BERNSTEIN-TYPE INEQUALITIES FOR $\mathcal{L}$-SPLINES

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New Bernstein-type inequalities are obtain for 2π-periodic $\mathcal{L}$-splines associated with a differential operator $\mathcal{L}_r(D)$ of degree r with fixed real coefficients.

1. Assume that C and L_p, $1 \le p \le \infty$, are spaces of real 2π-periodic functions with the corresponding norms; $\|\cdot\|_{L_p} = \|\cdot\|_p$. Let C^r, $r \in \mathbb{N}$, be a set of functions $f \in C$ such that $f^{(r)} \in C$; $C^0 = C$; let L_p^r be a set of functions $f \in C$ such that $f^{(r-1)}$ is locally absolutely continuous and $\|f^{(r)}\|_p < \infty$, $L_p^0 = L_p$; and let L_V^r be a set of functions $f \in C$ such that $f^{(r-1)}$ is locally absolutely continuous and $\bigvee_0^{2\pi}(f^{(r)}) < \infty$. We denote by $\mathcal{T}_{2n+1}$, $n \in \mathbb{N}$, the set of trigonometric polynomials whose degree is at most n. Finally, let $S_{2n,r}$, $r \in \mathbb{Z}_+$, denote the set of 2π-periodic polynomial splines of degree r with deficiency 1 and knots $l\pi/n$, $l \in \mathbb{Z}$.

In approximation theory, an important role is played by the Bernstein inequality [1] for trigonometric polynomials $\tau \in \mathcal{T}_{2n+1}$

$$\|\tau^{(k)}\|_\infty \le n^k \|\tau\|_\infty, \tag{1}$$

which turns into the equality for the polynomials of the form $\tau(x) = a\cos n(x - x_0)$, a, $x_0 \in \mathbb{R}$. This inequality was generalized in various manners. Thus, for $\tau \in \mathcal{T}_{2n+1}$, the unimprovable inequality

$$\frac{\|\tau^{(k)}\|_p}{\|\cos(\cdot)\|_p} \le n^k \frac{\|\tau\|_q}{\|\cos(\cdot)\|_q}, \quad k \in \mathbb{N}, \tag{2}$$

was established by Zigmund [2] in the case where $p = q \in [1, \infty)$; Tikov [3] and Ligun [4] proved this inequality in cases where $p \in [1, \infty)$, $q = \infty$, and $p = 1$, $q \in (1, \infty)$, respectively. There also exist other generalizations and specifications of inequalities (1) and (2). Omitting the details, we only note that most of these results can be found in [5–9].

There also exist analogs of inequalities (1) and (2) for splines. Denote by $\varphi_{n,r}$ an rth periodic integral of the function $\varphi_{n,0}(x) = \operatorname{sgn}\sin nx$ with a mean value zero on a period. Tikhomirov [10] (for $p = q = \infty$), Subbotin [11] (for $p = q = 1$), Ligun [4, 12] (for $p \in [1, \infty)$, $q = \infty$ and $p = 1$, $q \in (1, \infty)$), and Babenko and Pichugov [13] (for $p = q = 2$) have proved the validity of the unimprovable inequality

$$\frac{\|s^{(k)}\|_p}{\|\varphi_{n,r-k}\|_p} \le \frac{\|s\|_q}{\|\varphi_{n,r}\|_q}, \quad k = 1, \dots, r, \tag{3}$$

for $s \in S_{2n,r}$, $n, r \in \mathbb{N}$. For information concerning the other well-known Bernstein-type inequalities for splines from $S_{2n,r}$, we refer the reader to [9, 14].

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In this paper, we generalize the inequalities (3) for the case where the operator of k-times differentiation is replaced by a more general linear differential operator with constant coefficients (some of well-known results of this sort can be found in [9, 15, 16]). We also show the possibilities of the general approach based on the use of theorems on comparison of derivatives, permutations, and Σ-permutations. Note that Stechkin [5] was apparently the first who used the ideas connected with comparison of derivatives for solving problems of this sort. Information concerning the use of the theorems on comparison of derivatives and permutations for proving the Bernstein-type inequalities can be found, for example, in [8, 14, 17]. In this paper, we also obtain a new Kolmogorov-type inequality which, as we hope, is of interest itself.

2. Principal Results. First, we give some necessary definitions and notations. Let

$$\mathcal{L}_r(y) = a_0 y^r + a_1 y^{r-1} + \ldots + a_r, \tag{4}$$

where $r \in \mathbb{N}$, be an arbitrary algebraic polynomial of degree r with real coefficients, and let $\mathcal{L}_r(D) = a_0 D^r + a_1 D^{r-1} + \ldots + a_r$ be a differential operator associated with this polynomial $(D = d/dx)$.

If $\mathcal{L}_r(D)s \in S_{2n,0}$, a function $s \in C^{r-1}$ is called a periodic $\mathcal{L}$-spline with equidistant knots $l\pi/n$, $l \in \mathbb{Z}$, corresponding to the operator $\mathcal{L}_r(D)$. By $S_{2n,\mathcal{L}_r}$, we denote the set of all such $\mathcal{L}$-splines. It is clear that $S_{2n,(\cdot)^r} = S_{2n,r}$.

Denote by $\varphi_{n,\mathcal{L}_r}(x)$ a 2π-periodic function having mean value zero on the period and such that $\mathcal{L}_r(D)\varphi_{n,\mathcal{L}_r} = \varphi_{n,0}$.

If $f \in L_1$ and $f \geq 0$ almost everywhere, then by $P(f,t)$ we denote a decreasing permutation of the restriction of f to the period (see, e.g., [18, § 5.4]). In addition, if $g \in L_1$ and $g \geq 0$ almost everywhere and

$$\int_0^x P(f,t)\,dt \leq \int_0^x P(g,t)\,dt,$$

for any $x \in [0, 2\pi]$, then

$$f \prec g. \tag{5}$$

It is well-known (see, e.g., [18, p. 96]) that if $f, g \in L_p$, $1 \leq p \leq \infty$, then it follows from inequality (5) that $\|f\|_p \leq \|g\|_p$.

We now formulate the principal results.

Theorem 1. *Let $n, r \in \mathbb{N}$, and let $\mathcal{L}_r$ be a polynomial of the form* (4) *with roots $y_1, \ldots, y_r$ such that*

$$\mathcal{L}_r(y) = \mathcal{L}_k(y)\,\mathcal{L}_{r-k}(y), \tag{6}$$

where $\mathcal{L}_k(y)$ and $\mathcal{L}_{r-k}(y)$ are polynomials with real coefficients of degrees k and $r-k$, respectively. If $n > 2\max\{|\operatorname{Im} y_k| : k = 1, \ldots, r\}$, then, for any $\mathcal{L}$-spline $s \in S_{2n,\mathcal{L}_r}$, the following inequality holds

$$\frac{\|\mathcal{L}_k(D)s\|_p}{\|\varphi_{n,\mathcal{L}_{r-k}}\|_p} \leq \frac{\|s\|_p}{\|\varphi_{n,\mathcal{L}_r}\|_p}, \quad p = 1, 2, \infty. \tag{7}$$

In particular,

$$(2\pi)^{-1/p}\,\|\mathcal{L}_r(D)s\|_p \le \frac{\|s\|_p}{\|\varphi_{n,\mathcal{L}_r}\|_p}. \tag{8}$$

Furthermore, if $\mathcal{L}_k(y)$ *in* (6) *is such that* $\mathcal{L}_k(0) = 0$, *then for any* $s \in S_{2n,\mathcal{L}_r}$, *we have*

$$\left|\mathcal{L}_k(D)s - \lambda\right| \prec \left|\frac{\|s\|_\infty}{\|\varphi_{n,\mathcal{L}_r}\|_\infty}\varphi_{n,\mathcal{L}_{r-k}} - \lambda\right|, \tag{9}$$

and hence,

$$\frac{\|\mathcal{L}_k(D)s\|_p}{\|\varphi_{n,\mathcal{L}_{r-k}}\|_p} \le \frac{\|s\|_\infty}{\|\varphi_{n,\mathcal{L}_r}\|_\infty}, \tag{10}$$

for any $p \in [1,\infty]$.

If a polynomial $\mathcal{L}_r$ *of the form* (6) *has only real roots, then, for any* $s \in S_{2n,\mathcal{L}_r}$ *and* $q \in [1,\infty]$, *we have*

$$\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)}{\bigvee_0^{2\pi}(\varphi_{n,\mathcal{L}_{r-k}})} \le \frac{\|s\|_q}{\|\varphi_{n,\mathcal{L}_r}\|_q}. \tag{11}$$

Consequently, if $\mathcal{L}_k(0) = 0$, *then*

$$\frac{\|\mathcal{L}_k(D)s\|_1}{\|\varphi_{n,\mathcal{L}_{r-k}}\|_1} \le \frac{\|s\|_q}{\|\varphi_{n,\mathcal{L}_r}\|_q}. \tag{12}$$

Inequalities (9) and (10) have been announced in [9].

The analogs of the inequalities (6)–(12) also hold for trigonometric polynomials. They can easily be derived from the results obtained in [3, 4, 9]; one can also prove these inequalities in a manner similar to the proof of Theorem 1 given below (the proof is even simpler). To give the complete picture, we present these inequalities.

Theorem 2. *Let* $n, r \in \mathbb{N}$, *let* $\mathcal{L}_r$ *be a polynomial of the form* (4) *with roots* $y_1, \ldots, y_r$, *and let* $\tau \in \mathcal{T}_{2n+1}$. *If* $n > 2\max\{|\operatorname{Im} y_k| : k = 1, \ldots, r\}$, *then* $|\mathcal{L}_r(D)\tau| \prec |\mathcal{L}_r(in)|\,|\tau|$, *and hence, for any* $p \in [1,\infty]$, *we have* $\|\mathcal{L}_r(D)\tau\|_p \le |\mathcal{L}_r(in)|\,\|\tau\|_p$.

Furthermore, if $\mathcal{L}_r(0) = 0$, *then*

$$|\mathcal{L}_r(D)\tau - \lambda| \prec |\,\|\tau\|_\infty |\mathcal{L}_r(in)| \cos(\cdot) - \lambda\,|,$$

for any $\lambda \in \mathbb{R}$, *and*

$$\|\mathcal{L}_r(D)\tau\|_1\ |\cos(\cdot)| \prec 4\cdot|\mathcal{L}_r(in)|\,|\tau(\cdot)|,$$

and hence, for any $p, q \in [1,\infty]$,

$$\frac{\|\mathcal{L}_r(D)\tau\|_p}{\|\cos(\cdot)\|_p} \le |\mathcal{L}_r(in)|\,\|\tau\|_\infty$$

and

$$\frac{\|\mathcal{L}_k(D)\tau\|_1}{4|\mathcal{L}_r(in)|} \le \frac{\|\tau\|_q}{\|\cos(\cdot)\|_q}.$$

3. Let us present results which play a principal role in the proof of Theorem 1. The following theorem is a generalization of Kolmogorov's theorem on comparison of derivatives (see [19, § 5.4]). It can be found, for example, in [8] (statement 3.2.2).

Theorem 3. *Let functions f and φ be continuously differentiable everywhere on the real axis. Suppose that a function φ is $2l$-periodic and that on the interval $(a, a+2l)$, where a is the point of absolute extremum for the function φ, there exists a point c such that the function φ is strictly monotone on both intervals (a, c) and $(c, a+2l)$. Assume also that for any y, on each interval of monotonicity of the function φ, the sign of the difference $\varphi(\cdot)-\varphi(\cdot-y)$ either remains unchanged or changes only once (the sign changes from "+" to "–", if the function φ decreases, and from "–" to "+", if the function φ increases), and let $\min_t \varphi(t) \le f(x) \le \max_t \varphi(t)$. If, in addition, the points x and y are such that $\varphi(x) = f(y)$ and $\varphi'(x)f'(y) \ge 0$, then*

$$|f'(y)| \le |\varphi'(x)|.$$

The following theorem (see, e.g, [8, statement 3.2.7]) is a generalization of Korneichuk's theorem [19, Theorem 6.8.1].

Theorem 4. *Assume that functions f and φ satisfy the conditions of Theorem 3 for $l = 2\pi/n$, $n \in \mathbb{N}$, and the function f is periodic with period 2π. Suppose that the functions f' and φ' also satisfy the conditions of Theorem 3 for $l = 2\pi/n$. Then $|f'-\lambda| \prec |\varphi'-\lambda|$, for any $\lambda \in \mathbb{R}$.*

Denote by $\nu(q)$ the number of sign changes of a 2π-periodic function q on a period. If a function f is differentiable at the point x and $a \in \mathbb{R}$, then

$$\varphi'(x) + af(x) = e^{-ax}(f(x)\,e^{ax})'.$$

Together with the Rolle theorem, this implies that for any piecewise-continuous differentiable 2π-periodic function f and any $a \in \mathbb{R}$, we have

$$\nu(f'+af) \ge \nu(f). \tag{13}$$

Further, for any function $f \in C^2$, the following identity is valid (see, e.g., [15])

$$(D^2 - 2\gamma D + \gamma^2 + \alpha^2)f(x) = \frac{e^{\gamma(x-a)}}{\sin\alpha(x-a)} D\Big(\sin^2\alpha(x-a) D\big(e^{-\gamma(x-a)}\sin^{-1}\alpha(x-a) f(x)\big)\Big).$$

This identity implies that if $f \in C^1$, the function f'' is piecewise-continuous, $f(b) = f(a) = 0$, $b - a < \pi/\alpha$, and $f(x) \not\equiv 0$ on the interval (a, b), then there exists a point $\xi \in (a, b)$, such that $\mathcal{L}(D)f(x)\operatorname{sgn} f(\xi) < 0$, where $\mathcal{L}(D) = D^2 - 2\gamma D + \gamma^2 + \alpha^2$. Consequently, if the length of the largest interval of constant sign for the function f does not exceed π/α, then

$$\nu(\mathcal{L}(D)f) \ge \nu(f). \tag{14}$$

Let $\varphi_{\lambda,0}(x) = \operatorname{sgn}\sin\lambda x$, $\lambda > 0$, let the function $\varphi_{\lambda,\mathcal{L}_r}(x)$ be a $2\pi/\lambda$ periodic solution of the equation

$\mathcal{L}_r(D)f(x) = \varphi_{\lambda,0}(x)$, and let $g_{\lambda,\mathcal{L}_r}(x) = (4\lambda)^{-1}\varphi_{\lambda,\mathcal{L}_r}(x)$. It is clear that $\varphi_{\lambda,\mathcal{L}_r}(x+\pi/\lambda) = -\varphi_{\lambda,\mathcal{L}_r}(x)$ for any x. Taking into account inequalities (13) and (14), it is easy to check that if $\lambda > 2\max\{|\operatorname{Im} y_k| : k = 1, \ldots, r\}$ (here, y_k are the roots of the polynomial $\mathcal{L}_r$), and x_0 is the point of absolute minimum of the function $\varphi_{\lambda,\mathcal{L}_r}$, then a point y_0 can be indicated on the interval $(x_0, x_0 + 2\pi/\lambda)$ such that the function $\varphi_{\lambda,\mathcal{L}_r}(x)$ monotonically increases when $x \in (x_0, y_0)$ and monotonically decreases when $x \in (y_0, x_0 + 2\pi/\lambda)$.

By $\Pi(f; x)_{[0,2\pi/\lambda]}$, we denote Korneichuk's Σ-permutation of the restriction of a $2\pi/\lambda$-periodic function f to a period (see [19, § 6.4] for the definition of Σ-permutations and their properties). For $\lambda = 1$, we write $\Pi(f; x)$ instead of $\Pi(f; x)_{[0,2\pi]}$. For $\lambda > 0$, we set

$$\Phi_{\lambda,\mathcal{L}_r}(x) = \begin{cases} \lambda\Pi(g_{\lambda,\mathcal{L}_r}; x)_{[0,2\pi/\lambda]}, & 0 \le x \le \pi/\lambda; \\ 0, & x \ge \pi/\lambda. \end{cases}$$

For $\mathcal{L}_r(y) = y^r$, the following statement turns into the well-known Korneichuk theorem (see [19, § 6.7]) on comparison of Σ-permutations.

Theorem 5. *Assume that* $f \in L_V^r$, $r \in \mathbb{N}$, $\bigvee_0^{2\pi}(\mathcal{L}_r f) \le 1$,

$$\lambda > 2\max\{|\operatorname{Im} y_k| : k = 1,\ldots,r\}, \tag{15}$$

and

$$\bigvee_0^{2\pi}(f) \le 2\Phi_{\lambda,\mathcal{L}_r}(0). \tag{16}$$

If, for given $x \in (0, 2\pi)$, *we have*

$$\Pi(f, x) = \Phi_{\lambda,\mathcal{L}_r}(x), \tag{17}$$

and there exists $\Pi'(f, x)$, *then*

$$|\Pi'(f, x)| \le |\Phi'_{\lambda,\mathcal{L}_r}(x)|. \tag{18}$$

For $\lambda = n$, this statement was proved in [20].

Proof. First, we assume that $\lambda = m/n$, $m, n \in \mathbb{N}$. Condition (16) means that

$$\bigvee_0^{2\pi n}(f) \le \bigvee_0^{2\pi n}(g_{m/n,\mathcal{L}_r}). \tag{19}$$

Let $f = a + \sum_k f_k$ be an expansion in simple functions of the restriction of a function f to $[b, b + 2\pi n]$, where b is the point of absolute minimum of the function $|f|$ which is equal to $|a|$ (see, for example, [19, § 6.3]), let $[\alpha_k, \beta_k]$ be the support of the function f_k, and let $[\alpha'_k, \beta'_k] = \{t \in [\alpha_k, \beta_k] : |f_k(t)| = \max_u |f_k(u)|\}$. Denote by $A(x)$ a set of indices k such that $\beta'_k - \alpha'_k \le x \le \beta_k - \alpha_k$. If $k \in A(x)$, then, by t_k and τ_k, we denote points belonging to $[\alpha_k, \beta_k]$ such that $\tau_k - t_k = x$ and $f_k(t_k) = f_k(\tau_k)$ (and hence, $f(t_k) = f(\tau_k)$).

Let us enumerate the intervals (t_k, τ_k), $k = 1, \dots, l$, from the left to the right, and let $t_{l+1} = t_1 + 2\pi n$, $\tau_{l+1} = \tau_1 + 2\pi n$. If $\operatorname{sgn} f(t_k) = \operatorname{sgn} f(t_{k+1})$ for some k, and the function f does not change its sign on the interval $[\tau_k, t_{k+1}]$, then we denote, by ξ_k' and ξ_k'', the point of absolute minimum of $|f(t)|$ on the interval $[\tau_k, t_{k+1}]$. If $\operatorname{sgn} f(t_k) = \operatorname{sgn} f(t_{k+1})$, and the function f changes its sign in the interval $[\tau_k, t_{k+1}]$, then ξ_k' denotes a point belonging to the interval $[\tau_k, t_{k+1}]$ such that $f(\xi_k') = 0$ and $\operatorname{sgn} f(t) = -\operatorname{sgn} f(t_k)$ for all points $t > \xi_k'$ sufficiently close to the point ξ_k'; and ξ_k'' denotes a point belonging to the interval $[\tau_k, t_{k+1}]$ such that $f(\xi_k'') = 0$ and $\operatorname{sgn} f(t) = -\operatorname{sgn} f(t_k)$ for all points $t > \xi_k''$ sufficiently close to ξ_k''.

Denote, by $x_1 \le x_2 \le \dots \le x_{2j}$, the points t_k and ξ_k' enumerated in nondecreasing order, and let $y_1 \le y_2 \le \dots \le y_{2j}$ denote the points τ_k and ξ_k'' enumerated in the same manner. We set $f_*(t) = 2\sum_{k=1}^{2j}(-1)^k f(t + x_k - x_1)$. Then

$$f_*(t) = 2\sum_{k=1}^{2j}(-1)^k \left(f(t + x_k - x_1) - f(t + x_{k+1} - x_1)\right),$$

where $x_{2j+1} = x_1 + 2\pi n$. Taking (19) into account, we get

$$\| f_* \|_\infty \le \bigvee_0^{2\pi n}(f) \le \bigvee_0^{2\pi n}(g_{m/n, \mathcal{L}_r}) = n \| \varphi_{m/n, \mathcal{L}_r} \|_\infty; \tag{20}$$

moreover,

$$\| \mathcal{L}_r(D) f_* \|_\infty \le \bigvee_0^{2\pi n}(\mathcal{L}_r(D) f) \le n = n \| \mathcal{L}_r(D)\, \varphi_{m/n, \mathcal{L}_r} \|_\infty. \tag{21}$$

Taking (20), (21), and (13)–(15) into account, we obtain for any $c \in \mathbb{R}$,

$$\nu\left(f_*(\cdot + c) - n\, \varphi_{m/n, \mathcal{L}_r}(\cdot)\right) \le 2m$$

(here, $\nu(f)$ is the number of sign changes of the function f on the interval $[0, 2\pi n]$). Together with Theorem 3, this implies that if

$$f_*(t_*) = n\, \varphi_{m/n, \mathcal{L}_r}(\tau_*) \tag{22}$$

and

$$f_*'(t_*)\, \varphi_{m/n, \mathcal{L}_r}'(\tau_*) \ge 0, \tag{23}$$

then

$$| f_*'(t_*) | \le n\, | \varphi_{m/n, \mathcal{L}_r}'(\tau_*) |. \tag{24}$$

Assume that $t_* = x_1$ and τ_* is chosen so that conditions (22) and (23) are satisfied. Then inequality (24) holds, and, in addition,

$$| f_*'(x_1) | \le 2\sum_{k=1}^{2j} \| f'(x_k) \|. \tag{25}$$

Taking into account the definition of the function f_* and condition (17), we obtain

$$f_*(t_*) = f_*(x_1) = 2\Pi(f, x) = 2\Phi_{m/n, \mathcal{L}_r}(x) = n\,\varphi_{m/n, \mathcal{L}_r}(\tau_*).$$

Consequently, by virtue of (24) and (25),

$$\sum_{k=1}^{2j} \|f'(x_k)\| \le \tfrac{1}{2}\,|n\varphi'_{m/n, \mathcal{L}_r}(\tau_*)|. \tag{26}$$

Consider the interval of constancy of the sign of $\varphi_{m/n, \mathcal{L}_r}$ containing the point τ_*. On this interval, there exists only one point τ_{**} such that

$$\varphi_{m/n, \mathcal{L}_r}(\tau_{**}) = \varphi_{m/n, \mathcal{L}_r}(\tau_*),$$

$$\varphi'_{m/n, \mathcal{L}_r}(\tau_*)\varphi'_{m/n, \mathcal{L}_r}(\tau_{**}) < 0, \quad |\tau_{**} - \tau_*| = x,$$

$$n\,|\varphi_{m/n, \mathcal{L}_r}(\tau_*)| = n\,|\varphi_{m/n, \mathcal{L}_r}(\tau_{**})| = 2\Phi_{m/n, \mathcal{L}_r}(x).$$

By analogy with inequality (26), but using the function

$$f_{**}(t) = 2\sum_{k=1}^{2j}(-1)^k (f(t + y_k - y_1),$$

instead of the function $f_*(t)$, we get

$$\sum_{k=1}^{2j} \|f'(x_k)\| \le \tfrac{1}{2}\,|n\varphi'_{m/n, \mathcal{L}_r}(\tau_{**})|. \tag{27}$$

Since

$$|\Pi'(f, x)| = \sum_{k \in A(x)} \left(\frac{1}{|f'(t_k)|} + \frac{1}{|f'(\tau_k)|} \right)^{-1} \le \sum_{k=1}^{2j} \left(\frac{1}{|f'(x_k)|} + \frac{1}{|f'(y_k)|} \right)^{-1}$$

[18, Theorem 7.2.1], and

$$\sum_k \left(\frac{1}{a_k} + \frac{1}{b_k} \right)^{-1} \le \left(\left(\sum_k a_k \right)^{-1} + \left(\sum_k b_k \right)^{-1} \right)^{-1},$$

for any $a_k, b_k > 0$, we have

$$|\Pi'(f, x)| \le \left(\left(\sum_{k=1}^{2j} |f'(x_k)| \right)^{-1} + \left(\sum_{k=1}^{2j} |f'(y_k)| \right)^{-1} \right)^{-1}.$$

The last inequality, together with inequalities (26) and (27), yields

$$|\Pi'(f, x)| \le \tfrac{1}{2n}\,(|\varphi_{m/n, \mathcal{L}_r}(\tau_*)|^{-1} + |\varphi'_{m/n, \mathcal{L}_r}(\tau_{**})|^{-1})^{-1} = \left|\Phi'_{m/n, \mathcal{L}_r}(x)\right|.$$

Now let λ be an arbitrary number which satisfies condition (15). Assume that conditions (16), (17) are satisfied, and instead of (18) the inverse inequality holds. Then there exist $\lambda_0 = m/n < \lambda$ (and hence, $\Phi_{\lambda_0, \mathcal{L}_r}(0) > \Phi_{\lambda, \mathcal{L}_r}(0)$) sufficiently close to λ and a point x_0 such that

$$\Pi(f; x_0) = \Phi_{\lambda_0, \mathcal{L}_r}(x_0), \quad |\Pi'(f, x_0)| > \left|\Phi'_{\lambda_0, \mathcal{L}_r}(x_0)\right|,$$

but this contradicts the foregoing assumption. The theorem is proved.

4. Proof of Inequalities (7) – (10). Without loss of generality, we can assume that the polynomials $\mathcal{L}_r(y)$ and $\mathcal{L}_k(y)$ have the form

$$\mathcal{L}_r(y) = \prod_{l=1}^{m}(y^2 - 2\gamma_l y + \gamma_l^2 + \alpha_l^2)\prod_{l=1}^{r-2m}(x - \beta_l)$$

and

$$\mathcal{L}_k(y) = \prod_{l=1}^{m_1}(y^2 - 2\gamma_l y + \gamma_l^2 + \alpha_l^2)\prod_{l=1}^{k-2m_1}(y - \beta_l),$$

respectively, where β_l are the real zeros of the polynomial $\mathcal{L}_r(y)$ and $\gamma_l + i\alpha_l$ are its complex zeros.

Define the polynomials $\mathcal{L}_{j,r}$, $j = 1, 2, \ldots, r-k$, as follows

$$\mathcal{L}_{j,r}(y) = y^2 - 2\gamma_j y + \gamma_j^2 + \alpha_j^2, \quad j = 1, 2, \ldots, m_1,$$

$$\mathcal{L}_{j,r}(y) = y - \beta_{j-m_1}, \quad j = m_1 + 1, 2, \ldots, k - m_1,$$

$$\mathcal{L}_{j,r}(y) = y^2 - 2\gamma_{j-k+2m_1} y + \gamma_{j-k+2m_1}^2 + \alpha_{j-k+2m_1}^2, \quad j = k - m_1 + 1, \ldots, k + m - 2m_1,$$

$$\mathcal{L}_{j,r}(y) = y - \beta_{j-m}, \quad j = k + m - 2m_1 + 1, \ldots, r - m.$$

We set $f_0 = s$, $\varphi_0 = \left(\|s\|_\infty \|\varphi_{n,\mathcal{L}_r}\|_\infty^{-1}\right)\varphi_{n,\mathcal{L}_r}$ and, by induction, $f_j = \mathcal{L}_{j,r}(D) f_{j-1}$ and $\varphi_j = \mathcal{L}_{j,r}(D)\varphi_{j-1}$, for $j = 1, 2, \ldots, r-m$. If $\lambda_j = \|f_j\|_\infty \|\varphi_j\|_\infty^{-1}$, then, in order to prove inequality (7) for $p = \infty$, it suffices to show that

$$\lambda_j = 1, \quad j = 0, 1, \ldots, r - m. \tag{28}$$

It follows from the definitions of functions f_0 and φ_0 that $\lambda_0 = 1$. Assume that, for some $j \geq 1$, inequality (28) is not valid.

Let $j_0 = \min\{j : j \geq 1, \lambda_j > 1\}$. We now prove that in this case there exist numbers $\lambda^*_{j_0}, \lambda^*_{j_0+1}, \ldots, \lambda^*_{r-m-1}$ and $\tau_{j_0}, \tau_{j_0+1}, \ldots, \tau_{r-m-1}$ such that the inequalities $\nu(\varphi_j(\cdot) - \lambda^*_j f_j(\cdot - \tau_j)) \geq 2n + 2$ and $\|\lambda^*_j f_j\|_\infty < \|\varphi_j\|_\infty$ hold for $j_0 \leq j \leq r - m - 1$.

We proceed by induction on j. First, we prove that the required numbers λ^*_j and τ_j exist for $j = j_0$. Due to our choice of the index j_0, we have $\lambda_{j_0 - 1} < 1$. Since

$$\varphi_{n,\mathcal{L}_{j,m}}\left(x + \frac{l\pi}{n}\right) = (-1)^l \varphi_{n,\mathcal{L}_{j,m}}(x), \quad l \in \mathbb{Z}, \tag{29}$$

we get

$$\nu(\varphi_{j_0-1}(\cdot) - \lambda f_{j_0-1}(\cdot - \tau_j)) \geq 2n,$$

for any $\lambda(|\lambda| < 1)$ and τ. Thus, the length of the largest interval on which the function $\varphi_{j_0-1}(\cdot) - \lambda f_{j_0-1}(\cdot - \tau)$ does not change its sign is not greater than $2\pi / n$. Together with inequalities (13) and (14), this yields

$$\nu(\varphi_{j_0}(\cdot) - \lambda f_{j_0}(\cdot - \tau)) \geq 2n.$$

Let $\varphi_{j_0}(y_0) = \|\varphi_{j_0}\|_\infty$ and $|f_{j_0}(x_0)| = \|f_{j_0}\|_\infty$. Then there exist $\delta_1, \delta_2 \geq 0$ such that

$$\nu(\varphi_{j_0}(\cdot) - (\lambda_{j_0}^{-1} - \delta_1) f_{j_0}(\cdot - y_0 + x_0 - \delta_2)) \geq 2n + 2 \quad \text{if} \quad \varphi_{j_0}(y_0) f_{j_0}(x_0) > 0,$$
$$\nu(\varphi_{j_0}(\cdot) + (\lambda_{j_0}^{-1} - \delta_1) f_{j_0}(\cdot - y_0 + x_0 - \delta_2)) \geq 2n + 2 \quad \text{if} \quad \varphi_{j_0}(y_0) f_{j_0}(x_0) > 0, \tag{30}$$

and

$$\|(\lambda_{j_0}^{-1} - \delta_1) f_{j_0}\|_\infty < \|\varphi_{j_0}\|_\infty. \tag{31}$$

Therefore, the statement is proved for $j = j_0$.

We now assume that the required numbers λ_j and τ_j exist for $j = j_0, \ldots, l$, $l \geq j_0$ and prove their existence for $j = l + 1$. According to the assumption, we have

$$\|\lambda_l^* f_l\|_\infty < \|\varphi_l\|_\infty \tag{32}$$

and

$$\nu(\varphi_l(\cdot) - \lambda_l^* f_l(\cdot - \tau_l)) \geq 2n + 2.$$

By virtue of (29) and (32), the maximal length of the interval on which the sign of the difference $\varphi_l(\cdot) - \lambda_l^* f_l(\cdot - \tau_l)$ remains unchanged is not greater than $2\pi / n$. Therefore, taking into account inequalities (13) and (14), we get $\nu(\varphi_{l+1}(\cdot) - \lambda_l^* f_{l+1}(\cdot - \tau_l)) \geq 2n + 2$, and if $\|\lambda_l^* f_{l+1}(\cdot - \tau_l)\|_\infty < \|\varphi_{l+1}\|_\infty$, then the statement is proved. If the last inequality does not hold, then, by analogy with the proof of the inequalities (30) and (31), we find that there exist $\lambda_{l+1}^*(|\lambda_{l+1}^*| < 1)$ and τ_{l+1} such that

$$\nu(\varphi_{l+1}(\cdot) - \lambda_{l+1}^* f_{l+1}(\cdot - \tau_{l+1})) \geq 2n + 2$$

and

$$\|\lambda_{l+1}^* f_{l+1}\|_\infty < \|\varphi_{l+1}\|_\infty.$$

Hence, our statement is proved for all $j = j_0, \ldots, r - m - 1$, and, in particular, we have established the following inequalities

$$\nu(\varphi_{r-m-1}(\cdot) - \lambda_{r-m-1}^* f(\cdot - \tau_{r-m-1})) \geq 2n + 2, \tag{33}$$

$$\| \lambda^*_{r-m-1} f_{r-m-1} \|_\infty < \| \varphi_{r-m-1} \|_\infty . \tag{34}$$

It follows from inequalities (29) and (34) that the length of the largest interval on which the difference $\varphi_{r-m-1}(\cdot) - \lambda^*_{r-m-1} f(\cdot - \tau_{r-m-1})$ is a function of constant sign is not greater than $2\pi/n$. Together with inequalities (33) and (13) or (14), this yields the inequality

$$\nu(\varphi_{r-m}(\cdot) - \lambda^*_{r-m-1} f_{r-m}(\cdot - \tau_{r-m-1})) \geq 2n+2$$

which is impossible because $\varphi_{r-m}(x) = \varphi_{n,0}(x)$ and $f_{r-m} \in S_{2n,0}$.

This means that inequality (30) and, hence, inequality (7) are proved for $p = \infty$.

Employing inequality (7) with $p = \infty$ and the reasoning given in [13], it is now easy to derive inequality (7) for $p = 2$.

By using Stein's method (see, e.g, [18, pp. 117, 118]), the scheme developed in [11], and inequality (7) with $p = \infty$, we obtain the following inequality

$$\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)}{\bigvee_0^{2\pi}(\varphi_{n,\mathcal{L}_{r-k}})} \leq \frac{\bigvee_0^{2\pi}(s)}{\bigvee_0^{2\pi}(\varphi_{n,\mathcal{L}_r})}. \tag{35}$$

for $\mathcal{L}$ splines $s \in S_{2n,\mathcal{L}_r}$ (with n, satisfying the inequality (15)). Assume now that $s \in S_{2n,\mathcal{L}_r}$, n satisfy condition (15), and

$$\mathcal{L}_k(y) = y\mathcal{L}_{k-1}(y), \quad f_0 = \mathcal{L}_{k-1}(D)s, \quad \varphi_0 = \| s \|_\infty \| \varphi_{n,\mathcal{L}_r} \|_\infty^{-1} \varphi_{n,\mathcal{L}_r/\mathcal{L}_{k-1}} .$$

Then, by virtue of inequality (7) with $p = \infty$, we get

$$\| f_0 \|_\infty \leq \| \varphi_0 \|_\infty, \quad \| f_0' \|_\infty \leq \| \varphi_0' \|_\infty.$$

Moreover, inequalities (13) and (14) and the fact that $(\mathcal{L}_r / \mathcal{L}_{k-1})(D) f_0 \in S_{2n,0}$ imply that inequality $\nu\left(\varphi_0^{(j)}(\cdot) - f^{(j)}(\cdot - y)\right) = 2n$ holds for $j = 0, 1$ and an arbitrary y. Consequently, the functions f_0 and φ_0 satisfy the conditions in Theorem 4. Hence, $| f_0' - \lambda | \prec | \varphi_0' - \lambda |$ for any $\lambda \in \mathbb{R}$ which is equivalent to (9).

5. A Kolmogorov-Type Inequality. Let $\mathcal{L}_r = \mathcal{L}_k \mathcal{L}_{r-k}$, $\lambda > 0$, $p \in [1, \infty]$,

$$\Psi_{p,\mathcal{L}_r}(\lambda) = \tfrac{1}{4} \lambda^{-1/p} \| \varphi_{\lambda,\mathcal{L}_r}(t/\lambda) \|_p$$

and

$$\Theta_{p,\mathcal{L}_r,\mathcal{L}_{r-k}}(\lambda) = \Psi_{p,\mathcal{L}_r}(\Psi^{-1}_{\infty,\mathcal{L}_{r-k}}(\lambda)).$$

Theorem 6. *If a polynomial $\mathcal{L}_r(y)$ has only real roots, then the inequality*

$$\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)f)}{4\bigvee_0^{2\pi}(\mathcal{L}_r(D)f)} \leq \Theta^{-1}_{p,\mathcal{L}_r,\mathcal{L}_{r-k}}\left(\left(\frac{\nu(f')}{2}\right)^{1-1/p} \frac{\| f \|_p}{\bigvee_0^{2\pi}(\mathcal{L}_r(D)f)}\right). \tag{36}$$

holds for any function $f \in L_V^r$.

For $\mathcal{L}_r(y) = y^r$, this statement coincides with Theorem 1 in [4].

Proof. Let $f \in L_V^r$ and $h(x) = f(x) / \bigvee_0^{2\pi}(\mathcal{L}_r(D)f)$. Without loss of generality, we can assume that the function f has zeroes. By choosing a number $\lambda > 0$ for which the condition $\Pi(h; 0) = \Phi_{\lambda, \mathcal{L}_r}(0)$ is satisfied, we get

$$\bigvee_0^{2\pi}(h) = 2\Pi(h; 0) = 2\Phi_{\lambda, \mathcal{L}_r}(0) = \| \varphi_{\lambda, \mathcal{L}_r} \|_\infty = 4\Psi_{\infty, \mathcal{L}_r}(\lambda). \tag{37}$$

It was proved in [4] that any function $g \in C^1$ satisfies the inequality

$$\int_0^x P(|g|, t)\, dt \ge \int_0^{x/\nu(g')} \Pi(g; t)\, dt.$$

Taking into account this inequality, Theorem 5, condition (37), and the fact that $\nu(h') = \nu(f')$, we obtain

$$\int_0^x P(|h|, t)\, dt \ge \int_0^{x/\nu(f')} \Pi(h; t)\, dt \ge \int_0^{x/\nu(f')} \Phi_{\lambda, \mathcal{L}_r}(t)\, dt$$

$$= \frac{1}{\nu(f')} \int_0^x \Phi_{\lambda, \mathcal{L}_r}(t/\nu(f'))\, dt = \frac{1}{2\nu(f')} \int_0^x P\Big(|\varphi_{\lambda, \mathcal{L}_r}|_{[0, \pi/\lambda]}; t/\nu(f')\Big)\, dt,$$

for any $x \in [0, 2\pi]$. This implies that

$$\| h \|_p = \| P(h; \cdot) \|_{L_p[0, 2\pi]} \ge \frac{1}{2\nu(f')} \Big\| P\Big(|\varphi_{\lambda, \mathcal{L}_r}|_{[0, \pi/\lambda]}; (\cdot)/\nu(f')\Big) \Big\|_{L_p[0, \pi\nu(f')/\lambda]} = \left(\frac{\nu(f')}{2}\right)^{1/p-1} \Psi_{p, \mathcal{L}_r}(\lambda),$$

for any $p \in [1, \infty]$. Taking into account (37), we get

$$\|h\|_p \ge \left(\frac{\nu(f')}{2}\right)^{1/p-1} \Theta_{p; \mathcal{L}_r, \mathcal{L}_r}\left(\frac{1}{4} \bigvee_0^{2\pi}(h)\right). \tag{38}$$

For any function $f \in L_\infty^r$, the following inequality holds

$$\Theta_{\infty; \mathcal{L}_r, \mathcal{L}_{r-k}}\left(\frac{\|\mathcal{L}_k(D)f\|_\infty}{4\|\mathcal{L}_r(D)f\|_\infty}\right) \le \frac{1}{4}\|f\|_\infty / \|\mathcal{L}_r(D)f\|_\infty.$$

This inequality is a generalization of Kolmogorov's inequality (see, for example, [20, 21]). By using this inequality and Stein's method mentioned above, it is easy to obtain the inequality

$$\Theta_{\infty; \mathcal{L}_r, \mathcal{L}_{r-k}}\left(\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)f)}{4 \bigvee_0^{2\pi}(\mathcal{L}_r(D)f)}\right) \le \frac{\bigvee_0^{2\pi}(f)}{4 \bigvee_0^{2\pi}(\mathcal{L}_r(D)f)}, \tag{39}$$

which is valid for all functions $f \in L_V^r$. Taking into account inequalities (38) and (39) and the fact that the function $\Theta_{p; \mathcal{L}_r, \mathcal{L}_{r-k}}(x)$ monotonically increases on $[0, \infty)$, we obtain the inequality

$$\|h\|_p \geq \left(\frac{\nu(f')}{2}\right)^{1/p-1} \Theta_{p;\mathcal{L}_r,\mathcal{L}_{r-k}}\left(\frac{1}{4}\bigvee_0^{2\pi}(\mathcal{L}_k(D)h)\right),$$

which is equivalent to inequality (36).

6. Proof of Inequalities (11) and (12). Let $s \in S_{2n,\mathcal{L}_r}$. By virtue of Theorem 6,

$$\Theta_{p;\mathcal{L}_r,\mathcal{L}_{r-k}}\left(\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)}{4\bigvee_0^{2\pi}(\mathcal{L}_r(D)s)}\right) \leq \left(\frac{\nu(f')}{2}\right)^{1-1/p} \frac{\|s\|_p}{\bigvee_0^{2\pi}(\mathcal{L}_r(D)s)}.$$

In view of $\nu(s') \leq 2\pi$, we obtain

$$\|s\|_p \geq n^{1/p-1}\bigvee_0^{2\pi}(\mathcal{L}_r(D)s)\Theta_{p;\mathcal{L}_r,\mathcal{L}_{r-k}}\left(\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)}{4\bigvee_0^{2\pi}(\mathcal{L}_r(D)s)}\right)$$

$$= \frac{1}{4}n^{1/p-1}\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)\,\eta_{r,k}^{-1}\,\Theta_{p;\mathcal{L}_r,\mathcal{L}_{r-k}}(\eta_{r,k}), \tag{40}$$

where $\eta_{r,k} = \bigvee_0^{2\pi}(\mathcal{L}_k(D)s)\left(4\bigvee_0^{2\pi}(\mathcal{L}_r(s))\right)^{-1}$. It follows from inequality (35) that

$$\eta_{r,k} \geq \frac{\bigvee_0^{2\pi}(\varphi_{n,\mathcal{L}_{r-k}})}{4\bigvee_0^{2\pi}(\varphi_{n,0})} = \frac{1}{4}\|\varphi_{n,\mathcal{L}_{r-k}}\|_\infty = \Psi_{\infty,\mathcal{L}_{r-k}}(n).$$

Taking into account this inequality, (40), and the fact that the function $x^{-1}\Theta_{p,\mathcal{L}_r,\mathcal{L}_{r-k}}(x)$ monotonically increases on $[0,+\infty)$, we get

$$\|s\|_p \geq \frac{1}{4}n^{1/p-1}\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)}{\Psi_{\infty,\mathcal{L}_{r-k}}(n)}\Theta_{p;\mathcal{L}_r,\mathcal{L}_{r-k}}(\Psi_{\infty,\mathcal{L}_{r-k}}(n)).$$

By definition of the function $\Theta_{p,\mathcal{L}_r,\mathcal{L}_{r-k}}$, we obtain the relations

$$\|s\|_p \geq \frac{1}{4}n^{1/p-1}\frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)\Psi_{p,\mathcal{L}_r}(n)}{\Psi_{\infty,\mathcal{L}_{r-k}}(n)} = \frac{\bigvee_0^{2\pi}(\mathcal{L}_k(D)s)}{\bigvee_0^{2\pi}(\varphi_{n,\mathcal{L}_{r-k}})}\|\varphi_{n,\mathcal{L}_r}\|_p.$$

Thus, inequality (11) is proved. Inequality (12) follows immediately from inequality (11). Theorem 1 is proved.

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