounded [since $a(n + c_n) = c_n^{-1} + c_n$] and, in addition, T is a ρ-a.p. set [1]. We will show that A satisfies condition Γ_1. Fix an arbitrary number $\varepsilon < 0.25$. The sequence $\{q_n\}$, $q_n = \max\{c_n, \varepsilon\}$, will be a.p., and, since $q_n \geqslant \varepsilon > 0$, the sequence $\{q_n^{-1}\}$ will also be a.p. Let Ω_ε be the set of $\varepsilon/2$-a. periods common to $\{q_n^{-1}\}$ and $\{c_n\}$. We claim that if $p \in \Omega_\varepsilon$, then

$$|a(t + p) - a(t)| < \varepsilon/2 \quad \forall t \in R \setminus F_\varepsilon(T). \tag{7}$$

Indeed, if $t \in R \setminus F_\varepsilon(s(T))$, then either 1) $t \in (m + c_m + \varepsilon,\ m + 1 - \varepsilon]$ for some m and then, in view of the inequality $|c_{m+p} - c_m| < \varepsilon/2$, we have $t + p \in (m + p + c_m + \varepsilon,\ m + p + 1 - \varepsilon] \subset (m + p,\ m + p + c_{m+p}]$, so that (7) is true; or 2) $t \in (m + \varepsilon,\ m + c_m - \varepsilon]$ and therefore $c_m > 2\varepsilon$. In that case also $t + p \in (m + p + \varepsilon,\ m + p + c_m - \varepsilon] \subset (m + p,\ m + p + c_{m+p}]$ and $c_{m+p} \geqslant c_m - |c_m - c_{m+p}| > \varepsilon$, so that $q_m^{-1} = c_m^{-1}$, $q_{m+p}^{-1} = c_{m+p}^{-1}$. Finally, we obtain

$$|a(t) - a(t + p)| = |c_m^{-1} - c_{m+p}^{-1}| = |q_m^{-1} - q_{m+p}^{-1}| < \varepsilon/2.$$

Consequently, $A \in APH(R)$.

C. 1. The space APH(R) is not closed with respect to multiplication. 2. If $A, B \in APW(R)$, then $AB \in APW(R)$. 3. If $A \in APH(R)$, $|a(t)| \geqslant \mu > 0$ $\forall t$, then $A^{-1} \in APW(R)$.

Let us prove, say, 1). Let A be the function constructed in the previous example. Then $A^2 \notin \mathfrak{N}$, since $a^2(n + c_n) - a^2(n) = 2 + c_n^2 > 2$.; but $\inf_n c_n = 0$, $(n + c_n, n) \cap T = \varnothing$.

D. 1. Properties H_0, H_1, H_2, H_3 of [1] are independent of one another.

2. From a geometrical point of view, it is convenient to identify δ-a.p.p.c. functions $A = (a(t), T)$ and $B = (a(t), P)$ for which $s(T) = s(P)$. This identification is also important because, otherwise, the distributive law may not hold formally in $AP(\mathfrak{S}, \delta)$. 3. A matrix-valued function $M = \{M_{ij}\}$, $i = 1,\dots,n$; $j = 1,\dots,m$, is said to be δ-a.p.p.c. if $M_{ij} = (m_{ij}(t), T_{ij}) \in AP(\mathfrak{S}, \delta)$ $\forall i, j$. In view of Lemma 5, given any such function and an arbitrary $\varepsilon > 0$, there exists a set Ω_ε of numbers τ, relatively dense in R, such that

$$\delta(T, T + \tau) < \varepsilon, \ \|M(t + \tau) - M(t)\| < \varepsilon \ \ \forall t \in R \setminus F_\varepsilon(s(T)),$$

where $T = \bigcup_{i,j} T_{ij}$, $M(t) = \{m_{ij}(t)\}$, $\|\cdot\|$ is some matrix norm.

E. 1. Let $B = (b(t), T) \in AP(\mathfrak{S}, \delta)$; $\lambda_\varepsilon(t) = \operatorname{mes}(F_\varepsilon(s(T)) \cap [t, t + 1])$. Suppose that $b(t)$ is bounded, $|b(t)| \leqslant m$ $\forall t$, and $\lambda_\varepsilon(t) \to 0$ as $\varepsilon \to 0+$ uniformly in $t \in R$. Then $b(t)$ is an S-a.p. Stepanov function and therefore, if $b(t)$ is uniformly continuous over R, then $b(t)$ is an a.p. Bohr function. In particular, all functions in APW(R) are S-a.p.

2. APH(R) contains functions that have no mean (in the usual sense, and they are therefore not a.p. in Weyl's sense).

Proof. 1. Fix an arbitrary positive number $\varepsilon < 0.5$ and an ε-a. period τ of B. Then

$$|b(t) - b(t+\tau)| < \varepsilon \quad \forall t \in R \setminus F_\varepsilon(s(T)),$$

and if $\Phi' = F_\varepsilon(s(T)) \cap [t, t+1]$, $\Phi'' = [t, t+1] \setminus \Phi'$, then

$$\int_t^{t+1} |b(u) - b(u+\tau)|\, du = \int_{\Phi'} |\ldots|\, du + \int_{\Phi''} |\ldots|\, du \leqslant 2m\lambda_\varepsilon(t) + \varepsilon.$$

2. Let $T = \{n\} \cup \left\{n + \frac{1}{2}\cos^2 n\right\}$ and define a function f(t): R → R by

$$f(t) = \begin{cases} 0, & t \in (n + 1/2\cos^2 n,\ n+1]; \\ 2|\cos n|^{-4}, & t \in (n, n + 1/2\cos^2 n]. \end{cases}$$

As in the case of B, one shows that $(f(t), T) \in APH(R)$. Further,

$$\frac{1}{n+1} \int_0^{n+1} f(u)\, du = \frac{1}{n+1} \sum_{k=0}^{n} \cos^{-2} k,$$

but the limit of the expression on the right as $n \to \infty$ is $+\infty$ (this is readily proved with the help of Birkhoff's ergodic theorem).

3. In conclusion, we consider the question of a.p.p.c. solutions in APW(R) for pulsed systems.

The method we propose here, in principle, reduces the investigation of an a.p. pulsed system to the thoroughly studied continuous extension of a minimal system of the translation type on a metric compact space, while the results themselves are obtained in terms of pulsed systems. For simplicity's sake, we confine attention to linear systems

$$\begin{aligned} dx/dt &= A(t)x + a(t), \\ \Delta x|_{t_i} &= F_i x + f_i, \end{aligned} \tag{8}$$

where $i \in \mathbb{Z}$, $t_i \leqslant t_{i+1}\ \forall i$; f_i, $x \in R^n$; $F_i \in M_n(R)$, $T = \{t_i\} \in \mathfrak{B}$, $s(T) = \{\tau_j\}$; $\mathrm{pr}: T \to s(T)$ is the canonical map of T onto the quotient set s(T). It may happen that $T \neq s(T)$, in which case (8) will be an abbreviated notation for the pulsed system

$$\begin{aligned} dx/dt &= A(t)x + a(t), \\ \Delta x|_{\tau_j} &= \Phi_j x + \varphi_j, \end{aligned}$$

where $(E + \Phi_j)x + \varphi_j = H_{i_0+k} \circ \ldots \circ H_{i_0}(x)$, $H_i(x) = (E + F_i)x + f_i$, $\{t_{i_0+k}, \ldots, t_{i_0}\} = \mathrm{pr}^{-1}\{\tau_j\}$.

It is natural to consider the translation of (8) for arbitrary $s \in R$:

$$\begin{aligned} dx/dt &= A(t+s)x + a(t+s), \\ \Delta x|_{t_i - s} &= F_i x + f_i. \end{aligned} \tag{9}$$

The sequence $\{t_i - s\}$ is conveniently reindexed so that the resulting sequence $\{t_i(s)\}$ lies in $\mathfrak{B}$; in that case, for some integer k(s), we have $t_i(s) = t_{i+k(s)} - s\ \forall i$, and (9) becomes

$$\begin{aligned} dx/dt &= A(t+s)x + a(t+s), \\ \Delta x|_{t_i(s)} &= F_{i+k(s)} + f_{i+k(s)}. \end{aligned}$$

Definition 6. System (8) is said to be ρ-a.p. if A(t) and a(t) are S-a.p. functions, $\{f_i\}$ and $\{F_i\}$ are a.p. sequences, T is a ρ-a.p. set.

Given functions $(A(t), a(t), 0)$: $R \to M_n(R) \times R^n \times R$ and a vector $(F_n, f_n, 1)$, we introduce the notation L(t) and L_n.

If (8) is a ρ-a.p. system, then any sequence of real numbers $\{s'_k\}$ will contain a subsequence $\{s_m\} = \{s'_{k_m}\}$ such that in S and ℓ_∞, respectively,

$$\begin{aligned} &L(t + s_m) \to L^*(t) = (A^*(t), a^*(t), 0), \\ &\{L_{n+k(s_m)}\} \to \{L_n^*\} = \{(F_n^*, f_n^*, 1)\}, \ \{t_n(s_m) - t_n^*\} \to 0. \end{aligned} \tag{10}$$

The systems obtained by all such limit passages

$$dx/dt = A^*(t)x + a^*(t),$$

$$\Delta x|_{t_i^*} = F_i^* x + f_i^* \tag{11}$$

will be called the H-class of system (8) (after passing to the limit we may have $\{t_i^*\} \notin \mathfrak{B}$, so that it will be necessary to shift all the indices of the sequence again). It follows from the results of [1] and the properties of Stepanov a.p. functions that an H-class is uniquely defined by any of its representatives. After the necessary definitions, one can speak of compactness of an H-class and of the fact that (11) defines a linear extension over a minimal two-sided stable system of translations acting on H; it should be observed, however, that the linear extension will not define a continuous semigroup on $H \times R^n$, since the solutions of (11) need not be continuous ("semigroup" – because the matrices $E + F_i$ may be singular).

3.1. Let us assume from now on that system (8) is ρ-a.p. By [1], $t_n = an + c_n$, where $\{c_n\}$ is an a.p. sequence. Define $d_n = (a + 1)n + c_n$, then $D = \{d_n\}$ is a ρ-a.p. set, $d_{n+1} - d_n \geqslant 1 \ \forall n$.

Definition (see [6, 7]). A p.c. function $x(t)$: $R \to R$ with discontinuities of the first kind on T will be called an N – ρ-a.p.p.c. Levitan function if

1) $\forall \varepsilon > 0$ and $\forall N > 0$ there exists a relatively dense set $\Omega_{\varepsilon,N}$ of ε – N-a. periods τ (that is, numbers such that

$$|x(t \pm \tau) - x(t)| \leqslant \varepsilon \ \ \forall t \in (R \setminus F_\varepsilon(s(T))) \cap [-N, N]);$$

2) $\forall \varepsilon > 0 \ \ \forall N > 0 \ \ \exists \eta(\varepsilon, N) > 0$: $\Omega_{\eta,N} \pm \Omega_{\eta,N} \subset \Omega_{\varepsilon,N}$.

Our aim is to prove the following propositions, which are analogues for a.p. pulsed systems of theorems of Favard and Levitan, respectively.

THEOREM 5. If no homogeneous pulsed system in the H-class of system (8) has nontrivial bounded solutions, then a bounded solution $x^*(t)$ of any system (11) will be an element of APW(R) [to be precise: $(x^*(t), T^*) \in APW(R)$].

THEOREM 6. If the only bounded solution of a homogeneous pulsed system is trivial, then any bounded solution of the corresponding nonhomogeneous ρ-a.p. system (8) is an N – ρ-a.p.p.c. Levitan function.

To prove these theorems, we will need the following results, presented here without proof.

LEMMA 6. If f(x) is S-a.p., then $\forall \varepsilon > 0 \ \exists \delta > 0$:

$$\sup_{x \in R} \int_x^{x+\delta} |f(u)|\, du < \varepsilon.$$

LEMMA 7. Let $f(t)$: $R \to R$ be an a.p. function in Bohr's sense (an N-a.p. function in Levitan's sense), $\varphi(t) = f(t + n)$ for $t \in (t_n, t_{n+1}]$. Then $(\varphi(t), T) \in APW(R)$, $((\varphi(t), T)$ is an N – ρ-a.p.p.c. function in Levitan's sense).

Define a function $\mathscr{L}(t): R \to M_n(R) \times R^n \times R$:

$$\mathscr{L}(t) = \begin{cases} L_{n+1}, & t \in (d_{n+1} - 1, d_{n+1}]; \\ L(t - n), & t \in (d_n, d_{n+1} - 1]. \end{cases} \tag{12}$$

Obviously, given system (8) we can construct $\mathscr{L}(t)$, and conversely.

LEMMA 8. If (8) is a ρ-a.p. system, then $\mathscr{L}(t)$ is an S-a.p. function.

LEMMA 9. Let $\mathscr{L}^*(t)$ lie in the H-class $H(\mathscr{L})$ of $\mathscr{L}(t)$. Then (10) will be true for some sequence $\{s_m\}$, and if $d_n^* = t_n^* + n$ then, replacing $\mathscr{L}$, L and d in (12) by $\mathscr{L}^*$, L^* and d^*, we obtain an expression for $\mathscr{L}^*(t)$.

Lemma 9 shows that each function $\mathscr{L}^* \in H(\mathscr{L})$ uniquely determines a certain ρ-a.p. system in the H-class of system (8).

Proof of Theorems 5 and 6, following [7], Chap. VII. Consider the trivial bundle $(H(\mathscr{L}) \times R^n, \pi, H(\mathscr{L}))$. Let $\hat{x}(t)$ be a bounded solution of (8) and

$$x(t) = \begin{cases} \hat{x}(t - n), & t \in (d_n, d_{n+1} - 1]; \\ \hat{x}(t_{n+1} + 0) + \Delta\hat{x}|_{t_{n+1}}(t - d_{n+1}), \\ t \in (d_{n+1} - 1, \ \ d_{n+1}]. \end{cases}$$

A direct check will show that: x(t) is continuous and bounded over R; Lemma 6 and the inequality $\|\Delta\hat{x}|_{t_n}\| \leqslant \|\hat{x}(t)\|\,\|F_n\| + \|f_n\| \leqslant C < \infty$ imply that x(t) is also uniformly continuous over R. The set $Q_1 = \{(\mathscr{L}(t+s), x(s)),\ s \in R\}$ is bounded in $H(\mathscr{L}) \times R^n$ and therefore, if Q is its closure, then $\pi(Q) = H(L)$. Fix an arbitrary $L^* \in H(\mathscr{L})$ and let $\mathscr{L}(t+h_k) \to \mathscr{L}^*(t)$. Since x(t) is uniformly continuous, we can find a bounded continuous function $x^*(t): R \to R^n$ such that $x(t + h_k) \to x^*(t)$ as $k \to +\infty$ uniformly over compact subsets of R. In addition, by Lemma 9, the function $\hat{x}^*(t) = x^*(t + n)$, with $t \in (t_n, t_{n+1}]$, is bounded by a solution of system (11). 1. Suppose the assumptions of Theorem 5 are valid. Then $\hat{x}^*(t)$ is the unique solution of (11), and therefore $x^*(t)$ is the unique limit function of $x(t + h_k)$. Thus the set $\pi^{-1}(\mathscr{L}^*) \cap Q$ is a singleton and so the map $\pi: Q \to H(\mathscr{L})$ is a homeomorphism. But then $x^*(t)$ is a.p. in Bohr's sense, and by Lemma 7 $(\hat{x}^*(t), T^*) \in APW(R)$. 2. Suppose that the ρ-a.p. system (8) has a unique bounded solution $\hat{x}(t)$. Then $\forall s \in R$ the set $\pi^{-1}(\mathscr{L}(t+s)) \cap Q$ is a singleton and therefore, if we consider $\pi(Q_1)$ as a metric subspace of $H(\mathscr{L})$, it follows that the function

$$\mathscr{L}(t+s) \to (\mathscr{L}(t+s), x(s)),$$

defined in $\pi(Q_1)$, is continuous.

Consequently [7], x(t) is N-a.p. in Levitan's sense and the proof of Theorem 6 is completed by referring to Lemma 7.

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DUAL APPROXIMATION OF RANDOM EVOLUTIONS IN AN AVERAGING SCHEME

A. V. Svishchuk

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We consider a double approximation of semi-Markov random evolutions, namely, the averaging and diffusion approximation, when the balance condition is not fulfilled. Double approximation algorithms are applicable for reserve and transport processes and other stochastic systems in a semi-Markov random medium.

Algorithms of phase averaging of random evolutions, presented in [1] and developed in [2, 3], define an average evolution, which can be considered as a natural first approximation of the original evolution in a semi-Markov stochastic medium.

Algorithms of diffusion approximation of semi-Markov random evolutions (SMRE), presented in [1] and developed in [4, 5], define the second approximation of the original evolution, since the first approximation (the averaged evolution) is trivial when the balance condition is fulfilled.

In the present paper we use definitions, notation, and conditions introduced in papers [1-6].

Institute of Mathematics, Academy of Sciences of Ukraine, Kiev.

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