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Aerodynamic Design via Control Theory

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1. INTRODUCTION AND HISTORICAL SURVEY

Computers have had a twofold impact on the science of aerodynamics. On the one hand numerical simulation may be used to gain new insights into the physics of complex flows. On the other hand computational methods can be used by engineers to predict the aerodynamic characteristics of alternative designs. Assuming that one has the ability to predict the performance, the question then arises of how to modify the design to improve the performance. This paper is addressed to that question.

Prior to 1960 computational methods were hardly used in aerodynamic analysis. The primary tool for the development of aerodynamic configurations was the wind tunnel. Shapes were tested and modifications selected in the light of pressure and force measurements together with flow visualization techniques. Computational methods are now quite widely accepted in the aircraft industry. This has been brought about by a combination of radical improvements in numerical algorithms and continuing advances in both speed and memory of computers.

If a computational method is to be useful in the design process, it must be based on a mathematical model which provides an appropriate representation of the significant features of the flow, such as shock waves, vortices and boundary layers. The method must also be robust, not liable

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to fail when parameters are varied, and it must be able to treat useful configurations, ultimately the complete aircraft. Finally, reasonable accuracy should be attainable at reasonable cost. Much progress has been made in these directions (Murman and Cole, 1971; James on, 1974; 1987; Jameson and Caughey, 1977; Bristeau *et al.*, 1985; Jameson *et al.*, 1981, 1986; Ni, 1982; Pulliam and Steger, 1985; Mac Cormack, 1985). In many applications where the flow is unseparated, including designs for transonic flow with weak shock waves, useful predictions can be made quite inexpensively using the potential flow equation (Murman and Cole, 1971; Jameson, 1974; Jameson and Caughen, 1977; Bristeau *et al.*, 1985). Methods are also available for solving the Euler equations for two- and three-dimensional configurations up to a complete aircraft (Jameson *et al.*, 1981, 1986; Ni, 1982; Pulliam and Steger, 1985; MacCormack, 1985; Jameson, 1987). Viscous simulations are generally complicated by the need to allow for turbulence: while the Reynolds averaged equations can be solved by current methods, the results depend heavily on the choice of turbulence models.

Given the range of well-proven methods now available, one can distinguish objectives for computational aerodynamics at several levels:

1. Capability to predict the flow past an airplane or important components in different flight regimes such as take-off or cruise, and off design conditions such as flutter.
2. Interactive design calculations to allow rapid improvement of the design.
3. Automatic design optimization.

Substantial progress has been made toward the first objective, and in relatively simple cases such as an airfoil or wing in inviscid flow, calculations can be performed fast enough that the second objective is within reach. The third objective has also been addressed for various special cases. In particular it has been recognized that the designer generally has an idea of the kind of pressure distribution that will lead to the desired performance. Thus it is useful to consider the problem of calculating the shape that will lead to a given pressure distribution. Such a shape does not necessarily exist, unless the pressure distribution satisfies certain constraints, and the problem must therefore be very carefully formulated: no shape exists, for example, for which stagnation pressure is attained over the entire surface.

The problem of designing a two dimensional profile to attain a desired pressure distribution was first studied by Lighthill, who solved it for the case of incompressible flow by conformally mapping the profile to a unit circle (Lighthill, 1945). The speed over the profile is

$$q = \phi_\theta / h \tag{1.1}$$

where ϕ is the potential for flow past a circle, and h is the modulus of the mapping function. The solution for ϕ is known for incompressible flow. Let q_d be the desired surface speed. Then the surface value of h can be obtained by setting $q = q_d$ in Eq. (1.1), and since the mapping function is analytic, it is uniquely determined by the value of h on the boundary. A solution exists for a given speed q_∞ at infinity only if

$$\frac{1}{2\pi} \oint q \, d\theta = q_\infty \tag{1.2}$$

and there are additional constraints on q if the profile is required to be closed.

Lighthill's method was extended to compressible flow by McFadden (1979). Starting with a given shape, and a corresponding mapping function $h^{(0)}$, the flow equations can be solved for the potential $\phi^{(0)}$, which now depends on $h^{(0)}$. A new mapping function $h^{(1)}$ is then determined by setting $q = q_d$ in Eq. (1.1), and the process is repeated. In the limiting case of zero Mach number the method reduces to Lighthill's method, and McFadden gives a proof that the iterations will converge for small Mach numbers. He also extends the method to treat transonic flow through the introduction of artificial viscosity to suppress the appearance of shock waves, which would cause the updated mapping function to be discontinuous. This difficulty can also be overcome by smoothing the changes in the mapping function. Such an approach is used in a computer program written by the author for Grumman Aerospace. It allows the recovery of smooth profiles that generate flows containing shock waves, and it has been used to design improved blade sections for propellers (Taverna, 1983). A related method for three-dimensional design was devised by Garabedian and McFadden (1982). In their scheme the steady potential flow solution is obtained by solving an artificial time-dependent equation, and the surface is treated as a free boundary. This is shifted according to an auxiliary time-dependent equation in such a way that the flow evolves toward the specified pressure distribution.

Another way to formulate the problem of designing a profile for a given pressure distribution is to integrate the corresponding surface speed to obtain the surface potential. The potential flow equation is then solved with a Dirichlet boundary condition, and a shape correction is determined from the calculated normal velocity through the surface. This approach was first tried by Tranen (1974). Volpe and Melnik (1986) have shown how to allow for the constraints that must be satisfied by the pressure distribution if a solution is to exist. The same idea has been used by Henne (1980) for three-dimensional design calculations.

The hodograph transformation offers an alternative approach to the design of airfoils in transonic flows. Garabedian and Korn (1971) achieved a striking success in the design of airfoils to produce shock-free transonic flows by using the method of complex characteristics to solve the equations in the hodograph plane. Another design procedure has been proposed by Giles, Drela, and Thompkins (1985), who write the two-dimensional Euler equations for inviscid flow in a streamline coordinate system, and use a Newton iteration. An option is then provided to treat the surface coordinates as unknowns, while the pressure is fixed.

Finally, Hicks and Henne (1979) have explored the possibility of meeting desired design objectives by using constrained optimization. The configuration is specified by a set of parameters, and any suitable computer program for flow analysis is used to evaluate the aerodynamic characteristics. The optimization method then selects values of these parameters that maximize some criterion of merit, such as the lift-to-drag ratio, subject to other constraints such as required wing thickness and volume. In principle this method allows the designer to specify any reasonable design objectives. The method becomes extremely expensive, however, as the number of parameters is increased, and its successful application in practice depends heavily on the choice of a parametric representation of the configuration.

The purpose of this paper is to propose that there are benefits in regarding the design problem as a control problem in which the control is the shape of the boundary. A variety of alternative formulations of the design problem can then be treated systematically by using the mathematical theory for control of systems governed by partial differential equations (Lions, 1971). Suppose that the boundary is defined by a function $f(x)$, where x is the position vector. As in the case of optimization theory applied to the design problem, the desired objective is specified by a cost function I, which may, for example, measure the deviation from a desired surface pressure distribution, but could also represent other measures of performance such as lift and drag. The introduction of a cost function has the advantage that if the objective is unattainable, it is still possible to find a minimum of the cost function. Now a variation in the control δf leads to a variation δI in the cost. It is shown in the following sections that δI can be expressed to first order as an inner product of a gradient function g with δf:

$$\delta I = (g, df)$$

Here g is independent of the particular variation δf in the control, and can be determined by solving an adjoint equation. Now choose

$$\delta f = -\lambda g$$

where λ is a sufficiently small positive number. Then

$$\delta I = -\lambda(g, g) < 0$$

assuring a reduction in I. After making such a modification, the gradient can be recalculated and the process repeated to follow a path of steepest descent until a minimum is reached. In order to avoid violating constraints, such as a minimum acceptable wing thickness, the steps can be taken along the projection of the gradient into the allowable subspace of the control function. In this way one can devise design procedures that must necessarily converge at least to a local minimum, and which might be accelerated by the use of more sophisticated descent methods. While there is a possibility of more than one local minimum, the cost function can be chosen to reduce the likelihood of difficulties caused by such a contingency, and in any case the method will lead to an improvement over the initial design. The mathematical development resembles in many respects the method of calculating transonic potential flow proposed by Bristeau, Pironneau, Glowinski, Periaux, Perrier, and Poirier (1985), who reformulated the solution of the flow equations as a least-squares problem in control theory.

In order to illustrate the application of control theory to design problems in more detail, the following sections present design procedures for three examples. Section 2 discusses the design of two-dimensional profiles for compressible potential flow when the profile is generated by conformal mapping. This leads to a generalization of the methods of Lighthill and McFadden. Section 3 discusses the same problem when the flow is governed by the inviscid Euler equations. Finally, Section 4 addresses the three-dimensional design problem for a wing, assuming the flow to be governed by the inviscid Euler equations. The procedures that are presented require the solution of several partial differential equations at each step. The question of the most efficient discretization of these equations is deferred for future investigation.

2. DESIGN FOR POTENTIAL FLOW USING CONFORMAL MAPPING

Consider the case of two-dimensional compressible inviscid flow. In the absence of shock waves an initially irrotational flow will remain irrotational, and we can assume that the velocity vector $\mathbf{q}$ is the gradient of a potential ϕ. In the presence of weak shock waves this remains a fairly good approximation. Let ζ, T, and S denote vorticity, temperature, and entropy. Then according to Crocco's Theorem, vorticity in steady flow is associated with entropy production through the relation

$$\mathbf{q} \times \zeta + T\nabla S = 0$$

Thus, the introduction of a potential is consistent with the assumption of isentropic flow, and shock waves are modeled by isentropic jumps. Let p, ρ, c, and M be the pressure, density, speed of sound, and Mach number q/c. Then the potential flow equation is

$$\nabla \cdot \rho \nabla \phi = 0 \tag{2.1}$$

where the density is given by

$$\rho = \left[1 + \frac{\gamma - 1}{2} M_\infty^2 (1 - q^2) \right]^{1/\gamma - 1} \tag{2.2}$$

while

$$p = \frac{\rho^\gamma}{\gamma M_\infty^2}, \qquad c^2 = \frac{\gamma p}{\rho} \tag{2.3}$$

Here M_∞ is the Mach number in the free stream, and the units have been chosen so that p and q have the value unity in the far field. Equation (2.2) is a consequence of the energy equation in the form

$$\frac{c^2}{\gamma - 1} + \frac{q^2}{2} = \text{const}$$

Suppose that the domain D exterior to the profile C in the z plane is conformally mapped onto the domain exterior to a unit circle in the σ plane as sketched in Fig. 1. Let R and θ be polar coordinates in the σ plane, and let r be the inverted radial coordinate $1/R$. Also let h be the modulus of the derivative of the mapping function

$$h = \left| \frac{dz}{d\sigma} \right|$$

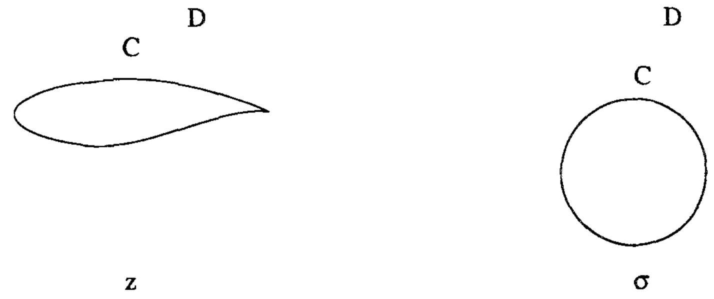

Figure 1

Now the potential flow equation becomes

$$\frac{\partial}{\partial\theta}(\rho\phi_\theta)+r\frac{\partial}{\partial r}(r\rho\phi_r)=0 \qquad \text{in } D \tag{2.4}$$

where the density is given by Eq. (2.1), and the circumferential and radial velocity components are

$$u=\frac{r\phi_\theta}{h}, \qquad v=\frac{r^2\phi_r}{h} \tag{2.5}$$

while

$$q^2=u^2+v^2 \tag{2.6}$$

The condition of flow tangency leads to the Neumann boundary condition

$$v=\frac{1}{h}\frac{\partial\phi}{\partial r}=0 \qquad \text{on } C \tag{2.7}$$

In the far field the potential is given by an asymptotic estimate, leading to a Dirichlet boundary condition at $r=0$ (Jameson, 1974).

Suppose that it is desired to achieve a specified velocity distribution q_d on C. Introduce the cost function

$$I=\frac{1}{2}\int_C (q-q_d)^2\,d\theta \tag{2.8}$$

The design problem is now treated as a control problem where the control function is the mapping modulus h, which is to be chosen to minimize I subject to the constraints defined by the flow equations (2.2)–(2.7).

A modification δh to the mapping modulus will result in variations $\delta\phi$, δu, δv, and $\delta\rho$ to the potential, velocity components and density. The resulting variation in the cost will be

$$\delta I = \int_C (q - q_d)\, \delta q\, d\theta \tag{2.9}$$

where on C $q = u$. Also

$$\delta u = r\frac{\delta\phi_\theta}{h} - u\frac{\delta h}{h}, \qquad \delta v = r^2\frac{\delta\phi_r}{h} - v\frac{\delta h}{h}$$

while according to Eq. (2.2) and (2.6)

$$\frac{\partial\rho}{\partial u} = -\frac{\rho u}{c^2}, \qquad \frac{\partial\rho}{\partial v} = -\frac{\rho v}{c^2}$$

Hence

$$\begin{aligned}\delta\rho &= -\frac{\rho}{c^2}(u\,\delta u + v\,\delta v)\\ &= \rho\frac{q^2}{c^2}\frac{\delta h}{h} - \frac{\rho}{c^2}\frac{r}{h}(u\,\delta\phi_\theta + vr\,\delta\phi_r)\end{aligned}$$

It follows that $\delta\phi$ satisfies

$$L\,\delta\phi = -\frac{\partial}{\partial\theta}\left(\rho M^2\phi_\theta\frac{\delta h}{h}\right) - r\frac{\partial}{\partial r}\left(\rho M^2 r\phi_r\frac{\delta h}{h}\right)$$

where

$$L \equiv \frac{\partial}{\partial\theta}\left[\rho\left(1-\frac{u^2}{c^2}\right)\frac{\partial}{\partial\theta} - \frac{\rho uv}{c^2}r\frac{\partial}{\partial r}\right] + r\frac{\partial}{\partial r}\left[\rho\left(1-\frac{v^2}{c^2}\right)r\frac{\partial}{\partial r} - \frac{\rho uv}{c^2}\frac{\partial}{\partial\theta}\right] \tag{2.10}$$

Then if ψ is any periodic differentiable function which vanishes in the far field

$$\int_D \frac{\psi}{r^2} L\,\delta\phi\, dS = \int_D \rho M^2\, \nabla\phi\cdot\nabla\psi\,\frac{\delta h}{h}\, dS \tag{2.11}$$

where dS is the area element $r\,dr\,d\theta$, and the right-hand side has been integrated by parts.

Now we can augment Eq. (2.9) by subtracting the constraint (2.11).

The auxiliary function ψ then plays the role of a Lagrange multiplier. Substituting for δq and integrating the term

$$\int_C (q-q_d)\, r \frac{\delta\phi}{h} \theta \, d\theta$$

by parts, we obtain

$$\delta I = \int_C (q-q_d)\, q \frac{\delta h}{h} \, d\theta - \int_C \delta\phi \frac{\partial}{\partial\theta}\left(\frac{q-q_d}{h}\right) d\theta$$
$$- \int_D \frac{\psi}{r^2} L \, \delta\phi \, dS + \int_D \rho M^2 \nabla\phi \cdot \nabla\psi \frac{\delta h}{h} \, dS$$

Now suppose that ψ satisfies the adjoint equation

$$L\psi = 0 \qquad \text{in } D \tag{2.12}$$

with the boundary condition

$$\frac{\partial\psi}{\partial r} = \frac{1}{\rho} \frac{\partial}{\partial\theta}\left(\frac{q-q_d}{h}\right) \qquad \text{on } C \tag{2.13}$$

Then integrating by parts

$$\int_D \frac{\psi}{r^2} L \, \delta\phi \, dS = -\int_C \rho\psi_r \, \delta\phi \, d\theta$$

and

$$\delta I = -\int_C (q-q_d)\, q \frac{\delta h}{h} \, d\theta + \int_D \rho M^2 \nabla\phi \cdot \nabla\psi \frac{\delta h}{h} \, dS \tag{2.14}$$

Here the first term represents the direct effect of the change in the metric, while the area integral represents a correction for the effect of compressibility.

Equation (2.14) can be further simplified to represent δI purely as a boundary integral because the mapping function is fully determined by the value of its modulus on the boundary. Set

$$\log \frac{dz}{d\sigma} = f + i\beta$$

where

$$f = \log \left| \frac{dz}{dr} \right| = \log h$$

and

$$\delta f = \frac{\delta h}{h}$$

Then f satisfies Laplace's equation

$$\Delta f = 0 \qquad \text{in } D$$

and if there is no stretching in the far field, $f \to 0$. Thus

$$\Delta\, df = 0 \qquad \text{in } D$$

and $\delta f \to 0$ in the far field.

Introduce another auxiliary function P that satisfies

$$\Delta P = \rho M^2 \nabla\phi \cdot \nabla\psi \qquad \text{in } D \tag{2.15}$$

and

$$P = 0 \qquad \text{on } C \tag{2.16}$$

Then the area integral in Eq. (2.14) is

$$\int_D \Delta P\, \delta f\, dS = \int_C \delta f \frac{\partial P}{\partial r}\, d\theta - \int_D P \Delta\, \delta f\, dS$$

and finally

$$\delta I = \int_C g\, \delta f\, d\theta \tag{2.17}$$

where

$$g = \frac{\partial P}{\partial r} - (q - q_d)\, q \tag{2.18}$$

This suggests setting

$$\delta f = -\lambda g$$

so that if λ is a sufficiently small positive number

$$\delta I = -\lambda \int_C g^2 \, d\theta < 0$$

Arbitrary variations δf cannot, however, be admitted. The condition that $f \to 0$ in the far field, and also the requirement that the profile should be closed, imply constraints that must be satisfied by f on the boundary C. Suppose that $\log(dz/d\sigma)$ is expanded as a power series

$$\log\left(\frac{dz}{d\sigma}\right) = \sum_{n=0}^{\infty} \frac{c_n}{\sigma^n} \tag{2.19}$$

where only negative powers are retained because otherwise $dz/d\sigma$ would become unbounded for large σ. The condition that $f \to 0$ as $\sigma \to \infty$ implies

$$c_0 = 0$$

Also the change in z on integration around a circuit is

$$\Delta z = \oint \frac{dz}{d\sigma} \, d\sigma = 2\pi i c_1$$

so the profile will be closed only if

$$c_1 = 0$$

On C Eq. (2.19) reduces to

$$f_C + i\beta_C = \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) + i \sum_{n=0}^{\infty} (b_n \cos n\theta - a_n \sin n\theta)$$

Thus a_n and b_n are the Fourier coefficients of f_C, and these constraints reduce to

$$a_0 = 0, \qquad a_1 = 0, \qquad b_1 = 0$$

In order to satisfy these constraints we can project g onto the admissible subspace for f_C by setting

$$\tilde{g} = g - A_0 - A_1 \cos\theta - B_1 \sin\theta \tag{2.20}$$

where

$$A_0 = \frac{1}{2\pi} \int_C g \, d\theta$$

$$A_1 = \frac{1}{\pi} \int_C g \cos\theta \, d\theta \tag{2.21}$$

$$B_1 = \frac{1}{\pi} \int_C g \sin\theta \, d\theta$$

Then

$$\int_C (g - \tilde{g}) \, \tilde{g} \, d\theta = 0$$

and if we take

$$\delta f = -\lambda \tilde{g}$$

it follows that to first order

$$\delta I - \lambda \int_C g\tilde{g} \, d\theta = -\lambda \int_C (\tilde{g} + g - \tilde{g}) \, \tilde{g} \, d\theta = -\lambda \int_C \tilde{g}^2 \, d\theta < 0$$

If the flow is subsonic this procedure should converge toward the desired speed distribution since the solution will remain smooth, and no unbounded derivatives will appear. If, however, the flow is transonic, one must allow for the appearance of shock waves in the trial solutions, even if q_d is smooth. Then $q - q_d$ is not differentiable. This difficulty can be circumvented by a more sophisticated choice of the cost function. Consider the choice

$$I = \frac{1}{2} \int_C \left[\lambda_1 S^2 + \lambda_2 \left(\frac{dS}{d\theta} \right)^2 \right] d\theta \tag{2.22}$$

where λ_1 and λ_2 are parameters, and the periodic function $S(\theta)$ satisfies the equation

$$\lambda_1 S - \lambda_2 \frac{d^2 S}{d\theta^2} = q - q_d \tag{2.23}$$

Then

$$\delta I=\int_C\left(\lambda_1 S\,\delta S+\lambda_2\frac{dS}{d\theta}\frac{d}{d\theta}\,\delta S\right)d\theta$$

$$=\int_C S\left(\lambda_1\,\delta S-\lambda_2\frac{d^2}{d\theta^2}\,\delta S\right)d\theta$$

$$=\int_C S\,\delta q\,d\theta$$

Thus S replaces $q-q_d$ in the previous formulas, and if one modifies the boundary condition (2.13) to

$$\frac{\partial\psi}{\partial r}=\frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{S}{h}\right)\qquad\text{on } C \tag{2.24}$$

the formula for the gradient becomes

$$g=\frac{\partial p}{\partial r}-Sq \tag{2.25}$$

instead of equation (2.18). Then one modifies f by a step $-\lambda\tilde{g}$ in the direction of the projected gradient as before.

The final design procedure is thus as follows. Choose an initial profile and corresponding mapping function f. Then

1. Solve the flow equations (2.2)–(2.7) for ϕ, u, v, q, ρ.
2. Solve the ordinary differential equation (2.23) for S.
3. Solve the adjoint equation (2.12) for ψ subject to the boundary condition (2.24).
4. Solve the auxiliary Poisson equation (2.15) for P.
5. Evaluate

$$g=\frac{\partial P}{\partial r}-Sq$$

on C, and find its projection $\tilde{g}$ onto the admissible subspace of variations according to equations (2.20) and (2.21).

6. Correct the boundary mapping function f_C by

$$\delta f=-\lambda\tilde{g}$$

and return to step 1.

3. DESIGN FOR THE EULER EQUATIONS USING CONFORMAL MAPPING

This section treats the case of two-dimensional compressible flow where the potential flow equation is replaced as a mathematical model by the inviscid Euler equations. Let p, ρ, u, v, E, and H denote the pressure, density, Cartesian velocity components, total energy, and total enthalpy. For a perfect gas

$$p = (\gamma - 1)\,\rho\left[E - \tfrac{1}{2}(u^2 + v^2)\right] \tag{3.1}$$

and

$$\rho H = \rho E + p \tag{3.2}$$

where γ is the ratio of specific heats. The Euler equations may then be written as

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 \tag{3.3}$$

where x and y are Cartesian coordinates, t is the time coordinate, and

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \qquad f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{bmatrix}, \qquad g = \begin{bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v H \end{bmatrix} \tag{3.4}$$

As in the previous section, suppose that the domain D exterior to the profile C in the z plane is mapped conformally onto the domain exterior to a unit circle in the σ plane (see Fig. 1). Assume also that the outer boundary B of the domain is very far from the profile. Let the derivative of the mapping function be

$$\frac{dz}{d\sigma} = h e^{i\beta} \tag{3.5}$$

Also let r and θ be polar coordinates in the σ plane, where in this case it is more convenient to take r as the true radial coordinate denoted by R in the previous section, and θ is measured in the clockwise direction. Define the rotation parameters

$$c = \cos(\beta - \theta), \qquad s = \sin(\beta - \theta) \tag{3.6}$$

and rotated velocity components

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} s & -c \\ c & s \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{3.7}$$

Then the Euler equations become

$$\frac{\partial}{\partial t}(rh^2 w) + \frac{\partial}{\partial \theta}(hF) + \frac{\partial}{\partial r}(rhG) = 0 \tag{3.8}$$

where

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \qquad F = \begin{bmatrix} \rho U \\ \rho U u + sp \\ \rho U v - cp \\ \rho U H \end{bmatrix}, \qquad G = \begin{bmatrix} \rho V \\ \rho V u + cp \\ \rho V v + sp \\ \rho V H \end{bmatrix} \tag{3.9}$$

Then the flow is determined as the steady state solution of Eqs. (3.8) and (3.9), subject to the flow tangency condition

$$V = 0 \qquad \text{on } C \tag{3.10}$$

At the far field boundary B conditions can be specified for incoming waves, while outgoing waves are determined by the solution.

In contrast to the case of potential flow, the pressure is not determined solely by the speed, and assuming that one wishes to control the surface pressure distribution, a suitable cost function is

$$I = \tfrac{1}{2} \int_C (p - p_d)^2 \, d\theta \tag{3.11}$$

where p_d is the desired pressure. A modification to the mapping function will influence Eqs. (3.8) and (3.9) through changes δh and $\delta \beta$ in both the modulus and argument of $dz/d\sigma$, finally leading to a variation in the cost function

$$\delta I = \int_C (p - p_d) \, \delta p \, d\theta \tag{3.12}$$

where δp is the variation of the pressure.

Now the mapping variations cause vriations in the rotation parameters

$$\begin{aligned} \delta s &= c\,\delta\beta, & \delta c &= -s\,\delta\beta \\ \delta(hs) &= s\,\delta h + hc\,\delta\beta, & \delta(hc) &= c\,\delta h - hs\,\delta\beta \end{aligned} \tag{3.13}$$

Define the Jacobian matrices

$$\begin{aligned} A &= \frac{\partial f}{\partial w}, & B &= \frac{\partial g}{\partial w} \\ C &= sA - cB, & D &= cA + sB \end{aligned} \tag{3.14}$$

Then the variation δw in w satisfies

$$\frac{\partial}{\partial \theta}(hC\,\delta w) + \frac{\partial}{\partial r}(rhD\,\delta w) = -\frac{\partial}{\partial \theta}(F\,\delta h + hG\,\delta\beta) - \frac{\partial}{\partial r} r(G\,\delta h - hF\,\delta\beta) \tag{3.15}$$

Also

$$\delta V = 0 \qquad \text{on } C \tag{3.16}$$

At the outer boundary there will be no variation in characteristic variables corresponding to incoming waves. If we take the outer boundary B at a fixed radius, incoming waves correspond to negative eigenvalues of D. Suppose that D is represented as $T\Lambda T^{-1}$, where Λ is a diagonal matrix containing its eigenvalues, and the columns of T are eigenvectors of D. Define

$$\delta\tilde{w} = T^{-1}\,\delta w$$

and $d\tilde{w}^-$ as the components of $\delta\tilde{w}$ corresponding to negative eigenvalues of T. Then

$$\delta\tilde{w}^- = 0 \qquad \text{on } B \tag{3.17}$$

Since δw satisfies the contraint (3.15), we can replace Eq. (3.12) by

$$\begin{aligned} \delta I = \int_C (p - p_d)\,\delta p\,d\theta &- \iint_D \psi^T \left[\frac{\partial}{\partial \theta}(hC\,\delta w) + \frac{\partial}{\partial r}(rhD\,\delta w)\right] dr\,d\theta \\ &- \iint_D \psi^T \left[\frac{\partial}{\partial \theta}(F\,\delta h + hG\,\delta\beta) + \frac{\partial}{\partial r} r(G\,\delta h - hF\,\delta\beta)\right] dr\,d\theta \end{aligned} \tag{3.18}$$

where the vector ψ is a Lagrange multiplier, and the superscript T denotes the transpose. Suppose that ψ is the steady state solution of the adjoint equation

$$\frac{\partial\psi}{\partial t} - C^T\frac{\partial\psi}{\partial\theta} - rD^T\frac{\partial\psi}{\partial r} = 0 \qquad \text{in } D \tag{3.19}$$

At the outer boundary B conditions can be specified for incoming waves, corresponding to positive eigenvalues of $D^T = T^{-1^T}\Lambda T^T$. Define

$$\tilde{\psi} = T^T\psi$$

and $\tilde{\psi}^+$ as the components of $\tilde{\psi}$ corresponding to positive eigenvalues of D. Then we can set

$$\tilde{\psi}^+ = 0 \qquad \text{on } B \tag{3.20}$$

If we integrate equation (3.18) by parts the contribution

$$\int_B rh\psi^T D\,\delta w\,d\theta = \int_B rh\tilde{\psi}\Lambda\,\delta\tilde{w}\,d\theta$$

vanishes because of the complementary boundary conditions (3.17) and (3.20) satisfied by $\delta\tilde{w}$ and $\tilde{\psi}$ at the outer boundary. If δh and $\delta\beta$ decay fast enough in the far field the contribution

$$\int_B r(G\,\delta h - hF\,\delta\beta)\,d\theta$$

will also be negligible. Thus we find that

$$\delta I = \int_C (p - p_d)\,\delta p\,d\theta + \int_C \psi^T(hD\,\delta w + G\,\delta h - hF\,\delta\beta)\,d\theta + J$$

where

$$J = \iint_D \{(F^T\psi_\theta + G^T r\psi_r)\,\delta h + (G^T\psi_\theta - F^T r\psi_r)\,h\,\delta\beta\}\,dr\,d\theta \tag{3.21}$$

Also

$$hD\,\delta w + G\,\delta h - hF\,\delta\beta = \delta(hG) = h\begin{bmatrix} 0 \\ c\,\delta p \\ s\,\delta p \\ 0 \end{bmatrix} + p\begin{bmatrix} 0 \\ \delta(hc) \\ \delta(hs) \\ 0 \end{bmatrix}$$

Thus using the relations (3.13)

$$\delta I = \int_C (p - p_d)\,\delta p\, d\theta + \int_C (c\psi_2 + s\psi_3)\,\delta p h\, d\theta + \int_C p(c\psi_2 + s\psi_3)\,\delta h\, d\theta$$

$$- \int_C p(s\psi_2 - c\psi_3)\,\delta\beta h\, d\theta + J$$

Now let ψ satisfy the boundary condition

$$h(c\psi_2 + s\psi_3) = -(p - p_d) \qquad \text{on } C \tag{3.22}$$

Then

$$\delta I = -\int_C (p - p_d)\, p\, \frac{\delta h}{h}\, d\theta - \int_C (s\psi_2 - c\psi_3)\, ph\, \delta\beta\, d\theta + J \tag{3.23}$$

Finally we can use the fact that the mapping function is fully determined by its boundary value to reduce J to a boundary integral. Set

$$\log \frac{dz}{d\sigma} = f + i\beta$$

where

$$f = \log \left| \frac{dz}{d\sigma} \right| = \log h$$

and

$$\delta f = \frac{\delta h}{h}$$

Also f and β separately satisfy Laplace's equation

$$\Delta f = 0, \qquad \Delta\beta = 0$$

and jointly they satisfy the Cauchy Riemann equations

$$f_\theta = r\beta_r, \qquad \beta_\theta = -rf_r$$

Let the auxiliary function P satisfy the equation

$$\Delta P = h(F^T\psi_\theta + G^T r\psi_r) \qquad \text{in } D \tag{3.24}$$

and the boundary condition

$$P=0 \qquad \text{on } C \tag{3.25}$$

Also let the auxiliary function Q satisfy the equation

$$\Delta Q=h(G^T\psi_\theta-F^T r\psi_r) \qquad \text{in } D \tag{3.26}$$

and the boundary condition

$$\frac{\partial Q}{\partial r}=hp(s\psi_2+c\psi_3) \qquad \text{on } C \tag{3.27}$$

Then

$$\begin{aligned} J&=\iint_D(\Delta P\,\delta f+\Delta Q\,\delta\beta)\,dr\,d\theta \\ &=\int_C\left[\delta f\frac{\partial P}{\partial r}+hp(s\psi_2+c\psi_3)-Q\frac{\partial}{\partial r}\delta\beta\right]d\theta \\ &=\int_C\left[\delta f\frac{\partial P}{\partial r}+hp(s\psi_2+c\psi_3)-Q\frac{\partial}{\partial\theta}\delta f\right]d\theta \end{aligned}$$

Thus finally

$$\delta I=\int_C g\,\delta f\,d\theta \tag{3.28}$$

where

$$g=\frac{\partial P}{\partial r}+\frac{\partial Q}{\partial\theta}-(p-p_d)\,p \tag{3.29}$$

As in the previous section, an appropriate modification of f is

$$\delta f=-\lambda\tilde{g}$$

where $\tilde{g}$ is the projection of g onto the admissible subspace of variations defined by Eqs. (2.20) and (2.21), and λ is a sufficiently small positive number. Then

$$\delta I=-\lambda\int_C\tilde{g}^2\,d\theta<0$$

If the flow is transonic, shock waves are likely to be formed, and again it may be desirable to use a more sophisticated cost function to produce a smooth shape change. In this case we can set

$$I = \frac{1}{2} \int_C \left[\lambda_1 S^2 + \lambda_2 \left(\frac{dS}{d\theta} \right)^2 \right] d\theta \tag{3.30}$$

where λ_1 and λ_2 are positive parameters, and the periodic function $S(\theta)$ satisfies the equation

$$\lambda_1 S - \lambda_2 \frac{d^2 S}{d\theta^2} = p - p_d \tag{3.31}$$

Then

$$\begin{aligned} \delta I &= \int_C \left(\lambda_1 S\, \delta S + \lambda_2 \frac{dS}{d\theta} \frac{d}{d\theta} \delta S \right) d\theta \\ &\times \int_C S \left(\lambda_1\, \delta S - \lambda_2 \frac{d^2}{d\theta^2} \delta S \right) d\theta \\ &= \int_C S\, \delta p\, d\theta \end{aligned}$$

Thus S replaces $p - p_d$ in the previous formulas. If one modifies the boundary condition (3.22) to

$$h(c\psi_2 + s\psi_3) = -S \qquad \text{on } C \tag{3.32}$$

the formula for the gradient becomes

$$g = \frac{\partial P}{\partial r} + \frac{\partial Q}{\partial \theta} - Sp \tag{3.33}$$

instead of Eq. (3.29), and an appropriate modification of f is again $-\lambda \bar{g}$.

The final design procedure using the Euler equations is thus as follows. Choose an initial profile and corresponding mapping function f. Then

1. Solve the flow equation (3.8) for w by integrating to a steady state.
2. Solve the ordinary differential equation (3.31) for S.
3. Solve the adjoint equation (3.19) with the boundary conditions (3.20) and (3.32) for ψ by integrating to a steady state.
4. Solve the auxiliary Poisson equations (3.24) and (3.26) for P and Q.

5. Evaluate

$$g = \frac{\partial P}{\partial r} + \frac{\partial Q}{\partial \theta} - Sp$$

on C, and find its projection $\tilde{g}$ onto the admissible subspace of variations according to Eqs. (2.20) and (2.21).

6. Correct the boundary mapping function f_C by

$$\delta f = -\lambda \tilde{g}$$

where $\lambda > 0$, and return to step 1.

4. WING DESIGN USING THE EULER EQUATIONS

In order to illustrate further the application of control theory to aerodynamic design problems, this section treats the case of three-dimensional wing design, again using the inviscid Euler equations as the mathematical model for compressible flow. In this case it proves convenient to denote the Cartesian coordinates and velocity components by x_1, x_2, x_3 and u_1, u_2, u_3, and to use the convention that summation over $i = 1$–3 is implied by a repeated index i. The three-dimensional Euler equations may then be written as

$$\frac{\partial w}{\partial t} + \frac{\partial f_i}{\partial x_i = 0} \tag{4.1}$$

where

$$w = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{bmatrix} \tag{4.2a}$$

and

$$f_1 = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ \rho u_1 u_3 \\ \rho u_1 E \end{bmatrix}, \quad f_2 = \begin{bmatrix} \rho u_2 \\ \rho u_2 u_1 \\ \rho u_2^2 + p \\ \rho u_2 u_3 \\ \rho u_2 E \end{bmatrix}, \quad f_3 = \begin{bmatrix} \rho u_3 \\ \rho u_3 u_1 \\ \rho u_3 u_2 \\ \rho u_3^2 + p \\ \rho u_3 E \end{bmatrix} \tag{4.2b}$$

Also

$$p=(\gamma-1)\,\rho(E-u_i^2/2), \qquad \rho H=\rho E+p \tag{4.3}$$

Consider a transformation to coordinates X_1, X_2, X_3 where

$$H_{ij}=\frac{\partial x_i}{\partial X_j}, \qquad J=\det(H), \qquad H_{ij}^{-1}=\frac{\partial X_i}{\partial x_j} \tag{4.4}$$

The Euler equations can now be written as

$$\frac{\partial W}{\partial t}+\frac{\partial F_i}{\partial X_i}=0 \tag{4.5}$$

where

$$W=J\mathrm{w}, \qquad F_i=J\frac{\partial X_i}{\partial x_j}f_j \tag{4.6}$$

Define the contravariant velocity vector

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}=H^{-1}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4.7}$$

Then

$$F_i=J\begin{bmatrix} \rho U_i \\ \rho U_i u_1+\dfrac{\partial X_i}{\partial x_1}p \\ \rho U_i u_2+\dfrac{\partial X_i}{\partial x_2}p \\ \rho U_i u_3+\dfrac{\partial X_i}{\partial x_3}p \\ \rho U \end{bmatrix} \tag{4.8}$$

Assume now that the new coordinate system conforms to the wing in such a way that the wing surface B_w is represented by $X_2=0$. Then the flow is determined as the steady state solution of Eq. (4.5) subject to the flow tangency condition

$$U_2=0 \qquad \text{on } B_w \tag{4.9}$$

At the far field boundary, conditions can be specified for incoming waves as in the two-dimensional case, while outgoing waves are determined by the solution.

Suppose now that it is desired to control the surface pressure by varying the wing shape. It is convenient to retain a fixed computational domain. Variations in the shape then result in corresponding variations in the mapping derivatives defined by H. Introduce the cost function

$$I=\frac{1}{2}\iint_{B_w}(p-p_d)^2\,dX_1\,dX_3 \tag{4.10}$$

where p_d is the desired pressure. A variation in the shape will cause a variation δp in the pressure and consequently a variation in the cost function

$$\delta I=\iint_{B_w}(p-p_d)\,\delta p\,dX_1\,dX_3 \tag{4.11}$$

Since p depends on w through the equation of state (4.3), the variation δp can be determined from the variation δw. Define the Jacobian matrices

$$A_i=\frac{\partial f_i}{\partial w}, \qquad C_i=H_{ij}A_j \tag{4.12}$$

Then

$$\frac{\partial}{\partial X_i}(\delta F_i)=0 \tag{4.13}$$

where

$$\delta F_i=C_i\,\delta w+\delta\left(J\frac{\partial X_i}{\partial x_j}\right)f_j \tag{4.14}$$

and for any differentiable vector ψ

$$\int_D \frac{\partial\psi^T}{\partial X_i}\,\delta F_i\,dv=\int_{\text{boundaries}} n_i\psi^T\,\delta F_i\,ds \tag{4.15}$$

where n_1, n_2, and n_3 are the components of a unit vector normal to the boundary. On the wing surface B_w, $n_1=n_3=0$ and it follows from Eq. (4.9) that

$$\delta F_2 = J \begin{bmatrix} 0 \\ \frac{\partial X_2}{\delta x_1}\delta p \\ \frac{\partial X_2}{\partial x_2}\delta p \\ \frac{\partial X_2}{\partial x_3}\delta p \\ 0 \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta\left(J\frac{\partial X_2}{\partial x_1}\right) \\ \delta\left(J\frac{\partial X_2}{\partial x_2}\right) \\ \delta\left(J\frac{\partial X_2}{\partial x_3}\right) \\ 0 \end{bmatrix} \tag{4.16}$$

Suppose now that ψ is the steady state solution of the adjoint equation

$$\frac{\partial \psi}{\partial t} - C_i^T \frac{\partial \psi}{\partial X_i} = 0 \qquad \text{in } D \tag{4.17}$$

At the outer boundary incoming characteristics for ψ correspond to outgoing characteristics for δw. Consequently, as in the two-dimensional case, one can choose boundary conditions for ψ such that

$$n_i \psi^T C_i \, \delta w = 0$$

If the coordinate transformation is such that $\delta(JH^{-1})$ is negligible in the far field, the only remaining boundary term is

$$-\iint_{B_w} \psi^T \, \delta F_2 \, dX_1 \, dX_3$$

Let ψ satisfy the boundary condition

$$J\left(\psi_2 \frac{\partial X_2}{\partial x_1} + \psi_3 \frac{\partial X_2}{\partial x_2} + \psi_4 \frac{\partial X_2}{\partial x_3}\right) = -(p - p_d) \qquad \text{on } B_w \tag{4.18}$$

Then, since it follows from Eq. (4.17) that

$$\int_D \frac{\partial \psi^T}{\partial X_i} C_i \, \delta w \, dV = 0$$

we find that

$$\delta I = \int_D \frac{\partial \psi^T}{\partial X_i} \delta\left(J\frac{\partial X_i}{\partial x_j}\right) f_j \, dV - \iint_{B_w} \left[\psi_2 \delta\left(J\frac{\partial X_2}{\partial x_1}\right) + \psi_3 \delta\left(J\frac{\partial X_2}{\partial x_2}\right) + \psi_4 \delta\left(J\frac{\partial X_2}{\partial x_3}\right)\right] p \, dX_1 \, dX_3 \tag{4.19}$$

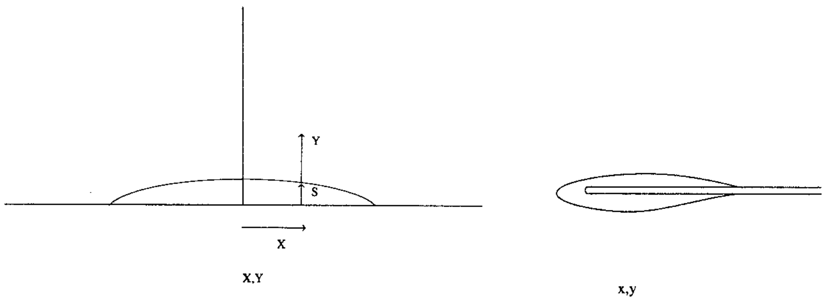

Figure 2

A convenient way to treat a wing is to introduce sheared parabolic coordinates through the transformation

$$x=\tfrac{1}{2}\{X^2-[Y+S(X,Z)]^2\}$$
$$y=X[Y+S(X,Z)]$$
$$z=Z$$

Here x, y, z are Cartesian coordinates, and X and $Y+S$ correspond to parabolic coordinates generated by the mapping

$$x+iy=\tfrac{1}{2}[X+i(Y+S)]^2$$

at a fixed span station Z. The surface $Y=0$ is a shallow bump corresponding to the wing surface, with a height $S(X,Z)$ determined by the equation

$$X+iS=[2(x_s+iy_s)]^{1/2}$$

where $x_s(z)$ and $y_s(z)$ are coordinates of points lying in the wing surface. We now treat $S(X,Z)$ as the control.

In this case

$$H=\begin{bmatrix} X-(Y+S)\,S_X & -(Y+S) & -(Y+S)\,S_Z \\ Y+S+XS_X & X & XS_Z \\ 0 & 0 & 1 \end{bmatrix}$$

while

$$J=X^2+(Y+S)^2$$

and

$$H^{-1}=\frac{1}{J}\begin{bmatrix} X & Y+S & 0 \\ -(Y+S+XS_X) & [X-(Y+S)\,S_X] & -JS_Z \\ 0 & 0 & J \end{bmatrix}$$

Also

$$\delta J = 2(Y+S)\,\delta S$$

and

$$\delta(JH^{-1})=\begin{bmatrix} 0 & \delta S & 0 \\ -(\delta S + X\,\delta S_X) & -(\delta S S_X + (Y+S)\,\delta S_X) & -(\delta J\, S_Z + J\,\delta S_Z) \\ 0 & 0 & 1 \end{bmatrix}$$

Inserting these formulas in Eq. (4.19) we find that the volume integral in δI is

$$\int_D \psi_X^T\,\delta S f_1\,dV - \int_D \psi_Y^T[(\delta S + X\,\delta S_X)\,f_1 + (\delta S S_X + (Y+S)\,\delta S_X)\,f_2$$

$$+\ (\delta J S_Z + J\,\delta S_Z\, f_3)]\,dV + \int_D \psi_Z^T\,\delta J\,dV$$

where S and δS are independent of Y. Therefore, integrating over Y, the variation of the cost function can be reduced to a surface integral of the form

$$\delta I = \iint_B [P(X,Z)\,\delta S + Q(X,Z)\,\delta S_X + R(X,Z)\,\delta S_Z]\,dX\,dZ$$

Also the shape change will be confined to a bounded region of the X–Z plane, so we can integrate by parts to obtain

$$\delta I = \iint_B \left(P - \frac{\partial Q}{\partial X} - \frac{\partial R}{\partial Z}\right)\delta S\,dX\,dZ$$

Thus to reduce I we can choose

$$\delta S = -\lambda\left(P - \frac{\partial Q}{\partial X} - \frac{\partial R}{\partial Z}\right)$$

where λ is sufficiently small and non-negative.

In order to impose a thickness contraint we can define a base-line surface $S_0(X, Z)$ below which $S(X, Z)$ is not allowed to fall. Now if we take $\lambda = \lambda(X, Z)$ as a non-negative function such that

$$S(X, Z) + \delta S(X, Z) \geqslant S_0(X, Z)$$

Then the constraint is satisfied, while

$$\delta I = -\iint_B \lambda \left(P - \frac{\partial Q}{\partial X} - \frac{\partial R}{\partial Z}\right)^2 dX\, dZ \leqslant 0$$

5. CONCLUSION

The purpose of the last three sections is to demonstrate by representative examples that control theory can be used to formulate computationally feasible procedures for aerodynamic design. The cost of each iteration is of the same order as two flow solutions, since the adjoint equation is of comparable complexity to the flow equation, and the remaining auxiliary equations could be solved quite inexpensively. Provided, therefore, that one can afford the cost of a moderate number of flow solutions, procedures of this type can be used to derive improved designs. The approach is quite general, not limited to particular choices of the coordinate transformation or cost function, which might in fact contain measures of other criteria of performance such as lift and drag. For the sake of simplicity certain complicating factors, such as the need to include a special term in the mapping function to generate a corner at the trailing edge, have been suppressed from the present analysis. Also it remains to explore the numerical implementation of the design procedures proposed in this paper.

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