Ukrainian Mathematical Journal, Vol. 46, No. 3, 1994

SPECTRAL THEORY OF A STRING

I. S. Kats

UDC 517.9

In this survey, we present the principal results of Krein's spectral theory of a string and describe its development by other authors.

In this survey, we outline Krein's spectral theory of a string and consider its further development in the papers of other authors. Unfortunately, we cannot dwell upon the spectral theory of strings with masses of different signs (Langer [64]), strings with dipoles, and strings with matrix masses (Klotz and Langer [37]). Naturally, the choice of the material for this survey reflects the interests of the author. For this, I would like to apologize to the reader.

1. Differential Operation of a String. Differential Equation of a String. Spectral Functions

1.1. Let I be an interval of one of the following four types: $I = (a, b)$, $I = [a, b)$, $I = (a, b]$, and $I = [a, b]$, where $-\infty \le a < b \le +\infty$ and, in the case of $a \in I$ $(b \in I)$, the equality $a = -\infty$ $(b = +\infty)$ is excluded. On I, we consider a finite nondecreasing function M which may have intervals of constancy and nonzero absolutely continuous, continuous singular, and discontinuous components. We put $a_0 := \inf \mathfrak{T}_M$ and $b_0 := \sup \mathfrak{T}_M$, where $\mathfrak{T}_M$ is a set of the points of increase of the function M. If $a \notin I$ $(b \notin I)$, we assume that

$$M(a) := \inf_{x \in I} M(x) \qquad \big(M(b) = \sup_{x \in I} M(x)\big).$$

With I and M, we associate a string $S(I, M)$ stretched by a unit force by I. For this string, M is a mass distribution function in the sense that, for any $x_1 \le x_2$, $x_1, x_2 \in I$, $M(x_2 + 0) - M(x_1 - 0)$ is the mass of its part located in the interval $[x_1, x_2]$; here, for $a \in I$ $(b \in I)$, we assume that $M(a-0) = M(a)$ $(M(b+0) = M(b))$. The left (right) end of the string $S(I, M)$ (as well as the interval I) is called *regular* if $a_0 > -\infty$, $M(a) > -\infty$ $(b_0 < +\infty$, $M(b) < +\infty)$. Otherwise, it is called *singular*. The end $x = a$ $(x = b)$ is called *completely regular* if $a \in I$ $(b \in I)$.

1.2. Extended Functions. In the case where $I = (a, b)$, the differential operation $l_M[\cdot]$ introduced below acts on the set of ordinary complex-valued functions $x \mapsto f(x)$ defined on I. But if the ends of the interval I are completely regular, then, in order to define the differential operation $l_M[\cdot]$, it is necessary to have extended functions. For example, if $I = [a, b]$, we need functions $f[\cdot]$ extended in both directions by adding to the ordinary functions $I \overset{f(\cdot)}{\mapsto} \mathbb{C}$ the numbers $f^-(a)$ and $f^+(b)$ (for convenience, they are called the left derivative at the point $x = a$ and the right derivative at the point $x = b$, respectively), i.e., $f[\cdot] = \{f(\cdot), f^-(a), f^+(b)\}$. In the case where only one end of the string $S(I, M)$ is completely regular, e.g., the left one, we introduce functions extended to the left $f[\cdot)$ that are obtained from ordinary ones by adding a single "derivative number" $f^-(a)$, i.e., $f[\cdot) = \{f(\cdot), f^-(a)\}$. In the case of $I = (a, b]$, we introduce functions extended to the right $f(\cdot] = \{f(\cdot), f^+(b)\}$. For all indicated types of extended functions $f(\cdot)$ is called their nonextended part. When speaking about extended functions, we shall, as a rule, omit the word "extended" replacing it by the corresponding brackets. The equality of extended functions, linear operations over them, and the operation of conjugation are defined in a natural way,

Odessa Technological Institute of Food Industry, Odessa. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 46, No. 3, pp. 155–176, March, 1994. Original article submitted June 17, 1993.

pointwise for a nonextended part and separately for the adjoined values (see [16], Sec. 1, Subsec. 1). Below, in the notations of the interval I and extended functions, we treat the bracket $\langle$ as $[$ if $a \in I$ and as $($ if $a \notin I$. The bracket $\rangle$ has the same meaning.

1.3. Differential Operation $l[\cdot] = l_{MI}[\cdot]$ of a String $S(I, M)$.

Definition 1. *Let $I = \langle a, b\rangle$. Then $D = D_M = D_{MI}$ is a set of all functions $f\langle\cdot\rangle$ such that*

(i) $f(\cdot)$ is locally absolutely continuous on $\langle a, b\rangle$;

(ii) at every point $x \in (a, b)$, there exist finite left and right derivatives $f^-(x)$ and $f^+(x)$;

(iii) there exists a function $\varphi(\cdot)$ M-measurable on I such that, for any two points $x_1, x_2 \in I$, the equality

$$f^{\pm}(x_2) - f^{\pm}(x_1) = -\int_{x_1 \pm 0}^{x_2 \pm 0} \varphi(s)\, dM(s) \tag{1}$$

holds for all four possible combinations of the sign (on the left-hand side, $f^{\pm}(x_j)$ is written with the same sign as $x_j \pm 0$ on the right-hand side; $j = 1, 2$).

Definition 2. *For any function $f\langle\cdot\rangle \in D_{MI}$, it is assumed that*

$$l[f](x) = l_M[f](x) = l_{MI}[f](x) = \varphi(x) \quad \forall x \in I,$$

where $\varphi(\cdot)$ is a function from Definition 1.

Remark. This relation defines $l_{MI}[f](x)$ to within an equivalence with respect to the measure M. Clearly, for $f\langle\cdot\rangle \in D_{MI}$ and M-almost all $x \in I$,

$$l_M[f](x) = -\frac{d}{(d)M(x)} f^+(x) = -\frac{d}{(d)M(x)} f^-(x), \tag{2}$$

where $\dfrac{d}{(d)M(x)}$ denotes the symmetric derivative with respect to the function M. In view of (2), we shall sometimes write the operation $l_M[\cdot]$ in the form $-\dfrac{d}{(d)M(x)}\dfrac{d}{dx}$, which reflects the "general idea" of its action.

In order to grasp the sense of the differential operation $l_{MI}[\cdot]$ better, we now formulate several properties of the functions $f\langle\cdot\rangle \in D_{MI}$ (see [23], Sec. 1, Subsec. 1; [36], Sec. 1, Subsec. 1):

(a) $f(\cdot)$ is linear on each interval of constancy of the function M;

(b) if x is a continuity point of the function M, then $f^-(x) = f^+(x)$, even in the case where $x = a \in I$ or $x = b \in I$;

(c) for every point $x_0 \in (I \setminus \{b\})$,

$$f^+(x_0) = \lim_{x \downarrow x_0} f^+(x) = \lim_{x \downarrow x_0} f^-(x),$$

and, for every point $x_0 \in (I \setminus \{a\})$,

$$f^-(x_0) = \lim_{x \uparrow x_0} f^+(x) = \lim_{x \uparrow x_0} f^-(x).$$

In view of the last property, we sometimes write $f'(x-0)$ and $f'(x+0)$ instead of $f^-(x)$ and $f^+(x)$, respectively (even if $f^-(x)$ and $f^+(x)$ are adjoined values).

Finally, note that for any $f\langle\cdot\rangle$, $g\langle\cdot\rangle \in D_{MI}$ and any $\alpha, \beta \in I$, the following "Lagrange identity" is true:

$$\int\limits_{\alpha\pm0}^{\beta\pm0} (l[f](x)\overline{g(x)} - f(x)\overline{l[g](x)})\, dM(x) = [f, g]_x \Big|_{\alpha \pm 0}^{\beta \pm 0}, \tag{3}$$

where $[f, g]_x = f(x)\overline{g'(x)} - f'(x)\overline{g(x)}$ (see [23], Sec. 2, Subsec. 1) for all four possible combinations of the signs $\pm$ of α and β which take the same values on the both sides of (3).

1.4. Differential Equation of a String.

Definition 3. *A function $u\langle\cdot\rangle$ is called a solution of the differential equation*

$$l_{MI}[y](x) = g(x) \tag{4}$$

if $u\langle\cdot\rangle \in D_{MI}$ and $l_{MI}[u](x) = g(x)$ for M-almost all $x \in I$.

The differential equation

$$l_{MI^c}[y] - \lambda y = 0 \tag{5}$$

is called the differential equation of a string $S(I, M)$. If $\omega > 0$, the differential equation (5) with $\lambda = \omega^2$ is satisfied by an amplitude function of the string $S(I, M)$ oscillating with the frequency ω.

We can now explain the mechanical reasons for introducing the adjoined values. For example, assume that a string $S([a, b], M)$ oscillates with a frequency ω and that the fixture of its left end is such that the latter can slide without friction in the direction normal to the equilibrium position. In this case, in view of the appearing inertial forces, its amplitude function satisfies the boundary condition $y^+(a) = -\omega^2 m_a y(a)$, where $m_a = M(a+0) - M(a)$ is a mass concentrated at the point $x = a$. This situation is unnatural: Indeed, the boundary condition depends not only on the type of the fixture but also on the distribution of mass in the string and on the frequency of its oscillations. By introducing an adjoined value $y^-(a)$, we eliminate this problem. The boundary condition now takes the form $y^-(a) = 0$.

In the particular case where M is absolutely continuous, Eq. (5) is equivalent to the equation

$$-y'' - \lambda\rho(x)y = 0, \qquad x \in I, \tag{6}$$

for the nonextended part. Here, $\rho(x) = M'(x)$ a.e. in I. The properties of the solutions of Eq. (5) are similar to the properties of the solutions of Eq. (6), and the same true for the inhomogeneous equation $l_{MI}[y] - \lambda y = g(x)$ and the equation $-y'' - \lambda\rho(x)y = \rho(x)g(x)$ (see [36, 23]).

1.5. Differential Operator L_0 of a String $S(I, M)$. Let $\mathfrak{H}$ be a Hilbert space $\mathfrak{L}_M^{(2)}(I)$ of complex-valued functions M-measurable on I and having an M-summable square. In this space, the scalar product is defined by the equality

$$(f, g)_{\mathfrak{H}} = \int_I f(x)\overline{g(x)}\, dM(x).$$

More precisely, an element $\boldsymbol{f}$ of the space $\mathfrak{H}$ is a family of its "representatives," i.e., of functions $f \in \mathfrak{L}_M^{(2)}$ that pairwise coincide M-almost everywhere on I. For these functions, we write $f \in \boldsymbol{f}$ or $f(\cdot) \in \boldsymbol{f}$ (sometimes, when this does not lead to misunderstanding, we shall say, simplifying the situation, that functions f are elements of the space $\mathfrak{H}$). By definition, $(\boldsymbol{f}, \boldsymbol{g})_{\mathfrak{H}} = (f, g)_{\mathfrak{H}}$, where $f \in \boldsymbol{f}$ and $g \in \boldsymbol{g}$.

An element $\boldsymbol{f} \in \mathfrak{H}$ belongs to $\mathfrak{H}'$ if there exists a function $f \in \boldsymbol{f}$ such that $f(x) = 0$ for all x in a certain neighborhood of each singular end of the interval I (if there are no singular ends, $\mathfrak{H}' = \mathfrak{H}$).

Definition 4. *We say that an ordered pair* $\{\boldsymbol{f}, \boldsymbol{g}\} \in \mathfrak{H}^2$ *belongs to* L_0' *if one can indicate* $f\langle\cdot\rangle \in \boldsymbol{f}$ *with the following properties:*[1]

(i) $f\langle\cdot\rangle \in D_{MI}$;

(ii) there exists $a_f \in I$ *such that* $f(x) = f^-(x) = 0$ *for any* $(x \in I,\ x \le a_f)$;

(iii) there exists $b_f \in I$ *such that* $f(x) = f^+(x) = 0$ *for any* $(x \in I,\ x \ge b_f)$;

(iv) $l_{MI}[f] \in \boldsymbol{g}$.

It is obvious that L_0' is a linear relation in $\mathfrak{H}$. Moreover, L_0' is an operator in $\mathfrak{H}$ and even in $\mathfrak{H}'$ — an analog of, e.g., the operator L_0' introduced by Naimark in [69]. As follows from (3), it is Hermitian. Furthermore, it is nonnegative. Unlike the operator in [69], its domain of definition $\mathfrak{D}'_0$ is not always dense in $\mathfrak{H}$. It is dense in $\mathfrak{H}$ if and only if the points a_0 and b_0 have M-measures of zero (in particular, if they do not belong to I). In any case, the operator L_0' possesses the operator closure L_0 and the operator L_0 is either self-adjoint itself or has self-adjoint extensions in $\mathfrak{H}$. The operator L_0 and its self-adjoint extensions are realized by the differential operation $l_{MI}[\cdot]$. Its deficiency index is[2] (p, p), where $p \le 2$.

1.6. Spectral Functions. Assume that each $\lambda \in \mathbb{R}$ is associated with a set G_λ of, generally speaking, extended functions. We say that a family $\mathcal{G} = \{G_\lambda \mid \lambda \in \mathbb{R}\}$ of such sets is determining if, for any $\lambda \in \mathbb{R}$, the problem

$$l_{MI}[y] - \lambda y = 0, \quad y \in G_\lambda \quad (G_\lambda \in \mathcal{G}) \tag{7}$$

admits a unique solution $u\langle\cdot, \lambda\rangle$. Assume that the functions $\lambda \mapsto u(x, \lambda)$ and $\lambda \mapsto u^-(x, \lambda)$ are B-measurable and bounded on every compact interval from $\mathbb{R}$ at least for one fixed $x \in I$ (and, hence, for all x). In this case, the determining family $\mathcal{G}$ is called an lB-family.

Definition 5. *A function* τ *nondecreasing on* $\mathbb{R}$ *and normalized by the conditions*

[1] We write $f\langle\cdot\rangle \in \boldsymbol{f}$ if $f(\cdot) \in \boldsymbol{f}$.

[2] The proofs of all results presented in this subsection can be found in [23].

$$2\tau(\lambda) = \tau(\lambda-0) + \tau(\lambda+0) \quad \forall \lambda \in \mathbb{R}, \quad \tau(0) = 0, \tag{8}$$

is called the spectral function of problem (7) *with a determining family* $\mathcal{G}$ *if the mapping* $U : f \mapsto \mathfrak{F}$, *where* $f \in \mathfrak{H}'$ *and*

$$\mathfrak{F}(\lambda) = \int_I f(x) u(x, \lambda)\, dM(x) \quad \forall \lambda \in \mathbb{R} \tag{9}$$

(if $f \in \mathfrak{H}'$, *the last integral converges for any fixed* $\lambda \in \mathbb{R}$*), maps* $\mathfrak{H}'$ *into* $\mathfrak{L}_\tau^{(2)}(\mathbb{R})$ *isometrically, i.e., for each* $f \in \mathfrak{H}'$,

$$\int_{\mathbb{R}} |\mathfrak{F}(\lambda)|^2 d\tau(\lambda) = \int_I |f(x)|^2 dM(x), \quad \mathfrak{F} = Uf. \tag{10}$$

A spectral function τ *of problem* (7) *is called orthogonal if* $\overline{U\mathfrak{H}'} = \mathfrak{L}_\tau^{(2)}(\mathbb{R})$. *The set of all points of increase of the spectral function* τ *is called its spectrum and is denoted by* $s[\tau]$.

Note that if, for a determining family $\mathcal{G}$, problem (7) possesses a spectral function, then the mapping U can be extended by continuity to the mapping $U_\tau : f \mapsto \mathfrak{F}$ that maps $\mathfrak{H}$ into $\mathfrak{L}_\tau^{(2)}(\mathbb{R})$ isometrically. This mapping is defined by (9) if the integral on the right-hand side is understood in the sense of convergence in the metric of $\mathfrak{L}_\tau^{(2)}(\mathbb{R})$. If, in addition, $\mathcal{G}$ is an *lB*-family, then, for any function $f \in \mathfrak{H}$,

$$f(x) = \int_{\mathbb{R}} \mathfrak{F}(\lambda) u(x, \lambda)\, d\tau(\lambda) \tag{11}$$

in the sense of convergence in $\mathfrak{L}_M^{(2)}(I)$, where $\mathfrak{F}(\cdot)$ is defined by equality (9).

Equality (11) realizes the mapping U_τ^{-1} on the elements $\mathfrak{F} \in U_\tau \mathfrak{H}$. Also note that U_τ maps the operator L_0 into the closed part of the operator Λ_τ of multiplication by an independent variable in $\mathfrak{L}_\tau^{(2)}(\mathbb{R})$ and, hence, possesses self-adjoint extensions with simple spectra. Therefore, if the operator L_0 is self-adjoint and its spectrum is not simple, then problem (7) does not have a spectral function for any determining family $\mathcal{G}$. A situation of this sort is observed, e.g., for a string $S(I, M)$ with $I = (-\infty, +\infty)$ and $M(x) = x$ for any $x \in I$.

Finally, note that the problem of finding spectral functions is a problem similar to the problems of moment theory. Indeed, there exists a family of functions $\lambda \mapsto |\mathfrak{F}(\lambda)|^2$ where $\mathfrak{F}$ runs through $U\mathfrak{H}'$ and, for each function from this family, equality (10) with $f = U^{-1}\mathfrak{F}$ determines the value of the integral on the left-hand side. It is necessary to find τ which will guarantee the validity of this equality for all these functions $|\mathfrak{F}(\cdot)|^2$.

2. Krein's Spectral Theory of a String S_1.

2.1. Strings S_1 and S_0 with Completely Regular Left Ends. For convenience, we assume that the left ends of the strings under consideration are located at the point $x = 0$. Thus, $I = [0, b\rangle$. The mass distribution function M is normalized by the condition $M(0) = 0$. If the left end of a string $S([0, b\rangle, M)$ is fixed so that it can slide without friction along a line perpendicular to the axis x, the string under consideration is denoted by $S_1([0, b\rangle, M)$. A string whose left end remains immobile is denoted by $S_0([0, b\rangle, M)$. The spectral (orthogonal spectral) functions of the boundary-value problems

$$l_{MI}[y] - \lambda y = 0, \quad y(0) = 1, \quad y^-(0) = 0, \tag{12}$$

$$l_{MI}[y] - \lambda y = 0, \quad y(0) = 0, \quad y^-(0) = 1, \tag{13}$$

are called the spectral (orthogonal spectral) functions of the strings $S_1([0, b\rangle, M)$ and $S_0([0, b\rangle, M)$, respectively. A (unique) solution of the boundary-value problem (12) is denoted by $\varphi[\cdot, \lambda\rangle$; for a solution of the boundary-value problem (13), we use the notation $\psi[\cdot, \lambda\rangle$. For fixed $x \in I$, the nonextended parts $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ of these solutions and the unilateral derivatives $\varphi^{\pm}(x, \lambda)$ and $\psi^{\pm}(x, \lambda)$ with respect to x are entire functions of the type

$$t_M(x) := \int_0^x \sqrt{M'(s)}\, ds \tag{14}$$

of order $1/2$ (here, M' is the derivative of the function M, which exists almost everywhere). This statement is also true for the adjoined values $\varphi^+(b, \lambda)$ and $\psi^+(b, \lambda)$ if $b \in I$.

In this survey, we restrict ourselves to the principles of Krein's spectral theory of strings S_1. First, we shall describe the set $\mathfrak{T}([0, b], M)$ of the spectral functions of the regular string $S_1([0, b], M)$ and the set $\mathfrak{T}_+([0, b], M)$ of the spectral functions of this string with nonnegative spectra. But before this, we present the following subsection.

2.2. *R*-functions. This is a brief review of the results obtained in [35].

Definition 6. *We say that a function f of a complex variable belongs to the class (R) and call it an R-function if it is defined and holomorphic in each half plane $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $\mathbb{C}_- := \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$, and, in addition,*

(i) $f(\bar{z}) = \overline{f(z)} \quad \forall z \in \mathbb{C}_+$;

(ii) $\operatorname{Im} z \operatorname{Im} f(z) \geq 0 \quad \forall z \in (\mathbb{C} \setminus \mathbb{R})$.

Any R-function $f(\cdot)$ is representable in the form

$$f(z) = \alpha + \beta z + \int_{-\infty}^{+\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\tau(\lambda), \qquad \operatorname{Im} z \neq 0, \tag{15}$$

where $\alpha \in \mathbb{R}$ and $\beta \geq 0$ are constants and $\tau(\cdot)$ is a nondecreasing function defined on $\mathbb{R}$. The constants α and β in (15) are uniquely determined by the R-function f. Under normalization (8), τ is also uniquely determined by this R-function. Normalized by (8), it is called the spectral function of the R-function f and satisfies the following Stieltjes inversion formula:

$$\tau(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_0^{\lambda} \operatorname{Im} f(\xi + i\varepsilon)\, d\xi. \tag{16}$$

By the way, if f is a function (not necessarily an R-function) such that the limit on the right-hand side of (16) exists for any $\lambda \in \mathbb{R}$, then, for the function τ defined by equality (16), we can write $\tau = \mathfrak{S}[f]$.

Below, we consider several subclasses of the class (R). A class (R_1) consists of the functions $f \in (R)$ whose representation (15) can be transformed as follows:

$$f(z) = \gamma + \int_{-\infty}^{+\infty} \frac{d\tau(\lambda)}{\lambda - z}, \qquad \operatorname{Im} z \neq 0,$$

where $\gamma \in \mathbb{R}$, and the spectral function τ is such that

$$\int_{-\infty}^{+\infty} (1 + |\lambda|)^{-1} d\tau(\lambda) < \infty.$$

A class (R_0) consists of the functions $f \in (R)$ for which representation (15) takes the form

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\tau(\lambda)}{\lambda - z}, \qquad \operatorname{Im} z \neq 0, \tag{17}$$

where the spectral function τ is a function of bounded variation on $(-\infty, +\infty)$.

A class (S) consists of the functions $f \in (R)$ for which representation (15) can be written as follows:

$$f(z) = \gamma + \int_{-0}^{+\infty} \frac{d\tau(\lambda)}{\lambda - z}, \qquad \operatorname{Im} z \neq 0, \tag{18}$$

where $\gamma \geq 0$ and, for the spectral function τ, we can write $\tau(\lambda) = \tau(-0)$ $\forall \lambda \in (-\infty, 0)$ and

$$\int_{-0}^{+\infty} (1 + \lambda)^{-1} d\tau(\lambda) < \infty.$$

The classes $(\tilde{R})$ and $(\tilde{S})$ are obtained from the classes (R) and (S), respectively, by adjoining the function identically equal to ∞. The necessary and sufficient conditions for the functions $f \in (R)$ to belong to the classes (R_1), (R_0), or (S) can be found in [35].

2.3. Description of the Set of Spectral Functions of the String $S_1([0, b], M)$. Assume that this string has a heavy right end, i.e., that $M(x) < M(b)$ for any $x \in (0, b)$.

For every function $h \in (\tilde{R})$, we define a function Ω_h by an equality

$$\Omega_h(z) = \frac{\psi^+(b, z)h(z) + \psi(b, z)}{\varphi^+(b, z)h(z) + \varphi(b, z)} \qquad \forall z \in (\mathbb{C} \setminus \mathbb{R}). \tag{19}$$

It was established (see [16], Lemma 3.1) that Ω_h belongs to (R_1) and, hence, admits an absolutely convergent representation

$$\Omega_h(z) = \gamma_h + \int_{-\infty}^{+\infty} \frac{d\tau_h(\lambda)}{\lambda - z} \qquad \forall z \in (\mathbb{C} \setminus \mathbb{R}), \tag{20}$$

where $\gamma_h \in \mathbb{R}$ and $\tau_h(\cdot)$ is a nondecreasing function normalized by conditions similar to (8) and, consequently,

$$\tau_h(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_0^{\lambda} \operatorname{Im} \Omega_h(\xi + i\varepsilon) d\xi \qquad \forall \lambda \in \mathbb{R}. \tag{21}$$

It turns out that $\gamma_h = a_0$ for any $h \in (\tilde{R})$ in (20).

Theorem 1. *Any function* $\tau \in \mathfrak{T}([0, b], M)$ *coincides with a function* τ_h *obtained by using relations (19) and (21) with some* $h \in (\tilde{R})$. *Conversely, in the case where the right end of the string* $S_1([0, b], M)$ *bears no concentrated mass* $(M(b) = M(b-0))$, *any function* τ_h *constructed according to these relations for* $h \in (\tilde{R})$ *belongs to* $\mathfrak{T}([0, b], M)$. *In the case where this end bears a concentrated mass*[3] $(M(b-0) < M(b))$, $\tau_h \in \mathfrak{T}([0, b], M)$ *if and only if* $h \in ((\tilde{R}) \setminus (R_0))$. *A spectral function* τ_h *of the string* $S_1([0, b], M)$ *is orthogonal if and only if* h *is a real constant that may be infinite.*

Note that, in the case where $M(b-0) < M(b)$, the function τ_h with $h \in (R_0)$ is a spectral function of a string obtained from the given one by extracting a concentrated mass at the point $x = b$ and of a string obtained from the given string $S_1([0, b], M)$ by adding a mass $(\sup_{\eta > 0} (\eta \operatorname{Im} h(i\eta)))^{-1}$ to the mass concentrated at this point but is not a spectral function of the original string.

Theorem 1 is complemented by the following theorem:

Theorem 2. *A spectral function* τ_h *of the string* $S_1([0, b], M)$ *belongs to* $\mathfrak{T}_+([0, b], M)$ *if and only if* $h \in (\tilde{S})$. *For these and only these* h, *the function* Ω_h *belongs to the class* $(\tilde{S})$.

Note that for $z < 0$, the inequalities

$$\frac{\psi(b, z)}{\varphi(b, z)} \le a_0 + \int_{-0}^{+\infty} \frac{d\tau(\lambda)}{\lambda - z} \le \frac{\psi^+(b, z)}{\varphi^+(b, z)} \tag{22}$$

hold for an arbitrary function $\tau \in \mathfrak{T}_+([0, b], M)$. The first inequality in (22) turns into the equality at least for one $z < 0$ (and, hence, for all z) only if $\tau = \tau_0$. The second inequality turns into the equality only for $\tau = \tau_\infty$.

In the case of a regular string $S_1([0, b), M)$ with $b = b_0 < \infty$, the set of its spectral functions coincides with the set of spectral functions of the string obtained from $S_1([0, b), M)$ by adjoining the point $x = b$ without a concentrated mass. In describing the set of the spectral functions of this string, we can replace $\varphi^+(b, z)$ and $\psi^+(b, z)$ by $\varphi^-(b, z)$ and $\psi^-(b, z)$, respectively.

2.4. Here, it is seems necessary to mention a generalization of the Chebyshev–Markov inequalities obtained by Krein (see [53, 62]). It establishes certain extreme properties of the orthogonal spectral functions of the string $S_1([0, b], M)$. The orthogonal spectral function τ_h $(h = \text{const}, h \in \mathbb{R})$ is a pure jump function and its spectrum coincides with the set of zeros of the entire function $z \mapsto \varphi^+(b, z)h + \varphi(b, z)$. For simplicity, we assume that the right end of the string $S_1([0, b], M)$ is heavy and the point $x = b$ does not bear a concentrated mass. In this case, any point ξ of the real axis is a spectral point of some orthogonal spectral function which is denoted by $\tau^{(\xi)}$. Krein proved [62] the following theorem:

Theorem 3. *For any function* $\tau \in \mathfrak{T}([0, b], M)$ *and any* $\xi \in \mathbb{R}$, *the inequalities*

[3] This part of the theorem was established by the author (see [36], Bibliographical and Historical Notes, Part 1; [29]).

$$\tau(\xi - 0) - \tau(-\infty) \geq \tau^{(\xi)}(\xi - 0) - \tau^{(\xi)}(-\infty);$$

$$\tau(\xi + 0) - \tau(-\infty) \leq \tau^{(\xi)}(\xi + 0) - \tau^{(\xi)}(-\infty)$$

hold. If at least one of these inequalities turns into a strict equality, then $\tau(\lambda) = \tau^{(\xi)}(\lambda)$ *for any* $\lambda \in \mathbb{R}$.

This theorem contains the following two assertions:

A. In the set of all spectral functions τ of the string $S_1([0, b], M)$, the orthogonal spectral function $\tau^{(\xi)}$ has the largest spectral mass $\tau(\xi + 0) - \tau(\xi - 0)$ at a fixed point $\xi \in \mathbb{R}$ and if $\tau(\xi + 0) - \tau(\xi - 0) = \tau^{(\xi)}(\xi + 0) - \tau^{(\xi)}(\xi - 0)$, then $\tau(\lambda) = \tau^{(\xi)}(\lambda)$ for any $\lambda \in \mathbb{R}$.

B. Let $\xi_1 < \xi_2$ be two adjacent points of the spectrum of the orthogonal spectral function $\tilde{\tau}$ of the string $S_1([0, b], M)$ (i.e., $\tilde{\tau}(\xi_1 - 0) < \tilde{\tau}(\xi_1 + 0) = \tilde{\tau}(\xi_2 - 0) < \tilde{\tau}(\xi_2 + 0)$). If a spectral function τ of this string has no points of increase on (ξ_1, ξ_2), then $\tau(\lambda) = \tilde{\tau}(\lambda)$ for any $\lambda \in \mathbb{R}$.

In connection with proposition A, note that

$$\tau^{(\xi)}(\xi + 0) - \tau^{(\xi)}(\xi - 0) = \left(\int_0^b (\varphi(x, \xi))^2 dM(x) \right)^{-1} \tag{23}$$

and, in particular,

$$\max_{\tau \in \mathfrak{T}([0,b],M)} (\tau(+0) - \tau(-0)) = (M(b))^{-1}. \tag{24}$$

For spectral functions with nonnegative spectra, one can formulate a similar theorem (see [62], Theorem 6) with the role of orthogonal spectral functions played by so-called canonical spectral functions. These functions are obtained from (19) and (21) if we choose $h(z)$ either in the form of a constant from $[0, +\infty]$ (these functions are called canonical spectral functions of the first type or orthogonal spectral functions) or in the form $-m/z$, where $m > 0$ (these are canonical spectral functions of the second type).

Note that for a boundary-value problem

$$-y'' + q(x)y - \lambda\rho(x)y = 0 \quad (0 \leq x < b), \quad y(0) = 1, \quad y'(0) = h, \tag{25}$$

with a loaded Sturm–Liouville equation, we have a theorem similar to Theorem 3 (see [53]). This enabled Krein to improve (in the case of $\rho(x) \equiv 1$) the remainder of the asymptotic relation

$$\tau(\lambda) = \frac{2}{\pi}\sqrt{\lambda} + o(\sqrt{\lambda}), \quad \lambda \to \infty, \tag{26}$$

for a spectral function due to Marchenko [67]. This was the second improvement; the first one was obtained by Levitan in [66].

2.5. Spectral Functions of Singular Strings $S_1([0, b), M)$. Recall that the left end of the string $S_1([0, b), M)$ is regular. Hence, the singularity of this string implies that its right end is singular, i.e., $M(b) + b_0 = \infty$, and, in this case, we necessarily have $b_0 = b$.

Any singular string $S_1([0, b), M)$ has exactly one spectral function τ_+ with nonnegative spectrum and this function is orthogonal. The function Ω_+ defined for $z \in \mathrm{Ext}\,[0, \infty)$ by an equality

$$\Omega_+(z) = a_0 + \int_{-0}^{+\infty} \frac{d\tau_+(\lambda)}{\lambda - z} \tag{27}$$

satisfies the following inequalities (see [36], Sec. 10, Subsec. 4):

$$\frac{\psi(x, z)}{\varphi(x, z)} < \Omega_+(z) \le \frac{\psi^+(x, z)}{\varphi^+(x, z)} \quad \forall\,(x \in [0, b),\ z < 0). \tag{28}$$

Note that for fixed $z < 0$, the function $x \mapsto \dfrac{\psi(x, z)}{\varphi(x, z)}$ monotonically increases while the function $x \mapsto \dfrac{\psi^+(x, z)}{\varphi^+(x, z)}$ exhibits no increase on $[0, b)$. In addition, $\dfrac{\psi^+(x, z)}{\varphi^+(x, z)} - \dfrac{\psi(x, z)}{\varphi(x, z)} \to 0$ for $x \uparrow b$. Thus,

$$a_0 + \int_{-0}^{+\infty} \frac{d\tau_+(\lambda)}{\lambda - z} = \lim_{x \uparrow b} \frac{\psi(x, z)}{\varphi(x, z)} := \mathring{\Gamma}(z). \tag{29}$$

According to the Vitali theorem, this equality holds not only for $z < 0$ but also for all $z \in \mathrm{Ext}\,[0, +\infty)$.

A spectral function τ_+ of a singular string $S_1([0, b), M)$ is unique if and only if

$$\int_{-0}^{b} x^2\, dM(x) = \infty. \tag{30}$$

Therefore, if a singular string $S_1([0, b), M)$ has finite length, it has only one spectral function τ_+. A description of the set of all spectral functions for the case where (30) does not hold can be found in the work [36, Sec. 10, Subsec. 7].

2.6. Principal Dynamic Compliance Coefficient of an Arbitrary String $S_1([0, b\rangle, M)$ and Its Principal Spectral Function. In Subsecs. 2.3 – 2.5, we have considered strings $S_1([0, b\rangle, M)$ with $b_0 = b$. These strings have no mass-free intervals on their right ends. Let us omit this requirement (for singular strings $S_1([0, b\rangle, M)$, it was satisfied automatically). For any string $S_1([0, b\rangle, M)$ (regular or singular), there exists a limit

$$\lim_{x \uparrow b} \frac{\psi(x, z)}{\varphi(x, z)} := \mathring{\Gamma}(z) \quad \forall\, z \in \mathrm{Ext}\,[0, +\infty) \tag{31}$$

(if $b \in I$, it is equal to $\psi(b, z)/\varphi(b, z)$). Here, $\mathring{\Gamma}$ is an $\tilde{S}$-function. Its spectral function $\mathring{\tau}$ has a nonnegative spectrum. Except for the case where $b \in I := [0, b\rangle$ and $M(b - 0) < M(b)$, the function $\mathring{\tau}$ is a spectral function of the string $S_1([0, b\rangle, M)$. It is called the *principal spectral function of this string,* while the function $\mathring{\Gamma}$ is called its *principal dynamic compliance coefficient.* This notion is explained by the fact that if $z = \omega^2$ with $\omega > 0$ and $z \notin s[\mathring{\tau}]$, then $\mathring{\Gamma}(z)$ is equal to the amplitude of forced oscillations of the left end of the string under a periodic force $\mathcal{F} = \sin \omega t$ applied to this end in the direction normal to the equilibrium position provided that its right end $x = b$ is fixed in the case where it is regular. The situation where the end $x = b$ is fixed and $b_0 < b \le \infty$ is equivalent to the deletion of the section $(b_0, b\rangle$ of the string allowing the end b_0 of the remaining part of the string to move without friction along a circle centered at the point b; in particular, for $b_0 < b = \infty$, the end moves along a straight line normal to the axis Ox.

In the case where the string $S_1([0, b\rangle, M)$ is weightless $(M(x) = 0$ for any $x \in [0, b\rangle)$, $\overset{\circ}{\Gamma}(z) \equiv b$ and, in particular, $\overset{\circ}{\Gamma}(z) \equiv \infty$ for $b = \infty$. For weightless strings, $\overset{\circ}{\tau}(\lambda) = 0$ for any $\lambda \in \mathbb{R}$.

If $b \in I$ and $m_b := M(b) - M(b-0) > 0$, then, as already noted, $\overset{\circ}{\tau}$ is not a spectral function of the string $S_1([0, b\rangle, M)$. In this case, the notion of principal dynamic compliance coefficient of a string is not introduced. "Mechanically," this is explained by the fact that the concentrated mass m_b does not participate in the oscillation process described above.

The principal dynamic compliance coefficient $\overset{\circ}{\Gamma}$ of a string $S_1([0, b\rangle, M)$ is connected with its principal spectral function $\overset{\circ}{\tau}$ as follows:

$$\overset{\circ}{\Gamma}(z) = \gamma + \int_{-0}^{+\infty} \frac{d\overset{\circ}{\tau}(\lambda)}{\lambda - z} \qquad \forall\, z \in \operatorname{Ext}[0, \infty), \tag{32}$$

where $\gamma = a_0$. The principal spectral function of a singular string $S_1([0, b), M)$ is its unique spectral function with nonnegative spectrum.

2.7. Krein's Principal Result in the Spectral Theory of Strings $S_1([0, b\rangle, M)$. This result is formulated as the following theorem:

Theorem 4. *Any function* $\overset{\circ}{\Gamma} \in (\tilde{S})$ *is the principal dynamic compliance coefficient of a (single) string* $S_1([0, b), M)$ *(singular or regular).*

Remark. If the string $S_1([0, b), M)$ appearing in Theorem 4 is regular and $b < \infty$, then the function $\overset{\circ}{\Gamma}$ is the principal dynamic compliance coefficient of just one extra string $S_1([0, b], M)$ obtained from the first one by adding the point $x = b$ which does not bear a concentrated mass.

Theorems 2 and 4 easily yield the following assertion:

Theorem 5. *Let* $S_1([0, b\rangle, M)$ *be a regular string with heavy right end. Any of its spectral functions* τ *with nonnegative spectrum is the principal spectral function either of this string or of a string* $S_1([0, B), \check{M})$ *obtained from* $S_1([0, b\rangle, M)$ *by a certain extension to the right.*

The problem of reconstruction of a string from its principal spectral function is connected with a certain ambiguity because the spectral function of a function from the class $(\tilde{S})$ determines this function only to within a certain additive nonnegative constant γ. In view of the mechanical meaning of this constant $(\gamma = a_0)$, Theorems 4 and 5 yield the following result:

Theorem 6. *In order that a function* τ *nondecreasing on* $(-\infty, +\infty)$, *normalized by conditions (8), and having no points of increase on the semiaxis* $(-\infty, 0)$ *be a spectral function of some string* S_1, *it is necessary and sufficient that the condition*

$$\int_{-0}^{+\infty} (1 + \lambda)^{-1}\, d\tau(\lambda) < \infty \tag{33}$$

be satisfied. Under this condition, τ is the principal spectral function of a (single) string $S_1([0, b), M)$ with heavy left end. Furthermore, any string $S_1([0, \tilde{b}], \tilde{M})$ with heavy ends, for which τ is a (nonprincipal) spectral function, can be obtained from $S_1([0, b), M)$ by deleting the interval $(\tilde{b}, b)$ with a possible additional removal of the concentrated mass at the point $x = \tilde{b}$ (without removing the point itself).

The set of functions τ satisfying the conditions of Theorem 6 is denoted by $\mathfrak{T}_+$.

We now present some properties of the bijective correspondence established by Theorem 4 between the set of strings $S_1([0, b\rangle, M)$ and the set of functions $\mathring{\Gamma} \in (\tilde{S})$ of the form (32) that are the principal dynamic compliance coefficients of these strings:

I⁰. $\mathring{\tau}(+\infty) - \mathring{\tau}(-\infty) = (M(a_0+0) - M(a_0-0))^{-1}$.

II⁰. $b = \gamma + \int_{-0}^{+\infty} \lambda^{-1}\, d\mathring{\tau}(\lambda)$.

III⁰. Under the assumption that the immobile fixing of the right end $x = b < \infty$ of a string $S_1([0, b\rangle, M)$ is equivalent to adjoining an infinite mass at the point $x = b$, we have

$$(\mathring{\tau}(+0) - \mathring{\tau}(-0))^{-1} = \bar{M}, \tag{34}$$

where $\bar{M}$ is the overall mass of the string. Consequently, if $\mathring{\tau}(+0) - \mathring{\tau}(-0) > 0$, then $b = \infty$, $\bar{M} < \infty$, and (34) is satisfied.

IV⁰. If $S_1 = S_1([0, b), M)$ is a singular string, then, for the function $\mathring{\Gamma}$ to be meromorphic, i.e., for the spectrum $s[\mathring{\tau}]$ of its spectral functions $\mathring{\tau}$ to be discrete, it is necessary and sufficient that one of the following two conditions be satisfied:

$$\lim_{x\uparrow b} x\,(M(b) - M(x)) = 0 \quad \text{and} \quad \lim_{x\uparrow b} M(x)(b-x) = 0.$$

It is assumed that $M(b) < \infty$ and $b = \infty$ in the first condition and $b < \infty$ and $M(b) = \infty$ in the second one (see [34]).

V⁰. If $S_1 = S_1([0, b), M)$ is a singular string, then, in order that the spectrum $s[\mathring{\tau}] = \{\lambda_j \mid j = 0, 1, 2, \ldots\}$ satisfy the condition

$$\sum_{\lambda_j \neq 0} \lambda_j^{-1} < \infty, \tag{35}$$

it is necessary and sufficient that one of the following two conditions hold:

$$\int^{b} (M(b) - M(x))\, dx < \infty \quad \text{and} \quad \int^{b} (b-x)\, dM(x) < \infty. \tag{36}$$

It is assumed that $M(b) < \infty$ in the first of these conditions and $b < \infty$ in the second one.

Note that in Feller's terminology accepted in the theory of diffusion processes, the right end of a string is called an entry end if the first condition in (36) is satisfied; under the second condition, it is called an exit end (clearly, an end which is both entry and exit is a regular end).

2.8. Classes $\mathfrak{B}_A$.

***Definition** 7. A continuous function Φ defined on $[0, A)$ is attributed to the class $\mathfrak{B}_A$ if*

(i) $\Phi(0) = 0$;

(ii) a kernel $\mathfrak{L}(s, t) := \Phi(s) + \Phi(t) - \Phi(|t-s|)$ *is positive definite on the square* $0 \le s,\ t < A$.

It is clear that the function Φ defined on $[0, +\infty)$ by the equality

$$\Phi(t) = \int_{-0}^{+\infty} \frac{1 - \cos\sqrt{\lambda} t}{\lambda}\, d\tau(\lambda) \tag{37}$$

with a function $\tau \in \mathfrak{T}_+$ belongs to $\mathfrak{B}_\infty$ and, hence, to the class $\mathfrak{B}_A$ with an arbitrary $A \in (0, +\infty)$. Krein proved in [57] that any function $t \mapsto \Phi(t)$ from $\mathfrak{B}_A$, where A is fixed, $0 < A \le +\infty$, is representable on $[0, A)$ in the form (37) with $\tau \in \mathfrak{T}_+$. Hence, for $A < \infty$, any function $\Phi \in \mathfrak{B}_A$ can be extended to $[0, +\infty)$ with an extension that remains in the class $\mathfrak{B}_\infty$. The problem of finding the indicated extension of a function $\Phi \in \mathfrak{B}_A$ with $A < \infty$ is equivalent to the problem of finding a function $\tau \in \mathfrak{T}_+$ for which equality (37) holds for any $t \in [0, A)$.

If $\Phi \in \mathfrak{B}_\infty$, then a number $T_\Phi > 0$ is called a separating point of the function Φ if the latter admits more than one extension from any interval $[0, A)$ with $A < T_\Phi$ and is uniquely extendable from any interval $[0, A)$ with $A > T_\Phi$. If Φ is uniquely extendable from any interval $[0, A)$ with $A > 0$, we set $T_\Phi = 0$. If it admits more than one extension from any interval $[0, A)$ with $A > 0$, we set $T_\Phi = \infty$.

In the case of $A < T_\Phi$ $(A > 0)$, M. Krein gave a description of the set $V_{\Phi, A} \subset \mathfrak{T}$ of all τ that deliver its representation (37) on the interval $0 \le t < A$. Moreover, he established an algorithm for computing the value of the so-called central function

$$\theta_\Phi(A) := \max_{\tau \in V_{\Phi, A}} (\tau(+0) - \tau(-0))$$

for any $A \in (0, T_\Phi)$. In many cases, this technique allows one to give an efficient solution of the inverse problem of finding a string if its principal spectral function $\tau \in \mathfrak{T}_+$ is given (for details, see the next section).

2.9. Transition Functions of Strings $S_1([0, b\rangle, M)$ with Heavy Left Ends. For every function $\tau \in \mathfrak{T}_+([0, b\rangle, M)$, we define a function Φ_τ on $[0, +\infty)$ by the equality

$$\Phi_\tau(t) = \int_{-0}^{+\infty} \frac{1 - \cos\sqrt{\lambda} t}{\lambda}\, d\tau(\lambda) \qquad \forall\, t \in [0, +\infty). \tag{38}$$

It is called a transition function of a string $S_1([0, b\rangle, M)$ and the function $\mathring{\Phi} := \Phi_{\mathring{\tau}}$ is called the principal transition function of this string. It follows from Theorem 6 that any function $\Phi \in \mathfrak{B}_\infty$ is the principal transition function of a single string S_1 with heavy left end; moreover, any transition function of a string $S_1([0, b\rangle, M)$ is the principal transition function either of this string or of the string $S_1([0, B\rangle, \check{M})$ obtained from $S_1([0, b\rangle, M)$ for $b_0 = b$ by an extension to the right ($\check{M}(x) = M(x)\ \forall x \in [0, b\rangle$) and, for $b_0 < b$, by removing its part that lies on $(b_0, b\rangle$ (and is weightless) followed by an extension to the right.

It is worth noting (see [57]) that for any $\tau \in \mathfrak{C}_+([0, b\rangle, M)$,

$$\Phi_\tau(t) = \mathring{\Phi}(t) \quad \forall t \in [0, 2T), \tag{39}$$

where $T = t_M(b)$ [see (14)]. It follows from (38) and (39) that $\mathfrak{C}_+([0, b), M) \subset V_{\mathring{\Phi}, 2T}$. Moreover, for the case where the right end of a string $S_1([0, b), M)$ is not discharged, i.e., one cannot indicate any interval $(b - \varepsilon, b)$ on which $M'(x) = 0$ a.e., Krein proved that $\mathfrak{C}_+([0, b), M) = V_{\mathring{\Phi}, 2T}$. In this case, $\theta_\Phi(2T)$ gives the overall mass of the string $S_1([0, b), M)$ [see (24)]. It is now clear that, for a given $\mathring{\Phi}$, one can find the mass of the part of the string that lies on $[0, x_t)$ for any $t \in (0, T)$, where x_t is the least root of the equation $x_M(x) = t$. In the case where the string S_1 contains no intervals on which $M'(x) = 0$ a.e., the idea indicated together with equality (14) enables one to find, for a given $\mathring{\Phi}$, the length b of the string and $M(x)$ for every $x \in (0, b)$. Therefore, these values can be reconstructed from the principal spectral function. In this connection, Krein proved two theorems presented below [57].

Theorem 7. *In order that a function $\tau \in \mathfrak{C}_+$ be the principal spectral function of a string $S_1([0, b), M)$ such that $t_M(b) = T$, where T is a given number from $[0, +\infty]$, it is necessary and sufficient that the equality $T_\Phi = 2T$, where $\Phi = \Phi_\tau$, hold.*

Theorem 8. *Assume that $\Phi \in \mathfrak{B}_\infty$, $T_\Phi > 0$, and $\Phi'(t)$ exists for any $t \in [0, T_\Phi/2)$. Furthermore, let Φ' be locally absolutely continuous in $[0, T_\Phi/2)$ and let $S_1([0, b), M)$ be a string whose principal transition function is Φ. If, for some $A \in [0, T_\Phi/2)$, an integral equation*

$$2\,\Phi'(0)\,q(t) + \int_{-A}^{A} \Phi''(|t - s|)\,q(s)\,ds = 1 \tag{40}$$

possesses a solution $q = q(t; A)$ summable on $[-A, A]$, then this solution is unique and, for any complex λ,

$$\int_{-A}^{A} q(t, A)\cos\lambda t\,dt = \int_{-0}^{x_A - 0} \varphi(x, \lambda^2)\,dM(x) = -\frac{1}{\lambda^2}\,\varphi^-(x_A, \lambda^2);$$

in particular,

$$\int_{-A}^{A} q(t, A)\,dt = M(x_A - 0).$$

Moreover, if $\Phi'(0) > 0$, then, for any $A \in (0, T_\Phi/2)$, Eq. (40) possesses a continuous solution on $[-A, A]$ and the functions $t \mapsto M(x_t)$ and $t \mapsto \varphi^-(x_t, \lambda)$ have absolutely continuous derivatives with respect to the variable t. Finally, the equality

$$\varphi(x_t, \lambda^2) = \frac{1}{p(t)} \frac{d}{dt} \int_{-t}^{t} q(s,t) \cos \lambda s \, ds$$

is true for an arbitrary complex λ; *here,* $p(t) = dM(x_t)/dt$.

This theorem, together with numerous assertions (see [54]) that describe transformations of a string caused by simple transformations of its principal spectral function, gave the possibility to indicate a broad class of functions $\tau \in \mathfrak{T}_+$ for which one can effectively construct a string S_1 with the principal spectral function τ. Furthermore, it opened the possibility to extend the class of strings S for which a solution of the string equation (5) can be expressed, for any λ, in terms of elementary and special (Bessel and Legendre) functions.

The development of the ideas of Theorem 8 lead M. Krein to many interesting results having no direct relation to the spectral theory of strings [58–61].

At the end of this subsection, we present a mechanical interpretation of the transition function of a string and make an attempt to give a mechanical explanation of equality (39). Let $\overset{\circ}{\Phi}$ be the principal transition function of the string $S_1([0, b\rangle, M)$. Then, as was shown by Krein, $\overset{\circ}{\Phi}(t)$ is equal to the shift of the left end of the string for time t under the action of the unit transverse force instantaneously applied to this end of the initially immobile string. If $\tau \in \mathfrak{T}_+([0, b\rangle, M)$, then, as mentioned above, Φ_τ is the principal transition function of a string $\check{S}_1 = S_1([0, B), \check{M})$ whose mass distribution on the interval $[0, b_0)$ coincides with that of the string $S_1([0, b\rangle, M)$. Note that, for any $x \in [0, B)$, $t_{\check{M}}(x)$ is the time for which a wave induced by a transverse shift of the left end of the string reaches the point x or an (inverse) wave induced by a transverse shift of the point x reaches the left end. The motion of the point $x = 0$ of the string $\check{S}_1$ is affected by the applied unit force, by the part of $\check{S}_1$ that moves, and by its immobile part. The effect of the immobile part manifests itself only in longitudinal tension which does not depend on the masses located in this segment of the string. For the points $x \in (b_0, B)$, we have $t_{\check{M}}(x) \geq T$ (note that $T = t_M(b) = t_M(b_0) = t_{\check{M}}(b_0)$). Therefore, for time $t < 2T$, the part of the string $\check{S}_1$ that lies to the right of the point b_0, i.e., the part of the string $\check{S}_1$ adjoined to the original string does not affect the motion of the left end (the wave cannot reach the points of this segment and return to the point $x = 0$). As a result, we observe the same effect as if the point b_0 is fixed. This explains the fact that the transition function Φ_τ of the original string (it is also the principal transition function of the string $\check{S}_1$) coincides with the principal transition function $\Phi_{\overset{\circ}{\tau}}$ on the interval $0 \leq t < 2T$.

I could never decide what had appeared earlier — the analytical approach to the proof of equality (39) and Theorem 7 or mechanical arguments. It seems to me that mechanics was the first. Krein had an extremely strong mechanical intuition which, being combined with his powerful analytical technique and ability to understand pheno mena in all their complexity, worked wonders.

2.10. Extrapolation Stationary Random Processes. Let σ be an odd function nondecreasing on $\mathbb{R}$ and such that

$$\int_{-\infty}^{+\infty} (1+\lambda^2)^{-1} \, d\sigma(\lambda) < \infty. \tag{41}$$

Denote by Λ_∞ a space $\mathfrak{L}_\sigma^{(2)}(-\infty, +\infty)$ with the standard norm. Let $J_\alpha = (-\alpha, +\alpha)$, $0 \leq \alpha < +\infty$, be an interval of the real axis and let Λ_α be the linear span of a family of functions

$$\lambda \mapsto \int_{t_1}^{t_2} \exp(i\lambda s)\, ds, \qquad t_1, t_2 \in J_\alpha,$$

closed in Λ_∞.

In [56], Krein solved the following two problems:

I. Establish a criterion for $\Lambda_\alpha = \Lambda_\infty$.

II. For $\Lambda_\alpha \neq \Lambda_\infty$, find an analytic expression of the orthogonal projection $P_\infty F$ of an arbitrary element $F \in \Lambda_\infty$ onto Λ_α.

These problems can be treated as the problems of the prediction (extrapolation) and filtration of stationary processes according to their observation on the interval $-\alpha < t < \alpha$. Without loss of generality, we can assume that σ is normalized by conditions of the form (8). We set $\tau(\lambda) = 2\sigma(\sqrt{\lambda}) - \sigma(+0)$ for all $\lambda > 0$, $\tau(0) = 0$, and $\tau(\lambda) = -\sigma(+0)$ for all $\lambda < 0$. The function thus defined is normalized by conditions (8) and, as follows from (41), condition (33) is satisfied. According to Theorem 6, there exists a string $S_1([0, b), M)$ with a heavy left end whose principal spectral function is τ. *A solution of Problem I* is given in terms of the string $S_1([0, b), M)$ by the following theorem:

Theorem 9. *In order that* $\Lambda_\alpha = \Lambda_\infty$, *it is necessary and sufficient that the following conditions be satisfied:*

(a) $t_M(x) \leq \alpha$ *for any* $x \in [0, b)$;

(b) $M(x_\alpha) = M(b)$, *where* x_α *is the least root of the equation* $t_M(x) = \alpha$.

Now let $\Lambda_\alpha \neq \Lambda_\infty$ and $F \in \Lambda_\infty$. We construct the following functions:

$$f(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{-N}^{N} F(\lambda)\,\varphi(x, \lambda^2)\, d\sigma(\lambda) \quad \text{and} \quad g(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{-N}^{N} F(\lambda)\,\varphi^-(x, \lambda^2)\, \frac{d\sigma(\lambda)}{\lambda},$$

where the first limit is understood in the sense of convergence in $\mathfrak{L}^{(2)}_M[0, b)$, and the second in the sense of convergence in $\mathfrak{L}^{(2)}[0, b)$. Here, $\varphi(\cdot, \lambda)$ is the nonextended part of a solution of problem (12).

A solution of Problem II is given by the relation

$$(P_\alpha F)(\lambda) = \int_{-0}^{x_\alpha - 0} f(x)\varphi(x, \lambda^2)\, dM(x) + \frac{1}{\lambda}\int_0^{x_\alpha} g(x)\varphi^-(x, \lambda^2)\, dx. \tag{42}$$

The squared distance between F and Λ_α is given by the equality

$$\int_{-\infty}^{+\infty} |F(\lambda) - (P_\alpha F)(\lambda)|^2\, d\sigma(\lambda) = \int_{x_\alpha - 0}^{b} |f(x)|^2\, dM(x) + \int_{x_\alpha}^{b} |g(x)|^2\, dx.$$

In parallel with the solution of these problems, Krein proved that entire functions of the form

$$\mathcal{F}(\lambda) = \int_{-0}^{x_\alpha - 0} f(x)\varphi(x, \lambda^2)\, dM(x),$$

where $f \in \mathfrak{L}_M^{(2)}[0, x_\alpha)$, exhaust the set of all even entire functions in Λ_α; the set of all odd entire functions from Λ_α is exhausted by entire functions of the form

$$\mathcal{G}(\lambda) = \frac{1}{\lambda} \int_0^{x_\alpha} g(x)\varphi^-(x, \lambda^2)\, dx,$$

where $g \in \mathfrak{L}^{(2)}[0, x_\alpha)$. This yields (42).

3. Further Development of the Spectral Theory of Strings with Nonnegative Masses

3.1. Multiplicity of a Spectrum. Consider a string $S((a, b), M)$ with two singular ends. As in Subsection 1.5, L_0 is the differential operator of the string. It was established that if this operator is not self-adjoint [i.e., if at least one of the following conditions is violated:

$$\int_a^{k-0} (x - k)^2 dM(x) = \infty, \qquad \int_{k-0}^{b} (x - k)^2 dM(x) = \infty, \tag{43}$$

where k is a fixed point from (a, b)], then it admits self-adjoint extensions with simple spectrum. In the case where its deficiency index is (1, 1), i.e., only one condition in (43) is not satisfied, all self-adjoint extensions of this operator have simple spectra.

The theorem formulated below proves to be the most interesting; it deals with the case where L_0 is a self-adjoint operator [both conditions in (43) are satisfied].

Let us introduce an additional notation. For simplicity, let c be a continuity point of the function M. Consider two S_1-type strings $S_1((a, c], M)$ and $S_1([c, b), M)$. By definition, their spectral functions τ_l and τ_r are unique spectral functions of the boundary-value problems

$$l_{M,(a,c]}[y] - \lambda y = 0, \quad y(c) = 1, \quad y^+(c) = 0,$$

$$l_{M,[c,b)}[y] - \lambda y = 0, \quad y(c) = 1, \quad y^-(c) = 0,$$

respectively. Let Γ_l and Γ_r be their dynamic compliance coefficients, i.e.,

$$\Gamma_l(z) = \gamma_l + \int_{-0}^{+\infty} \frac{d\tau_l(\lambda)}{\lambda - z}, \quad \Gamma_r(z) = \gamma_r + \int_{-0}^{+\infty} \frac{d\tau_r(\lambda)}{\lambda - z},$$

where $\gamma_l(\gamma_r)$ is the length of the largest interval without masses with the right (left) end at the point c. Denote by $P_a[\tau_l]$ a set of points $\lambda \in \mathbb{R}$ at which the symmetric derivative $\tau_l^{(\prime)}(\lambda)$ exists and is finite and nonzero. Let $P_{a+}[\tau_l]$ be a set of points $\lambda \in \mathbb{R}$ where the finite nonreal limit $\lim_{\varepsilon \downarrow 0} \Gamma_l(\lambda + i\varepsilon)$ exists. (We have $P_{a+}[\tau_l] \subset P_a[\tau_l]$, and $P_a[\tau_l] \setminus P_{a+}[\tau_l]$ is a set of Lebesgue measure zero.) For τ_r, we introduce a similar notation.

Theorem 10 (Kats [20, 21]). *If the operator* L_0 *is self-adjoint, then it possesses a multiple spectrum if and only if a set* $K_+ := P_{a+}[\tau_l] \cap P_{a+}[\tau_r]$ *(a set* $K := P_a[\tau_l] \cap P_a[\tau_r]$*) has a positive Lebesgue measure. In this case, the set* K_+ *is the maximal homogeneous part of the spectrum of the operator* L_0 *with multiplicity two (to within sets of spectral measure zero). On the set* K_+*, the spectrum of the operator* L_0 *is absolutely continuous. Furthermore, it is a Lebesgue-type spectrum, i.e., any set* $A \subset K_+$ *has spectral measure zero if and only if its Lebesgue measure is zero.*

Thus, the *multiple part of the spectrum* of the operator L_0 *cannot have a singular component* (if this operator is self-adjoint).

Note that in cases where the operator L_0 is not self-adjoint or where it is self-adjoint and has a simple spectrum, there exists an lB-family $\mathcal{G}$ such that problem (7) possesses a spectral function.

All these results remain valid for the Sturm–Liouville operator and the operators generated in $\mathfrak{L}_M^{(2)}(I)$ by the differential operation

$$-\frac{d}{dM(x)}\left(y^-(x) - \int_{c-0}^{x-0} y(s)\,dQ(s)\right)$$

introduced by the author in [14].

3.2. Denseness of a Spectrum. The spectrum of a soft nonnegative self-adjoint extension of the operator L_0 (or the spectrum of the operator L_0 itself if the latter is self-adjoint) is called the spectrum of a string $S(\langle a, b\rangle, M)$. Note that the spectrum of a string $S([0, b\rangle, M)$ with the regular left end coincides with the spectrum of the principal spectral function of the string $S_1([0, b\rangle, M)$ provided that the right end of the former is singular; in the case where the right end is regular, the required spectrum coincides with the spectrum of the principal spectral function of the string $S_1([0, +\infty\rangle, \check{M})$ obtained from $S_1([0, b\rangle, M)$ by adjoining an infinite (if $b < +\infty$) mass-free interval to its right end. It follows from Krein's Theorem 6 that an arbitrary closed subset of the interval $[0, +\infty)$ may be a spectrum of a string. It is thus interesting to clarify the existing relations between the location of the spectrum and the behavior of the function M. For strings $S([0, b\rangle, M)$, this problem is partially solved by properties IV^0 and V^0 presented in Subsec. 2.7.

A string $S(\langle a, b\rangle, M)$ is attributed to the class $\mathfrak{S}_\alpha$ if its spectrum consists of numbers $(0 \le)\, \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ and the series

$$\sum_{j=1}^{\infty} (\lambda_j)^{-\alpha}$$

converges. The indicated property V^0 implies that $S(\langle a, b\rangle, M) \in \mathfrak{S}_1$ if and only if each of its ends is either an entry or an exit.

The spectra of regular strings $S([a, b], M)$ are always discrete and, as was showed by Krein (see [46, 50, 6]),

$$\lim_{n\to\infty} \frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi}\int_a^b \sqrt{M'(x)}\,dx \ (< \infty). \tag{44}$$

Thus, for a regular string, the question of whether it belongs to the class $\mathfrak{S}_\alpha$ is meaningful only for $\alpha \le 1/2$ and only in the case where $M'(x) = 0$ a.e. in $[a, b]$. If the string $S(\langle a, b\rangle, M)$ is singular and $M'(x) > 0$ on a set of positive measure, then one should consider only the case where $\alpha > 1/2$.

In [17], Kats presented sufficient conditions for a string S to belong to the class $\mathfrak{S}_\alpha$ with $\lambda \in (0, 1)$. This was probably the first time when the problem of growth of λ_n was discussed in the case where $M'(x) = 0$ a.e. These results were later improved (first, in [28] and then in [30, 32]) and, finally, formulated as the following theorem:

Theorem 11. *Let* $S(\langle a, b\rangle, M) \in \mathfrak{S}_1$ *and let* $r \mapsto \chi(r)$ *be a convex function nondecreasing on* $[0, +\infty)$ *and such that* $\chi(+0) = \chi(0) = 0$. *Then*

$$\left(\sum_{\lambda_j \neq 0} \chi(\lambda_j^{-1}) < \infty\right) \Leftrightarrow \left(\int_a^b dM(s) \int_0^{x_s(1)} \chi^+(x(M(s+x) - M(s-x)))dx < \infty\right),$$

where

$$x_s(1) = \sup\{x \in \mathbb{R}_+ \mid x(M(s+x) - M(s-x)) < 1,\ s - x > a,\ s + x < b\}.$$

Relation (44) was generalized to the case of singular strings $S([0, +\infty), M)$ by Birman and Borzov in [1]. They proved that (44) is true if there exists a function p decreasing on $[0, +\infty)$ and such that $p(x) > 0$ for any $x \in [0, +\infty)$ and

$$\int_0^{+\infty} p(x)dx < \infty, \quad \int_0^{+\infty} (p(x))^{-1} dM(x) < \infty.$$

This result was first formulated by McKean and Ray [68] but their proof contained an error. In [7], Dym and McKean gave a correct proof different from that presented in [1].

In the work [71] that appeared several months later than [17], Uno and Hong proved that, for the eigenvalues λ_n of a string $S([0, 1], M)$, where M is a Cantor singular function (the "Cantor ladder"), the inequalities $C_1 \leq n\lambda_n^{-\gamma} \leq C_2$ hold (here, C_1 and C_2 are positive constants and $\gamma = \log_6 2$).

In the work [2], for the case where $b - a < \infty$ and the function M is bounded and constant on intervals Δ_1, Δ_2, ... enumerated in the order of decreasing of their lengths, i.e., $|\Delta_1| \geq |\Delta_2| \geq \ldots$, and such that the sum of their lengths is equal to $b - a$, Borzov proved that the asymptotic equality $|\Delta_n| = O(n^{-\delta})$ as $n \to \infty$ implies that $\lambda_n \geq Cn^{1+\delta}$ for all $n \in \mathbb{N}$, where C is a positive constant (this result admits generalizations). This fact and Theorem 11 yield a series of assertions concerning the local properties of singular functions of bounded variation with the intervals of constancy similar to those described above.

In [28], Kats formulated a theorem that gave bilateral estimates for $\limsup_{n\to\infty} n\lambda_n^{-\gamma}$ with $\gamma \in (0, 1/2)$ depending on the behavior of the relations

$$\frac{M(s+h) - M(s+0)}{h^\beta}, \quad \frac{M(s+0) - M(s-h)}{h^\beta}$$

as $h \downarrow 0$; here, $\beta = \gamma/(1-\gamma)$ at all points s of the support of the M-measure.

3.3. Growth of Spectral Functions. Here, we present two author's results concerning the growth of spectral functions of strings $S_1([0, b\rangle, M)$ as $\lambda \to +\infty$. Recall that Krein established the fact that for any spectral function τ of a string $S_1([0, b\rangle, M)$, the integral

$$\int_{1}^{+\infty} \lambda^{-\alpha}\, d\tau(\lambda) \tag{45}$$

converges for $\alpha = 1$; in the case of $\alpha = 0$, it converges if and only if $M(+0) > M(0)$.

Theorem 12 [31]. *Let τ be a spectral function of a string $S_1([0, b\rangle, M)$ with heavy left end and let $M(0) = 0$. Assume that a function ξ is nondecreasing on $[k, +\infty)$, where $k > 0$ and $\xi(k) > 0$. Then, for any fixed $l \in (0, b)$ such that $lM(l) < k^{-1}$,*

$$\left(\int_{k}^{+\infty} \xi(\lambda)\lambda^{-2}\tau(\lambda)\,d\lambda < \infty\right) \Leftrightarrow \left(\int_{0}^{l} \xi\left(\frac{1}{xM(x)}\right)dx < \infty\right).$$

This theorem yields, in particular, the necessary and sufficient conditions for the convergence of integral (45) with $\alpha \in (0, 1)$.

Theorem 13 [26, 27]. *Under the conditions of Theorem 12, if*

$$\lim_{x \downarrow 0} x^{-\alpha} M(x) = A,$$

where A and α are constants, $0 < A \le +\infty$, $\alpha \in (0, +\infty)$, then

$$\tau(\lambda) = A^{\frac{-1}{\alpha+1}}\, \mathrm{B}(\alpha)\, \lambda^{\frac{\alpha}{\alpha+1}} + o\left(\lambda^{\frac{\alpha}{\alpha+1}}\right), \quad \lambda \to +\infty,$$

where

$$\mathrm{B}(\beta) = \left(\frac{\beta}{(\beta+1)^2}\right)^{\beta/(\beta+1)} \Gamma^{-2}\left(\frac{2\beta+1}{\beta+1}\right).$$

It follows from Theorem 13 that the asymptotic equality (26) established by Marchenko [67] for the spectral functions of the boundary-value problem

$$-y'' + q(x)y - \lambda y = 0, \quad 0 \le x < b, \quad y(0) = 1, \quad y'(0) = h, \quad h \in \mathbb{R},$$

holds provided that M possesses the right derivative equal to one at the point $x = 0$ but this is possible even in the case where M is a pure jump function.

Theorem 13 was generalized and completely inverted by Kasahara in [1].[4] Later, he used this result in the theory of one-dimensional quasidiffusion processes (see also [12, 13]).

3.4. Strings from the Class $\mathfrak{M}_s$ with Boundary Conditions Given at the Entry End that May Be Singular. A nondecreasing function M defined on the interval $I = (-\infty, b)$ with $b \le +\infty$ (or $I = (-\infty, b]$ with $b < +\infty$) is attributed to the class $\mathfrak{M}$ if $M \in \mathfrak{L}^{(1)}(-\infty, c)$, where $c < b$. A string $S(I, M)$ whose mass distribution function M belongs to $\mathfrak{M}$ is attributed to the class $\mathfrak{M}_s$. A spectral function of a boundary-value problem

$$l_{MI}[y] - \lambda y = 0, \qquad \lim_{x \downarrow -\infty} y(x) = 1 \tag{46}$$

[4] In [27], the author obtained a partial inversion of this theorem.

is called a spectral function of a string $S(I, M) \in \mathfrak{M}_s$. The existence of a spectral function of this string was announced in [14] and proved in [23]. Note that any string S_1 is, in fact, a string from $\mathfrak{M}_s$. Thus, the spectral theory of strings from the class $\mathfrak{M}_s$ developed in [22, 24] is a generalization of the spectral theory of strings S_1. The next theorem belongs to Kats and describes the set of all spectral functions of the string $S((-\infty, b], M) \in \mathfrak{M}_s$ with heavy right end. In this theorem, $\Phi(\cdot, z]$ is a (unique) solution of the boundary-value problem (46) and $\Phi(\cdot, z)$ is its nonextended part.

Theorem 14. *Under the indicated conditions, a function τ nondecreasing on $(-\infty, +\infty)$ is a spectral function of a string $S((-\infty, b], M) \in \mathfrak{M}_s$ if and only if it coincides with a function τ_h given by the relations*

$$\Omega_h(z) = \frac{1}{(\Phi(b, z))^2 + (\Phi^+(b, z))^2} \frac{\Phi(b, z)\, h(z) - \Phi^+(b, z)}{\Phi^+(b, z)\, h(z) + \Phi(b, z)}, \quad \operatorname{Im} z > 0,$$

$$\tau_h(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_0^\lambda \operatorname{Im} \Omega_h(\xi + i\varepsilon)\, d\xi \quad \forall \lambda \in (-\infty, +\infty)$$

with $h \in (\tilde{R})$ in the case where $M(b) - M(b-0) = 0$ and with $h \in ((\tilde{R}) \setminus (R_0))$ in the case where $M(b) - M(b-0) > 0$. The spectral function τ_n is orthogonal if and only if h is a real constant (maybe infinite). The spectral function τ_h possesses a nonnegative spectrum if and only if $h \in (\tilde{S})$.

In achieving this description, it was necessary to overcome the difficulty connected with the fact that here, unlike the description of the set of spectral functions of a regular string S_1 (Theorem 1), only one solution Φ of the string equation was available.

It was shown that $\Phi(b, z) = D(z)$ and $\Phi^+(b, z) = \mathcal{E}(z)$, where

$$D(z) = \prod_j \left(1 - \frac{z}{\lambda_j}\right), \quad \mathcal{E}(z) = -Mz \prod_j \left(1 - \frac{z}{\mu_j}\right), \tag{47}$$

and $M > 0$, $0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < \ldots$. Moreover, any pair of functions $D(z)$ and $\mathcal{E}(z)$ representable in the form (47) under the indicated conditions can be treated as $\Phi(b, z)$ and $\Phi^+(b, z)$ for some string $S((-\infty, b], M) \in \mathfrak{M}_s$. This enabled the author to prove Theorem 2 in [22] that gave a description of the set $T_{\mathfrak{M}}$ of all functions τ which may serve as spectral functions of the strings from $\mathfrak{M}_s$.

This theorem yields unexpected corollaries:

I. For any $\alpha < 1$, the set $T_{\mathfrak{M}}$ contains a continuously differentiable function τ such that $\tau'(\lambda)\, e^{-\lambda^\alpha} \to +\infty$ as $\lambda \to +\infty$.

II. For any function ρ nondecreasing on $[1, +\infty)$, there exists a function $\tau \in T_{\mathfrak{M}}$ such that $\tau(\lambda) > \rho(\lambda)$ for any $\lambda \in [1, +\infty)$.

There are many theorems clarifying relations between the growth of spectral functions τ of strings $S \in \mathfrak{M}_s$ and the behavior of mass distribution functions M of these strings in a right neighborhood of the point $-\infty$. For example (see the corollary of Theorem 1 in [22]), a function τ can be majorized by a polynomial if and only if $M(x) = o(|x|^{-1-\varepsilon})$ as $x \downarrow -\infty$ for some $\varepsilon > 0$. Sufficient conditions were established for the validity of the following relation: $\log \tau(\lambda) = o(\lambda)$ as $\lambda \to +\infty$; e.g., this is true if $|x| \log |x| M(x) = o(1)$ as $x \downarrow -\infty$.

Numerous works are devoted to the inverse problem. Thus, it was established ([24], Theorem 1) that an arbitrary nondecreasing function τ normalized by conditions (8), having no points of increase on $(-\infty, 0)$, and admitting a majorization by a polynomial on $[0, +\infty)$ belongs to $T_{\mathfrak{M}}$. Furthermore, there exists a string from the class $\mathfrak{M}_s$ for which τ serves as a spectral function and this string is unique to within a natural ambiguity. The theory of strings from the class $\mathfrak{M}_s$ was used by Kotani in [38]. He gave another description of the set $T_{\mathfrak{M}}$ based on the Krein–de Branges spaces. However, all significant results of this work repeated the corresponding results obtained in [22] and [24] as was, in fact, noted by Kotani in the introduction to [38].

Finally, note that Dym and McKean [7] solved the problem of interpolation of stationary Gaussian random processes by using strings whose spectral functions increase sufficiently rapidly. This was just the reason for considering the strings from the class $\mathfrak{M}_s$ which were called a new class of strings (see [7], Secs. 6.12 and 6.13).

REFERENCES

1. M. S. Birman and V. V. Borzov, "On the asymptotics of discrete spectra of some singular differential operators," *Probl. Mat. Fiz.,* Issue 5, 24–38 (1971).
2. V. V. Borzov, "On the quantitative characteristics of singular measures," *Probl. Mat. Fiz.,* Issue 4, 42–47 (1970).
3. L. De Branges, *Hilbert Spaces of Entire Functions,* Prentice-Hall, London (1968).
4. S. Watanabe, "On time inversion of one-dimensional diffusion processes," *Z. Wahrscheinlichkeitstheor. verw. Geb.,* **31**, 115–124 (1975).
5. F. R. Gantmakher and M. G. Krein, *Oscillation Matrices and Kernels. Small Oscillations of Mechanical Systems* [in Russian], Gostekhteorizdat, Moscow-Leningrad (1950).
6. I. Ts. Gokhberg and M. G. Krein, *Theory of Volterra Operators in Hilbert Spaces and Its Applications* [in Russian], Nauka, Moscow (1967).
7. H. Dym and H. P. McKean, *Gaussian Processes, Function Theory, and the Inverse Spectral Problem,* Academic Press, New York (1976).
8. E. B. Dynkin, *Markov Processes* [in Russian], Fizmatgiz, Moscow (1963).
9. K. Itô, *Probability Processes* [Russian translation], Issue 2, Inostr. Lit., Moscow (1963).
10. K. Itô and H. P. McKean, *Diffusion Processes and Their Sample Paths,* Springer-Verlag, Berlin-Heidelberg-New York (1965).
11. Y. Kasahara, "Spectral theory of generalized second order differential operators and its applications to Markov processes," *Jpn. J. Math.,* **1**, No. 1, 67–84 (1975).
12. Y. Kasahara, "Limit theorems of occupation times for Markov processes," *Publ. RIMS, Kyoto Univ.,* **12**, 801–818 (1977).
13. Y. Kasahara, S. Kotani, and H. Watanabe, "On the Green functions of 1-dimensional diffusion processes," *Publ. RIMS, Kyoto Univ.,* **16**, 175–188 (1980).
14. I. S. Kats, "On the existence of spectral functions of singular second-order differential systems," *Dokl. Akad. Nauk SSSR*, **106**, No. 1, 15–18 (1956).
15. I. S. Kats, "On the behavior of spectral functions of second-order differential systems," *Dokl. Akad. Nauk SSSR,* **106**, No. 2, 183–186 (1956).
16. I. S. Kats, "General theorems on the behavior of spectral functions of second-order differential systems," *Dokl. Akad. Nauk SSSR,* **122**, No. 6, 974–977 (1958).
17. I. S. Kats, "On the denseness of the spectrum of a string," *Dokl. Akad. Nauk SSSR,* **126**, No. 6, 1180–1182 (1959).
18. I. S. Kats, "On the type of the spectrum of a singular string," *Izv. Vyssh. Uchebn. Zaved., Mat.,* No. 1 (26), 57–64 (1962).
19. I. S. Kats, "Two general theorems on the asymptotic behavior of spectral functions of second-order differential systems," *Izv. Akad. Nauk SSSR, Ser. Mat.,* **26**, No. 1, 53–78 (1962).
20. I. S. Kats, "On the multiplicity of the spectrum of a second-order differential operator," *Dokl. Akad. Nauk SSSR,* **145**, No. 3, 510–513 (1962).
21. I. S. Kats, "Multiplicity of the spectrum of a second-order differential operator and expansion in eigenfunctions," *Izv. Akad. Nauk SSSR, Ser. Mat.,* **27**, No. 5, 1081–1112 (1963).
22. I. S. Kats, "On the behavior of spectral functions of second-order differential systems with a boundary condition at a singular end," *Dokl. Akad. Nauk SSSR,* **157**, No. 1, 34–37 (1964).
23. I. S. Kats, "Existence of spectral functions of generalized second-order differential systems with boundary conditions at a singular end," *Mat. Sb.,* **68 (110)**, No. 2, 174–227 (1965).
24. I. S. Kats, "Some cases of the uniqueness of a solution of the inverse problem for strings with a boundary condition at a singular end," *Dokl. Akad. Nauk SSSR,* **164**, No. 5, 975–978 (1965).

25. I. S. Kats, "Integral characteristics of the growth of spectral functions of the generalized second-order boundary-value problems with boundary conditions at a regular end," *Izv. Akad. Nauk. SSSR, Ser. Mat.*, **35**, No. 1, 175–205 (1971).

26. I. S. Kats, "Power asymptotics of the spectral functions of generalized second-order boundary-value problems," *Dokl. Akad. Nauk SSSR,* **203**, No. 4, 752–755 (1972).

27. I. S. Kats, "Generalization of the Marchenko asymptotic formula for spectral functions of a second-order boundary-value problem," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **37**, No. 2, 422–436 (1973).

28. I. S. Kats, "Denseness of the spectrum of a string," *Dokl. Akad. Nauk SSSR,* **221**, No. 3, 16–19 (1973).

29. I. S. Kats, "Description of the set of spectral functions of a regular string bearing a concentrated mass at its end which is free of boundary conditions," *Izv. Vyssh. Uchebn. Zaved., Mat.*, No. 7 (146), 27–33 (1974).

30. I. S. Kats, "Some general theorems on the denseness of the spectrum of a string," *Dokl. Akad. Nauk SSSR,* **238**, No. 4, 785–788 (1978).

31. I. S. Kats, "Integral estimates for the growth of spectral functions of a string," *Ukr. Mat. Zh.*, **34**, No. 3, 296–302 (1982).

32. I. S. Kats, "Integral estimates for the distribution of the spectrum of a string," *Sib. Mat. Zh.*, **27**, No. 2, 62–74 (1986).

33. I. S. Kats, "Denseness of the spectrum of a singular string," *Izv. Vyssh. Uchebn. Zaved., Mat.*, No. 3, 23–30 (1990).

34. I. S. Kats and M. G. Krein, "A criterion of the discreteness of the spectrum of a singular string," *Izv. Vyssh. Uchebn. Zaved., Mat.*, No. 2 (3), 136–153 (1958).

35. I. S. Kats and M. G. Krein, "*R*-functions are analytic functions that map the upper half plane onto itself," Appendix 1 in: F. Atkinson, *Discrete and Continuous Boundary-Value Problems* [Russian translation], Mir, Moscow (1968).

36. I. S. Kats and M. G. Krein, "On spectral functions of a string," Appendix 2 in: F. Atkinson, *Discrete and Continuous Boundary-Value Problems* [Russian translation], Mir, Moscow (1968).

37. L. P. Klotz and H. Langer, "Generalized resolvents and spectral functions of a matrix generalization of the Krein – Feller second order derivative," *Math. Nachr.*, **100**, 163–186 (1981).

38. S. Kotani, "On a generalized Sturm – Liouville operator with a singular boundary," *J. Math. Kyoto Univ.*, **15**, No. 2, 423–454 (1975).

39. S. Kotani and S. Watanabe, "Krein's spectral theory of strings and generalized diffusion processes," Springer Lecture Notes Math., **923**, 235–259 (1982).

40. M. G. Krein, "On the representations of functions by Fourier – Stieltjes integrals," *Uchen. Zapiski Kuibyshev. Ped. Inst.*, Issue 7, 123–148 (1943).

41. M. G. Krein, "On the problem of extension of spiral arcs in Hilbert spaces," *Dokl. Akad. Nauk SSSR,* **45**, No. 4, 147–150 (1944).

42. M. G. Krein, "General method for the decomposition of positive definite kernels into elementary products," *Dokl. Akad. Nauk SSSR,* **53**, No. 1, 3–6 (1946).

43. M. G. Krein, "To the theory of entire functions of exponential type," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **11**, 309–326 (1947).

44. M. G. Krein, "On Hermitian operators with guiding functionals," *Sb. Tr. Inst. Mat. Akad. Nauk Ukr. SSR,* **10**, 83–106 (1948).

45. M. G. Krein, "Solution of the inverse Sturm – Liouville problem," *Dokl. Akad. Nauk SSSR,* **76**, No. 1, 21–24 (1951).

46. M. G. Krein, "Determination of the density of an inhomogeneous symmetric string from the spectrum of its frequencies," *Dokl. Akad. Nauk SSSR,* **76**, No. 3, 345–348 (1951).

47. M. G. Krein, "On certain problems of maximum and minimum for characteristic numbers and on the Lyapunov regions of stability," *Prikl. Mat. Mekh.*, **15**, Issue 3, 323–348 (1951).

48. M. G. Krein, "On the indefinite case of the Sturm – Liouville boundary-value problem on the interval (0, •)," *Izv. Akad. Nauk. SSSR, Ser. Mat.*, **16**, No. 2, 293–324 (1952).

49. M. G. Krein, "On inverse problems for inhomogeneous strings," *Dokl. Akad. Nauk SSSR,* **82**, No. 5, 669–672 (1952).

50. M. G. Krein, "On a generalization of Stieltjes results," *Dokl. Akad. Nauk SSSR,* **87**, No. 6, 881–884 (1952).

51. M. G. Krein, "New problems in the theory of oscillations of Sturm systems," *Prikl. Mat. Mekh.*, **16**, Issue 5, 555–568 (1952).

52. M. G. Krein, "On the transition function of a second-order one-dimensional boundary-value problem," *Dokl. Akad. Nauk SSSR,* **88**, No. 3, 405–408 (1953).

53. M. G. Krein, "An analog of the Chebyshev – Markov inequality in a one-dimensional boundary-value problem," *Dokl. Akad. Nauk SSSR,* **89**, No. 1, 5–8 (1953).

54. M. G. Krein, "Some cases of effective determination of the density of an inhomogeneous string from its spectral function," *Dokl. Akad. Nauk SSSR,* **93**, No. 4, 617–620 (1953).

55. M. G. Krein, "On the inverse problems in filter theory and λ-domains of stability," *Dokl. Akad. Nauk SSSR,* **93**, No. 5, 767–770 (1953).

56. M. G. Krein, "Principal approximation problem in the theory of extrapolation and filtration of stationary random processes," *Dokl. Akad. Nauk SSSR,* **94**, No. 1, 13–16 (1954).

57. M. G. Krein, "An effective method for the solution of inverse boundary-value problems," *Dokl. Akad. Nauk SSSR,* **97**, No. 6, 987–990 (1954).

58. M. G. Krein, "Integral equations that generate second-order differential equations," *Dokl. Akad. Nauk SSSR,* **97**, No. 4, 21–24 (1954).

59. M. G. Krein, "Determination of the potential of a particle according to its S-function," *Dokl. Akad. Nauk SSSR,* **105**, No. 3, 433–436 (1955).
60. M. G. Krein, "Continual analogs of the assertions on polynomials orthogonal on the unit circle," *Dokl. Akad. Nauk SSSR,* **105**, No. 4, 637–640 (1955).
61. M. G. Krein, "On the theory of accelerants and S-matrices of canonical differential systems," *Dokl. Akad. Nauk SSSR,* **111**, No. 6, 1167–1170 (1956).
62. M. G. Krein, "Chebyshev – Markov inequalities in the theory of spectral functions of a string," *Mat. Issled.,* **5**, No. 1, 77–101 (1970).
63. U. Küchler, "Some asymptotic properties of the transition densities of one-dimensional quasidiffusions," *Publ. RIMS, Kyoto Univ.,* **16**, No. 1, 245–268 (1980).
64. H. Langer, *Zur Spectraltheorie Verallgemeinerter Gewönnlicher Differenzialoperatoren Zweiter Ordung mit Einer Nichtmonotonen Gewichtfunktion,* University of Jyväskylä, Department of Mathematics, Report 14 (1972).
65. H. Langer, L. Partzsch, and D. Schütze, "Über verallgemeinerte gewönliche Differentialoperatoren mit nichtlokalen Randbedingungen und die von ihnen erzeugten Markov–Processe," *Publ. RIMS, Kyoto Univ.,* **7**, No. 3, 660–702 (1972).
66. B. M. Levitan, "On the asymptotic behavior of the spectral function of a self-adjoint second-order differential equation," *Izv. Akad. Nauk SSSR, Ser. Mat.,* **16**, 325–352 (1952).
67. V. A. Marchenko, "Some problems in the theory of one-dimensional second-order linear differential operators. I," *Tr. Mosk. Mat. Obshch.,* **1**, 381–422 (1952).
68. H. P. McKean and D. B. Ray, "Spectral distribution of a differential operator," *Duke Math. J.,* **29**, 281–292 (1962).
69. M. A. Naimark, *Linear Differential Operators* [in Russian], Nauka, Moscow (1969).
70. M. Tomisaki, "On the asymptotic behavior of transition probability densities of one-dimensional diffusion processes," *Publ. RIMS Kyoto Univ.,* **12**, 819–834 (1977).
71. T. Uno and I. Hong, "Some consideration of eigenvalues for the equation $d^2u / dx^2 + \lambda\rho(x)\, u = 0$," *Jpn. J. Math.,* **29**, 152–164 (1959).
72. W. Feller, "On second order differential operators," *Ann. Math.,* **61**, 90–105 (1955).