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ON A FORMULA OF THE GENERALIZED RESOLVENTS OF A NONDENSELY DEFINED HERMITIAN OPERATOR

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The Weyl function and the prohibited lineal, corresponding to a given space of boundary values of a nondensely defined Hermitian operator, are introduced and investigated. The prohibited lineal is characterized in terms of the limiting values of the Weyl function. An analogue of M. G. Krein's formula for the resolvent is obtained and its connection with the space of boundary values is found.

This paper is a detailed presentation of [1]. Here, from the point of view of spaces of boundary values (SBV), i.e., an abstract variant of Green's second formula, we investigate some questions regarding the extension of a Hermitian operator with a nondense domain of definition D(A). We introduce and investigate the Weyl function M(λ) and the characteristic function C(λ), corresponding to a given SBV. It is shown that the Weyl function is a Q-function of the operator A; the relationship between the angular limiting value M(i∞) of the Weyl function at infinity and the prohibited lineal V_Γ is found. An analogue of Krein's formula for the resolvent is obtained and its application to the moment problem is given.

We shall adhere to the following notations: $\mathfrak{H}, \mathcal{H}$ are separable Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ ($\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$) is the set of bounded (closed) linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$; if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, then $[\mathcal{H}_1, \mathcal{H}_2] = [\mathcal{H}]$, $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{C}(\mathcal{H})$; $\tilde{\mathcal{C}}(\mathcal{H})$ is the collection of closed linear relations in $\mathcal{H}$ and, moreover, $\mathcal{C}(\mathcal{H}) \subset \tilde{\mathcal{C}}(\mathcal{H})$ by identifying an operator with its graph; $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are the domain of definition and the range of the relation $T \in \tilde{\mathcal{C}}(\mathcal{H})$, $T(f) = \{g \in \mathcal{H}; \{f, g\} \in T\}$ and, in particular, $T(0) = \{g \in \mathcal{H}; \{0, g\} \in T\}$; $T^{-1} = \{\{f, g\} \in \mathcal{H} \oplus \mathcal{H}; \{f, g\} \in T\}$; $\alpha T = \{\{f, \alpha g\}; \{f, g\} \in T\}$; $\rho(T)$ and $\sigma(T)$ are the resolvent set and the spectrum of the relation $T \in \tilde{\mathcal{C}}(\mathcal{H})$; $\hat{\rho}(A)$ is the regularity field of the operator A; $\sigma_p(A)$, $\sigma_c(T)$, and $\sigma_r(T)$ are the point,

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continuous, and residual spectra of the relation T; A is a Hermitian operator in $\mathfrak{H}$, $\mathfrak{M}_\lambda = (A-\lambda)\mathcal{D}(A)$, $\mathfrak{N}_\lambda = \mathfrak{M}_{\bar\lambda}^\perp$, $n_\pm(A) = \dim \mathfrak{N}_{\pm i}$; $E_{\tilde A(\lambda)}$ is the resolution of the identity for the relation $\tilde A = \tilde A^* \in \tilde{\mathcal{C}}(\mathfrak{H})$, i.e., the resolution of the identity for its operator part $\tilde A' \in \mathcal{C}(\mathfrak{H})$; P_L is the orthoprojection in $\mathfrak{H}$ on the subspace L; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the open upper (lower) half-plane, $T \upharpoonright L$ is the restriction of the operator T to the lineal L.

1. Preliminary Information. We recall briefly the fundamental aspects of the theory of the extensions of a non-densely defined Hermitian operator A in a separable Hilbert space $\mathfrak{H}$. Let $\mathfrak{H}_0 \doteq \overline{\mathcal{D}(A)}$, $\mathfrak{N} = \mathfrak{H}_0^\perp$, let P_0 be the orthoprojection onto $\mathfrak{H}_0$, $\mathfrak{M}_\lambda \doteq (A-\lambda)\,\mathcal{D}(A)$, $\mathfrak{N}_\lambda \doteq \mathfrak{M}_{\bar\lambda}^\perp$ is the defect subspace. For the operator A, acting from $\mathfrak{H}_0$ into $\mathfrak{H}$ ($A \in \mathcal{C}(\mathfrak{H}_0, \mathfrak{H})$), the adjoint operator $A^* \in \mathcal{C}(\mathfrak{H}, \mathfrak{H}_0)$. is well defined. Clearly, $\overline{\mathcal{D}(A^*)} = \mathfrak{H}$, $\mathcal{D}(A) \subset \mathcal{D}(A^*)$, $\mathfrak{N}_\lambda \subset \mathcal{D}(A^*)$ $\forall \lambda \in \mathbb{C}\setminus\mathbb{R}$ and the following equalities hold:

$$A^* f_A = P_0 A f_A \quad \forall f_A \in \mathcal{D}(A), \; A^* f_\lambda = \lambda P_0 f_\lambda \quad \forall f_\lambda \in \mathfrak{N}_\lambda. \tag{1}$$

Statement 1. If $\mathfrak{N} = \mathfrak{H}_0^\perp$, then $\mathfrak{M}_\lambda \cap \mathfrak{N} = \{0\}$ for $\lambda = \alpha + i\beta \neq \bar\lambda$.

Proof. If $f_A \in \mathcal{D}(A)$, $(A-\lambda) f_A = n \in \mathfrak{N}$, then

$$0 = (n, f_A) = = \operatorname{Im}((A-\lambda) f_A, f_A) = -\operatorname{Im}(\lambda f_A, f_A) = -\operatorname{Im}\lambda \, \| f_A \|^2 \Rightarrow f_A = 0.$$

Statement 2 [2]. The lineals $\mathcal{D}(A)$ and $\mathfrak{N}_\lambda$ are linearly independent.

Proof. If $f_A + f_\lambda = 0$ for some $f_A \in \mathcal{D}(A)$, $f_\lambda \in \mathfrak{N}_\lambda$, , then $Af_A + \lambda f_\lambda = n \in \mathfrak{N}$.. From here $(A - \lambda) f_A = n$ and, consequently, $f_A = n = 0$. But then $f_\lambda = 0$.

The following proposition is a generalization of the known Neumann formula [3] to the case $\overline{\mathcal{D}(A)} \neq \mathfrak{H}$.

Proposition 1 [4-6]. Let $\overline{\mathcal{D}(A)} = \mathfrak{H}_0 \subseteq \mathfrak{H}$.. Then

$$\mathcal{D}(A^*) = \mathcal{D}(A) + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar\lambda}. \tag{2}$$

In addition, for each pair of vectors $f \in \mathcal{D}(A^*)$, $n \in \mathfrak{N}$ we have the unique decomposition

$$f = f_A + f_\lambda + f_{\bar\lambda}, \quad A^* f + n = A f_A + \lambda f_\lambda + \bar\lambda f_{\bar\lambda}, \tag{3}$$

where $f_A \in \mathcal{D}(A)$, $f_\lambda \in \mathfrak{N}_\lambda$, $f_{\bar\lambda} \in \mathfrak{N}_{\bar\lambda}$.

Proof (compare with [3]). Since $\mathfrak{H} = (A-\lambda)\,\mathcal{D}(A) \oplus \mathfrak{N}_{\bar\lambda}$ $(\lambda \neq \bar\lambda)$, it follows that $\forall \{f, n\} \in \mathcal{D}(A^*) \times \mathfrak{N}$ there exist $f_A \in \mathcal{D}(A)$ and $f_{\bar\lambda} \in \mathfrak{N}_{\bar\lambda}$ such that

$$A^* f - \lambda f + n = (A - \lambda) f_A + (\bar\lambda - \lambda) f_\lambda. \tag{4}$$

Applying the projection P_0 to (4) and taking into account (1), we obtain the equality $A^*(f - f_A - f_{\bar\lambda}) = \lambda P_0 (f - f_A - f_{\bar\lambda})$, meaning that $f_\lambda \doteq f - f_A - f_{\bar\lambda} \in \mathfrak{N}_\lambda$. From here we obtain the first of the equalities (3) and, with the aid of (4), also the second one.

Assuming the lack of uniqueness in (3), we obtain the equalities

$$f_A' + f_\lambda' + f_{\bar\lambda}' = 0, \; A f_A' + \lambda f_\lambda' + \bar\lambda f_{\bar\lambda}' = 0, \; f_A' \in \mathcal{D}(A), f_\lambda \in \mathfrak{N}_\lambda. \tag{5}$$

Multiplying the first of them by λ and subtracting it from the second one, we obtain $(A-\lambda) f_A' + (\bar\lambda - \lambda) f_{\bar\lambda}' = 0$. Since $\mathfrak{M}_\lambda \perp \mathfrak{N}_{\bar\lambda}$, we have $f_A' = f_{\bar\lambda}' = 0$. But then from (5) there follows that $f_\lambda' = 0$.

COROLLARY 1 [2]. The relation $\overline{\mathcal{D}(A)} = \mathfrak{H}$ s equivalent to the linear independence of the lineals $\mathcal{D}(A)$, $\mathfrak{N}_\lambda$, $\mathfrak{N}_{\bar\lambda}$ in the decomposition (2).

The reason for the nonuniqueness of the decomposition (2) is revealed by the following proposition.

Proposition 2 [4]. The vectors $f_\lambda \in \mathfrak{N}$, $-f_{\bar\lambda} \in \mathfrak{N}_{\bar\lambda}$ are congruent modulo ♦ ♦ (i.e., $f_\lambda + f_{\bar\lambda} \in \mathcal{D}(A)$), if and only if there exists a (unique) vector $n \in \mathfrak{N}$ such that

$$f_\lambda = P_{\mathfrak{N}_\lambda} n, \; f_{\bar\lambda} = -P_{\mathfrak{N}_{\bar\lambda}} n, \; n \in \mathfrak{N} = \mathfrak{H} \ominus \mathfrak{H}_0. \tag{6}$$

In this case $\| f_\lambda \| = \| f_{\bar\lambda} \|$ and $n = (\lambda - \bar\lambda)\left(A f_A + \lambda f_\lambda + \bar\lambda f_{\bar\lambda}\right)$.

Proof. Necessity. Let f_λ and $-f_{\bar\lambda}$ be congruent modulo $\mathcal{D}(A)$, i.e., there exists a vector $f_A \in \mathcal{D}(A)$ such that

$$f_A + f_\lambda + f_{\bar\lambda} = 0,\ f_A \in \mathcal{D}(A);\ f_\lambda \in \mathfrak{N}_\lambda;\ f_{\bar\lambda} \in \mathfrak{N}_{\bar\lambda}\ . \tag{7}$$

Applying the operator A^* to (7) and taking into account (1), we have $l \doteq A f_A + \lambda f_\lambda + \bar\lambda\ f_{\bar\lambda} \in \mathfrak{N}$. From here and from (7) we find

$$l = (A - \bar\lambda) f_A + (\lambda - \bar\lambda) f_\lambda = (A - \lambda) f_A + (\lambda - \bar\lambda) f_{\bar\lambda}. \tag{8}$$

If in (8) we set $n \doteq l/(\lambda - \bar\lambda)$, then we obtain the equalities (6). In addition, from (8) there follow the relations

$$\begin{aligned}\left\|P_{\mathfrak{N}_\lambda} l\right\|^2 &= \left\|(\lambda - \bar\lambda) f_\lambda\right\|^2 = \|l\|^2 - \left\|(A - \bar\lambda) f_A\right\|^2 = \|l\|^2 - \|(A - \alpha) f_A\|^2 - \\ &- \beta^2 \|f_A\|^2 = \|l\|^2 - \|(A - \lambda) f_A\|^2 = \left\|(\lambda - \bar\lambda) f_{\bar\lambda}\right\|^2 = \left\|P_{\mathfrak{N}_{\bar\lambda}} l\right\|^2,\end{aligned} \tag{9}$$

meaning that $\|f_\lambda\| = \|f_{\bar\lambda}\|$.

Sufficiency. We show that $\forall n \in \mathfrak{N}$ the vectors $P_{\mathfrak{N}_\lambda} n$ and $-P_{\mathfrak{N}_{\bar\lambda}} n$ are congruent modulo $\mathcal{D}(A)$. Setting $f = 0$, $n = n$ in (3), from the second equality we subtract the first one, multiplied by λ and $\bar\lambda$. As a result we obtain the relations (8), in which $l = n$. From here

$$P_{\mathfrak{N}_\lambda} n = (\lambda - \bar\lambda) f_\lambda,\ P_{\mathfrak{N}_{\bar\lambda}} n = (\lambda - \bar\lambda) f_{\bar\lambda},\ n = A f_A + \lambda f_\lambda + \bar\lambda f_{\bar\lambda}. \tag{10}$$

Now from (10) and from the equality $f_A + f_\lambda + f_{\bar\lambda} = 0$ there follows the required relation $P_{\mathfrak{N}_\lambda} n - P_{\mathfrak{N}_{\bar\lambda}} n = (\bar\lambda - \lambda) f_A \in \mathcal{D}(A)$.

COROLLARY 2 [4, 7]. Let $U_{\lambda\bar\lambda} \doteq (A - \bar\lambda)(A - \lambda)^{-1}$. Then

$$\mathfrak{N} = \{n \in \mathfrak{H};\ P_{\mathfrak{M}_{\bar\lambda}} n = U_{\lambda\bar\lambda} P_{\mathfrak{M}_\lambda} n\}. \tag{11}$$

Proof. If $n \in \mathfrak{N}$, then, by virtue of (8), there exists a vector $f_A \in \mathcal{D}(A)$ such that $P_{\mathfrak{M}_\lambda} n = (A - \lambda) f_A$, $P_{\mathfrak{M}_{\bar\lambda}} n = (A - \bar\lambda) f_A$. From here $P_{\mathfrak{M}_\lambda} n = U_{\bar\lambda\lambda} P_{\mathfrak{M}_{\bar\lambda}} n$. Conversely, if this equality is satisfied, then the equalities (8) hold (with n instead of l) and, therefore, so does (3); from here $n \in \mathfrak{N}$. Relation (11) is proved.

Proposition 2 enables us to introduce the following definition.

Definition 1. Let $\mathfrak{N}''_\lambda \doteq P_{\mathfrak{N}_\lambda} \mathfrak{N}$. By the equality

$$V_э P_{\mathfrak{N}_\lambda} n = P_{\mathfrak{N}_{\bar\lambda}} n,\ n \in \mathfrak{N}, \tag{12}$$

the isometric operator $V_e \in [\mathfrak{N}''_\lambda, \mathfrak{N}''_{\bar\lambda}]$, called the exclusion operator, is well defined.

The exclusion operator V_e has been introduced in [2], while in [4] one has elucidated its role in the description of the self-adjoint extensions of the operator A. In [4] it has been shown that the lineals $\mathfrak{N}''_\lambda$ are closed or not, simultaneously for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. In the first case the operator A is said to be regular, in the second case it is called singular [5].

Definition 2. The subspaces $\mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \ominus \mathfrak{N}''_\lambda$ are called the semidefect subspaces of the operator A, while the numbers $n'_\pm(A) = \dim \mathfrak{N}'_{\pm i}$ are called the semidefect numbers.

As it can be easily seen, they are the defect subspaces (numbers) of the operator $A' = P_0 A(\mathcal{D}(A') = \mathcal{D}(A))$ in $\mathfrak{H}_0$.

2. Spaces of Boundary Values and Proper Extensions. Here the operator A is identified with the graph: $A \leftrightarrow \operatorname{gr} A = \{\{f, Af\}; f \in \mathcal{D}(A)\}$, while the symbol A^* denotes the adjoint relation. In order to avoid confusion, we denote the adjoint operator from Sec. 1 by $A^*_{op} (\in \mathfrak{C}(\mathfrak{H}, \mathfrak{H}_0))$, and A^* and A^*_{op} are connected by the obvious equality

$$A^* = \{\{f, f'\}; f \in \mathcal{D}(A^*) = \mathcal{D}(A^*_{op}), f' = A^*_{op} f + n,\ n \in \mathfrak{N}\}. \tag{13}$$

We note that, by virtue of (1) and (13), the Neumann formulas (3) are equivalent to the direct sum decomposition (see [6, 8])

$$A^* = A \dotplus \hat{\mathfrak{N}}_\lambda \dotplus \hat{\mathfrak{N}}_{\bar\lambda} \quad \left(\hat{\mathfrak{N}}_\lambda \doteq \{\{f_\lambda, \lambda f_\lambda\}; f_\lambda \in \mathfrak{N}_\lambda\}\right), \tag{14}$$

and Proposition 2 describes the components of the vectors from the indeterminate part $\hat{\mathfrak{N}} = \{0, \mathfrak{N}\}$ of the relation A^* with respect to the decomposition (14).

The lineal A^* is a Hilbert space with respect to the norm

$$\|\hat{f}\|^2 = \|f\|^2 + \|f'\|^2 = \|f\|^2 + \left\|A^*_{\text{op}} f\right\|^2 + \|n\|^2 \quad (\hat{f} = \{f, f'\}). \tag{15}$$

Definition 3. A collection $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$, where $\mathcal{H}$ is a separable Hilbert space, while $\Gamma_i \in [A^*, \mathcal{H}]$, $i = 1, 2$, is called a space of boundary values of the relation A^* if

1) $(f', g) - (f, g') = \left(\Gamma_1 \hat{f}, \Gamma_2 \hat{g}\right)_{\mathcal{H}} - \left(\Gamma_2 \hat{f}, \Gamma_1 \hat{g}\right)_{\mathcal{H}} \quad \forall \hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in A^*;$ (16)

2) the mapping $\Gamma : \hat{f} \to \{\Gamma_2 \hat{f}, \Gamma_1 \hat{f}\}$ from A^* into $\mathcal{H} \oplus \mathcal{H}$ is surjective.

It is easy to see that, under the conditions 1, 2, we have $\ker \Gamma = A$.

Statement 3. For a Hermitian operator A with equal defect numbers $n_+(A) = n_-(A) \leq \infty$ there exists a SBV.

Proof. Let U_0 be an isometry from $\mathfrak{N}_{-i}$ onto $\mathfrak{N}_i$, $\tilde{U}_0 \doteq U_0 \oplus U_0$, let $P_{\pm i}$ be the oblique projections onto $\hat{\mathfrak{N}}_{\pm i}$ in the decomposition (14) parallel to $A \dotplus \hat{\mathfrak{N}}_{\mp i}$, and let π_1 be the orthoprojection onto the first term in $\hat{\mathfrak{N}}_i$. Setting

$$\mathcal{H} = \mathfrak{N}_i, \Gamma_1 = \pi_1(P_i + \tilde{U}_0 P_{-i}), \Gamma_2 = -i\,\pi_1(P_i + \tilde{U}_0 P_{-i}), \tag{17}$$

we obtain the equality

$$(f', g) - (f, g') = 2i\,(f_i, g_i) - 2i\,(f_{-i}, g_{-i}) = (\Gamma_1 \hat{f}, \Gamma_2 \hat{g})_{\mathcal{H}} - (\Gamma_2 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{H}}$$

which coincides with (16). The epimorphism property of the mapping Γ is obvious.

Definition 4. An extension $\tilde{A} (\in \tilde{\mathfrak{C}}(\mathfrak{H}))$ of the operator A is said to be proper if it is closed and $A \subset \tilde{A} \subset A^*$.

Definition 5. Two proper extensions are said to be disjoint if $\tilde{A}' \cap \tilde{A}'' = A$, and transversal if, in addition, $\tilde{A}' + \tilde{A}'' = A^*$.

Let $\mathcal{H}_i = \Gamma_i \hat{\mathfrak{N}} = \Gamma_i \{0, \mathfrak{N}\}$ ($i = 1, 2$) be lineals in $\mathcal{H}$ (in general, not closed), where, as before, $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{H}_0$.

Definition 6. By the exclusion relation V_Γ, corresponding to the SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$, we mean the lineal $V_\Gamma \doteq \Gamma \hat{\mathfrak{N}} = \Gamma\{0, \mathfrak{N}\}$, i.e.,

$$\{h_2, h_1\} \in V_\Gamma \Leftrightarrow \exists\, n \in \mathfrak{N} : h_i = \Gamma_i \hat{n},\ \hat{n} = \{0, n\},\ i = 1, 2. \tag{18}$$

If in (16) we set $\hat{f} = \hat{n} = \{0, n\}$, $\hat{g} = \hat{l} = \{0, l\} \in \hat{\mathfrak{N}}$, then we see that the exclusion relation V_Γ is Hermitian: $V_\Gamma \subset V_\Gamma^*$.

With each SBV there are connected two transversal selfadjoint extensions $\tilde{A}_i = \tilde{A}_i^*$, for which

$$\tilde{A}_i = \ker \Gamma_i,\ i = 1, 2 \quad (\Gamma \tilde{A}_1 = \mathcal{H} \oplus \mathbf{0},\ \Gamma \tilde{A}_2 = \mathbf{0} \oplus \mathcal{H}). \tag{19}$$

From Definitions 3-6 there follows at once the following lemma.

LEMMA 1. The mapping Γ: $\hat{f} \to \{\Gamma_2 \hat{f}, \Gamma_1 \hat{f}\}$ of the Hilbert space A^* (with the norm (15)) into $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ is surjective and defines a topological isomorphism between A^*/A and $\tilde{\mathcal{H}}$ such that:

1) between the proper extensions of $\tilde{A}$ and the closed linear relations in $\mathcal{H}$ (i.e., the subspaces in $\tilde{\mathcal{H}}$) we have the bijective correspondence

$$\tilde{A} = \tilde{A}_\theta \leftrightarrow \theta = \Gamma \tilde{A} = \left\{\{\Gamma_2 \hat{f}, \Gamma_1 \hat{f}\}; \hat{f} \in \tilde{A}\right\}; \tag{20}$$

2) $\left(\tilde{A}_\theta\right)^* = \tilde{A}_{\theta^*}$;

3) the inclusion relation is preserved: $\tilde{A}_{\theta_1} \subset \tilde{A}_{\theta_2} \Leftrightarrow \theta_1 \subset \theta_2$;

4) the extensions $\tilde{A}_{\theta_1}$ and $\tilde{A}_{\theta_2}$ are disjoint $\Leftrightarrow \theta_1 \cap \theta_2 = \{0\}$;

5) the extensions $\tilde{A}_{\theta_1}$ and $\tilde{A}_{\theta_2}$ are transversal $\Leftrightarrow \theta_1 + \theta_2 = \mathcal{H} \oplus \mathcal{H}$;

6) for $\theta = V_\Gamma$, the extension $\tilde{A}_{V_\Gamma}$ is Hermitian and has the form

$$\tilde{A}_{V_\Gamma} = A \dotplus \mathfrak{N} = \{\{f, Af + n\}; f \in \mathcal{D}(A), n \in \mathfrak{N}\}; \tag{21}$$

7) $\tilde{A}_\theta \in \mathfrak{C}(\mathfrak{H})$ (i.e., $\tilde{A}_\theta$ is an operator) $\Leftrightarrow \theta \cap V_\Gamma = \{0\}$.

Statements 1-6 are consequences of Definitions 3-6. Statement 7 follows from statements 4 and 6 if we note that $\tilde{A}_\theta$ is an operator exactly then when $\tilde{A}_\theta$ and $\tilde{A}_{V_\Gamma}$ are disjoint since $\tilde{A}_{V_\Gamma}(0) = \mathfrak{N}$.

Proposition 3. Let $\overline{\mathcal{D}(A)} = \mathfrak{H}_0 \neq \mathfrak{H}$, and let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a SBV of A^*. Then:

1) a proper extension $\tilde{A}$ is transversal (disjoint) to the extension $\tilde{A}_2$ if and only if there exists an operator $B \in [\mathcal{H}]$ ($B \in \mathcal{C}(\mathcal{H})$) such that

$$\tilde{A} = \tilde{A}_B = \ker(\Gamma_1 - B\,\Gamma_2); \tag{22}$$

2) if $\theta_i \in \tilde{\mathcal{C}}(\mathcal{H})$, i = 1, 2, and $\rho(\theta_1) \cap \rho(\theta_2) \neq \varnothing$, then the transversality (disjointness) of the extensions $\tilde{A}\theta_1$ and $\tilde{A}\theta_2$ is equivalent to the condition

$$0 \in \rho[(\theta_1 - z)^{-1} - (\theta_2 - z)^{-1}] \ (0 \notin \sigma_p((\theta_1 - z)^{-1} - (\theta_2 - z)^{-1})) \quad \forall z \in \rho(\theta_1) \cap \rho(\theta_2); \tag{23}$$

3) if $\theta_1 \in \tilde{\mathcal{C}}(\mathcal{H})$, while $\theta_2 = B \in [\mathcal{H}]$, then we have the equivalences

$$\ker(\theta_1 - B) = \{0\} \Leftrightarrow \theta_1 \cap \operatorname{gr} B = \{0\}$$
$$0 \in \rho(\theta_1 - B) \Leftrightarrow \theta_1 \dotplus \operatorname{gr} B = \mathcal{H} \oplus \mathcal{H} \tag{23'}$$

(i.e., θ_1 and gr B are transversal);

4) each dissipative extension $\tilde{A}$ is proper;

5) an extension $\tilde{A} = \tilde{A}_\theta$ is dissipative (accumulative) if and only if so is θ. In this case $n_-(\tilde{A}_\theta) = n_-(\theta)$ ($n_+(\tilde{A}_\theta) = n_+(\theta)$) and, in particular, $\tilde{A}_\theta$ is maximal dissipative (accumulative) exactly when so is θ;

6) the extension $\tilde{A}_\theta$ is Hermitian $\Leftrightarrow$ the relation θ is Hermitian. In this case $n_\pm(\tilde{A}_\theta) = n_\pm(\theta)$.

Proof. 1. By virtue of (19), the disjointness of the extensions $\tilde{A} = \tilde{A}_\theta$ and $\tilde{A}_2$ is equivalent to the condition $\theta \cap (0 \oplus \mathcal{H}) = \{0\}$, meaning that $\theta(0) = \{0\}$, i.e., θ is a (closed) operator: $\theta = B \in \mathcal{C}(\mathcal{H})$. In this case condition (20) assumes the form (22). The transversality of the extensions $\tilde{A}_\theta$ and $\tilde{A}_2$ is equivalent now to the condition

$$\theta \dotplus (0 \oplus \mathcal{H}) = \tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} \Leftrightarrow \operatorname{gr}B \dotplus (0 \oplus \mathcal{H}) = \tilde{\mathcal{H}}. \tag{24}$$

From (24) we conclude that $\mathcal{D}(B) = \mathcal{H}$, i.e., $B \in [\mathcal{H}]$. ♦

2. If $z_0 \in \rho(\theta_1) \cap \rho(\theta_2)$, then the relations θ_i can be represented in the form $\theta_i = \{\{(\theta_1 - z_0)^{-1}f, f\}; f \in \mathcal{H}\}$, i = 1, 2. From here it is clear that their transversality (disjointness) is equivalent to the condition (23).

3. The equivalence (23′) is obvious. Assume further that $\theta_1 \dotplus \operatorname{gr} B = \mathcal{H} \oplus \mathcal{H}$. Then for all $\{h_1, h_2\} \in \mathcal{H} \oplus \mathcal{H}$ there exist vectors $\{f, f'\} \in \theta_1$ and $\{g, Bg\} \in \operatorname{gr} B$ such that $f + g = h_1$, $f' + Bg = h_2$. From here for $h_1 = 0$ we obtain $f' - Bf = h_2$ ($\forall h_2 \in \mathcal{H}$), i.e., $\mathcal{R}(\theta_1 - B) = \mathcal{H}$, and, consequently, taking into account (23′), $0 \in \rho(\theta_1 - B)$. Conversely, let $0 \in \rho(\theta_1 - B)$ and $\{h_1, h_2\} \in \mathcal{H} \oplus \mathcal{H}$. Then there exist vectors $\{f, f'\} \in \theta_1$ such that $f' - Bf = h_2 - Bh_1$. Setting $g = h_1 - f$, we obtain the equality $\{f, f'\} + \{g, Bg\} = \{h_1, h_2\}$, proving, by taking into account (23′), the decomposition $\theta_1 \dotplus \operatorname{gr} B = \mathcal{H} \oplus \mathcal{H}$.

4. Assume that $\tilde{A}$ ($\supset A$) is dissipative. We show that $A \subset A^*$. Without loss of generality, we assume that $\rho(\tilde{A}) \neq \varnothing$. Setting

$$T = (A - i)(A + i)^{-1} \in [\mathfrak{M}_{-i}, \mathfrak{M}_i],\ \tilde{T} = I - 2i(\tilde{A} + i)^{-1} \in [\mathfrak{H}, \mathfrak{H}], \tag{25}$$

we can see that the operator T is isometric from $\mathfrak{M}_{-i}$ onto $\mathfrak{M}_{i}$, while $\tilde{T}$ is a contraction in $\mathfrak{H}$. The operator $\tilde{T}$ ($\supset T$) has the block-matrix representation

$$\tilde{T} = \begin{pmatrix} T & M \\ 0 & U \end{pmatrix}, M \in [\mathfrak{N}_i, \mathfrak{M}_i], U \in [\mathfrak{N}_i, \mathfrak{N}_{-i}].$$

Since $TT^* = I_{\mathfrak{M}_{-i}}$, we have

$$I - \tilde{T}\tilde{T}^* = \begin{pmatrix} -MM^* & -MU^* \\ -UM^* & I - UU^* \end{pmatrix} \geq 0.$$

From here we conclude that M = 0. Consequently, $\tilde{T} = T \oplus U$ (compare with [7], Lemma 1.2). Returning to the relation $\tilde{A}$, from (25) we obtain

$$\tilde{A} = \{\{(I - \tilde{T})f, i(I + \tilde{T})f\}; f = (A + i)f_A + f_i \in \mathfrak{M}_{-i} \oplus \mathfrak{N}_i = \mathfrak{H}\} =$$
$$= \{\{2if_A + (I - U)f_i, 2iAf_A + i(I + U)f_i\}; f_A \in \mathcal{D}(A), f_i \in \mathfrak{N}_i\} =$$

$$= \left\{2i\begin{pmatrix} f_A \\ Af_A \end{pmatrix} + \begin{pmatrix} f_i \\ if_i \end{pmatrix} + \begin{pmatrix} -Uf_i \\ iUf_i \end{pmatrix}; f_A \in \mathcal{D}(A),\ f_i \in \mathfrak{N}_i \right\} \subset A^*. \tag{26}$$

Relation (26) means that the extension $\tilde{A}$ is proper.

5. Assume that $\tilde{A} \supset A$ and that $\Im\tilde{A} \geq 0$. According to statement 4, we have $\tilde{A} = \tilde{A}_\theta$, where (see (20)) $\theta = \Gamma\tilde{A}$. From (16) it is clear that θ and $\tilde{A}_\theta$ can be dissipative only simultaneously. Since the correspondence (20) preserves the inclusion (Lemma 1), it follows that $\tilde{A}_\theta$ is a maximal dissipative relation exactly when so is θ. If, however, we have $n \doteq n_-(\tilde{A}_\theta) > 0$ ($\Leftrightarrow \rho(\tilde{A}_\theta) = \varnothing$) and $\tilde{A}' = \tilde{A}_{\theta'}$ is its maximal dissipative extension, then $n_-(\tilde{A}_\theta) = \dim(\tilde{A}'/\tilde{A})$, which follows from the existence of the "tower" $\tilde{A}_0 \doteq \tilde{A}_\theta \subset \tilde{A}_{(1)} \subset \tilde{A}_{(2)} \subset \dots \subset \tilde{A}_{(n)} \doteq \tilde{A}'$ of dissipative extensions, each "tier" of which is one-dimensional. Similarly, $n_-(\theta) = \dim(\theta'/\theta)$. Now, the validity of statement 4 follows from the isomorphism

$$\theta'/\theta \cong \tilde{A}'/\tilde{A} = (\tilde{A}'/A)/(\tilde{A}/A) \ (\rho(\theta') \neq \varnothing,\ \rho(\tilde{A}') \neq \varnothing).$$

6. Since the relation $\tilde{A}_\theta$ is Hermitian if it is simultaneously dissipative and accumulative, statement 5 follows from the previous one. We note also that a maximal Hermitian relation $\tilde{A}$ is maximal dissipative only in the case when $n_-(\tilde{A}) = 0$. Otherwise ($n_-(\tilde{A}) > 0$) it admits a maximal dissipative extension $\tilde{A}'$, defined, for example, by extending by zero the operator T of the form (25): $\tilde{T} \restriction \mathfrak{N}_{-i} = 0$.

COROLLARY 3. The semidefect numbers of the operator A are equal to the defect numbers of the relation V_Γ, i.e.,

$$n_\pm(V_\Gamma) = n'_\pm(A) \doteq (\dim \mathfrak{N}_{\pm i} \cap \mathfrak{H}_0). \tag{27}$$

Proof. From formula (21) for $\tilde{A}_{V_\Gamma}$ we conclude that $n_\pm(\tilde{A}_{V_\Gamma}) = n'_\pm(A)$ $(= \dim \mathfrak{N}_{\pm i} \cap \mathfrak{H}_0)$. On the other hand, according to statement 6, we have $n_\pm(V_\Gamma) = n_\pm(\tilde{A}_{V_\Gamma})$. Combining these equalities, we obtain (27).

COROLLARY 4. If the self-adjoint extensions $\tilde{A}_1$ and $\tilde{A}_2$ of an operator are transversal, then there exists a SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ for which $\tilde{A}_i = \ker \Gamma_i$.

Proof. We consider some SBV $\{\mathcal{H}, \Gamma_1', \Gamma_2'\}$ for which $\tilde{A}_2 = \ker \Gamma_2'$. Since $\tilde{A}_1$ is transversal to $\tilde{A}_2$, it follows, according to Proposition 2, that $\tilde{A}_1 = \ker(\Gamma_2' - B\Gamma_2')$, $B \in [\mathcal{H}]$. Setting $\Gamma_2 \doteq \Gamma_2'$, $\Gamma_1 \doteq \Gamma_1' - B\Gamma_2'$ we obtain the desired SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$.

Remark 1. Other approaches to the definition of a SBV for a nondensely defined Hermitian operator can be found in [9-11]. The case $\overline{\mathcal{D}(A)} = \mathfrak{H}$ is discussed in detail in [12].

3. The Weyl Function. 1. Assume, as before, that $\hat{\mathfrak{N}}_\lambda = \{\hat{f}_\lambda = \{\hat{f}_\lambda, \lambda f_\lambda\}; f_\lambda \in \mathfrak{N}_\lambda\}$, π_1 is the orthoprojection in $\hat{\mathfrak{N}}_\lambda$ onto $\mathfrak{N}_\lambda \oplus 0$, $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{H}_0$.

LEMMA 2. Let $\tilde{A} = \tilde{A}^*$ be an extension of operator A, let $\tilde{A}'$ be its operator part, $\mathfrak{N}_\lambda(\tilde{A}) \doteq \mathfrak{N}_\lambda \cap \mathcal{D}(\tilde{A})$, $\mathfrak{N}'(\tilde{A}) \doteq \tilde{A}(0)$, $\mathfrak{N}''(\tilde{A}) = \mathfrak{N} \ominus \mathfrak{N}'(\tilde{A})$,

$$U_{\zeta\lambda} \doteq I + (\lambda - \zeta)(\tilde{A} - \lambda)^{-1},\ \lambda, \zeta \in \rho(\tilde{A}). \tag{28}$$

Then the following assertions hold:

1) $U_{\zeta\lambda}\mathfrak{N}_\zeta = \mathfrak{N}_\lambda$, $U_{\zeta\lambda} \restriction \mathfrak{N}'(\tilde{A}) = I$ (i.e., $U_{\zeta\lambda}n = n \ \forall n \in \mathfrak{N}''(\tilde{A})$);

2) $\mathfrak{N}_\lambda(\tilde{A}) = (\tilde{A}' - \lambda)^{-1}\mathfrak{N}''(\tilde{A}) = (\tilde{A} - \lambda)^{-1}\mathfrak{N}''(\tilde{A}) = (\tilde{A} - \lambda)^{-1}\mathfrak{N}$; (29)

3) $U_{\zeta\lambda}\mathfrak{N}''_\xi(\tilde{A}) = \mathfrak{N}''_\lambda(\tilde{A}) \ \forall \lambda, \zeta \in \rho(\tilde{A})$. (30)

Proof. Let $f_\zeta \in \mathfrak{N}_\zeta$. Then $\forall f_A \in \mathcal{D}(A)$, we have:

$$((A - \bar{\lambda})f_A, U_{\zeta\lambda}f_\zeta) = ((A - \bar{\lambda})f_A, f_\zeta + (\lambda - \zeta)(\tilde{A} - \lambda)^{-1}f_\zeta) = ((A - \bar{\lambda})f_A, f_\zeta) + \\ + (\lambda - \bar{\zeta})(f_A, f_\zeta) = ((A - \bar{\zeta})f_A, f_\zeta) = 0.$$

Consequently, $U_{\zeta\lambda}\mathfrak{N}_\zeta \subset \mathfrak{N}_\lambda$ and $U_{\lambda\zeta}\mathfrak{N}_\lambda \subset \mathfrak{N}_\zeta$. Since for all $\lambda, \zeta \in \rho(\tilde{A})$ the operators $U_{\zeta\lambda}$ are invertible and $(U_{\zeta\lambda})^{-1} = U_{\lambda\zeta}$, it follows that the first assertion is proved, while the equality $U_{\zeta\lambda} \restriction \mathfrak{N}'(\tilde{A}) = I$ follows from the fact that $(\tilde{A} - \lambda)^{-1}n = 0 \ \forall n \in \mathfrak{N}'(\tilde{A})$.

Further, $(\tilde{A} - \lambda)^{-1}\mathfrak{N}'(\tilde{A}) = 0$, while $\forall h \in \mathfrak{N}''(\tilde{A})$ and $\forall f_A \in \mathcal{D}(A)$, we have

$$((\tilde{A}' - \lambda)^{-1}h, (A - \bar{\lambda})f_A) = ((\tilde{A} - \lambda)^{-1}h, (A - \bar{\lambda})f_A) = (h, f_A) = 0, \tag{31}$$

i.e., $(\tilde{A}' - \lambda)^{-1}\mathfrak{N}''(\tilde{A}) \subset \mathfrak{N}''_\lambda(\tilde{A})$. On the other hand, $\mathcal{D}(\tilde{A}) = (\tilde{A}' - \lambda)^{-1}\mathfrak{H}$. Therefore, by virtue of (31) $(\tilde{A}' - \lambda)^{-1}h \in \mathfrak{N}_\lambda$, only if $h \perp \mathcal{D}(A)$, i.e., $h \in \mathfrak{N}$.

Finally, the third assertion follows from the equalities

$$U_{\zeta\lambda}\mathfrak{N}''_\zeta(A) = [I + (\lambda-\zeta)(\tilde{A}-\lambda)^{-1}](\tilde{A}'-\zeta)^{-1}\mathfrak{N}''(\tilde{A}) = [(\tilde{A}'-\zeta)^{-1} + (\lambda - \zeta)(\tilde{A}'-\lambda)^{-1}(\tilde{A}'-\zeta)^{-1}]\mathfrak{N}''(\tilde{A}) = (\tilde{A}'-\lambda)^{-1}\mathfrak{N}''(\tilde{A}) = \mathfrak{N}''_\lambda(\tilde{A}).$$

The lemma is proved.

LEMMA 3. Let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a SBV of the relation A^*. Then:

1) for all $\lambda \in \rho(\tilde{A}_2)$ the operator-valued functions

$$\hat{\gamma}(\lambda) = (\Gamma_2 \restriction \hat{\mathfrak{N}}_\lambda)^{-1}, \gamma(\lambda) = \pi_1\hat{\gamma}(\lambda), \tag{32}$$

are well defined, are holomorphic in $\rho(\tilde{A}_2)$, assume values in $[\mathcal{H}, \hat{\mathfrak{N}}_\lambda]$ and $[\mathcal{H}, \mathfrak{N}_\lambda]$ respectively, and $\exists\, \gamma(\lambda)^{-1} \in [\mathfrak{N}_\lambda, \mathcal{H}]$;

2) the function $\gamma(\lambda)$ is the γ-field of the extension $\tilde{A}_2$, i.e.,

$$\gamma(\lambda) = U_{\zeta\lambda}\gamma(\zeta) = \gamma(\zeta) + (\lambda-\zeta)(A_2-\lambda)^{-1}\gamma(\zeta) \quad \forall \lambda, \zeta \in \rho(\tilde{A}_2). \tag{33}$$

Proof. 1) For each $\lambda \in \rho(\tilde{A}_2)$ we have the direct decomposition

$$A^* = \tilde{A}_2 \dotplus \hat{\mathfrak{N}}_\lambda. \tag{34}$$

Indeed, in terms of the vector $\{f, f'\} \in A^*$ we define a vector $g \in \mathcal{D}(\tilde{A}_2)$ by the equality $g = (A_2 - \lambda)^{-1}(f' - \lambda f)$. Then $\forall f_A \in \mathcal{D}(A)$ we have

$$((A-\bar{\lambda})f_A, f - g) = ((A-\bar{\lambda})f_A, f) - ((A-\bar{\lambda})f_A, (\tilde{A}_2-\lambda)^{-1}(f'-\lambda f)) = ((A-\bar{\lambda})f_A, f) - (f_A, f'-\lambda f) = (Af_A, f) - (f_A, f') = 0,$$

i.e., $f_\lambda \doteq f - g \in \mathfrak{N}_\lambda$. Therefore, if $g' \doteq [I + \lambda(\tilde{A}_2-\lambda)^{-1}](f'-\lambda f)$, then

$$\{f, f'\} = \{g, g'\} + \{f_\lambda, \lambda f_\lambda\} \in \tilde{A}_2 + \hat{\mathfrak{N}}_\lambda. \tag{35}$$

Equality (35) proves the decomposition (34) since the uniqueness in (34) and (35) is obvious (although, in general, $\mathcal{D}(A^*) = \mathcal{D}(\tilde{A}_2) + \mathfrak{N}_\lambda$ is not a direct decomposition).

By virtue of (34) we have $(\Gamma_2 \restriction \hat{\mathfrak{N}}_\lambda) = \{0\}$ and $\Gamma_2 A^* = \Gamma_2\hat{\mathfrak{N}}_\lambda = \mathcal{H}$. Consequently, Γ_2 maps $\hat{\mathfrak{N}}_\lambda$ isomorphically onto $\mathcal{H}$ and the operator-valued functions $\hat{\gamma}(\lambda)$ and $\gamma(\lambda)$ of the form (32) are well defined.

2) Since $\hat{\gamma}(\zeta)$ is an isomorphism from $\mathcal{H}$ onto $\hat{\mathfrak{N}}_\zeta$, it follows that for each $\hat{f}_\zeta = \{f_\zeta, \zeta f_\zeta\} \in \hat{\mathfrak{N}}_\zeta$ there exists $h \in \mathcal{H}$ such that $\hat{f}_\zeta = \hat{\gamma}(\zeta)h$. Therefore, setting $\tilde{U}_{\zeta\lambda} = U_{\zeta\lambda} \oplus U_{\zeta\lambda}$, from Lemma 2 we obtain

$$\Gamma_2(\tilde{U}_{\zeta\lambda}\hat{f}_\zeta) = \Gamma_2\begin{pmatrix} U_{\zeta\lambda}f_\zeta \\ \lambda U_{\zeta\lambda}f_\zeta \end{pmatrix} = \Gamma_2\begin{pmatrix} f_\zeta \\ \zeta f_\zeta \end{pmatrix} + (\lambda-\zeta)\,\Gamma_2\begin{pmatrix} (\hat{A}_2-\lambda)^{-1}f_\zeta \\ f_\zeta + \lambda(\hat{A}_2-\lambda)^{-1}f_\zeta \end{pmatrix} = \Gamma_2\hat{f}_\zeta = \Gamma_2\hat{\gamma}(\zeta)\,h = h. \tag{36}$$

Since $U_{\zeta\lambda}\hat{f}_\zeta \in \hat{\mathfrak{N}}_\lambda$, $\Gamma_2\hat{\gamma}(\lambda)\,h = h$, while Γ_2 is an isomorphism from $\hat{\mathfrak{N}}_\lambda$ onto $\mathcal{H}$, the equality $\hat{\gamma}(\lambda) = \tilde{U}_{\zeta\lambda}\hat{\gamma}(\zeta)$, following from (36), proves (33).

Definition 7. The operator-valued functions $M(\lambda)$ and $C(\lambda)$, defined by the equalities

$$M(\lambda)\,\Gamma_2\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda, \ (\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \hat{\mathfrak{N}}_\lambda, \lambda \in \rho(\tilde{A}_2)), \tag{37}$$

$$C(\lambda)\,(\Gamma_1 + i\,\Gamma_2)\hat{f}_\lambda = (\Gamma_1 - i\,\Gamma_2)\hat{f}_\lambda, \lambda \in \mathbb{C}_+, \tag{38}$$

are called the Weyl function and the characteristic function of the Hermitian operator A, corresponding to the SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$.

Definition 8 [7]. An operator-valued function $Q(z)$ with values in $[\mathcal{H}]$ is called a Q-function of the Hermitian operator A, belonging to its self-adjoint extension $\tilde{A}$, if

$$Q(z) - Q^*(\zeta) = (z - \bar{\zeta})\,\gamma^*(\zeta)\,\gamma(z) \quad \forall z, \zeta \in \rho(\tilde{A}_z). \tag{39}$$

Here $\gamma(z)$ is the γ-field of the extension $\tilde{A} = \tilde{A}^*$.

Proposition 4. The operator-valued functions M(z) and C(z), corresponding to the SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$, are well defined, are holomorphic in $\rho(\tilde{A}_2)$, and assume values in $[\mathcal{H}]$. In addition, M(z) is a Q-function of the operator A, corresponding to the extension $\tilde{A}_2$ (= ker Γ_2).

Proof. The fact that M(z) in $\rho(\tilde{A}_2)$ is well defined and holomorphic follows from the equality

$$M(z) = \Gamma_1 \hat{\gamma}(z) \ \ \forall z \in \rho(\tilde{A}_2), \tag{40}$$

which, taking into account (32), is equivalent to the definition (37).

Now we apply Green's formula to the vectors $\hat{f}_z \in \hat{\mathfrak{N}}_z, \hat{f}_\zeta \in \hat{\mathfrak{N}}_\zeta$

$$(z-\bar{\zeta})(f_z, f_\zeta) = (\Gamma_1\hat{f}_z, \Gamma_2\hat{f}_\zeta)_{\mathcal{H}} - (\Gamma_2\hat{f}_z, \Gamma_1\hat{f}_\zeta)_{\mathcal{H}} = (M(z)\Gamma_2\hat{f}_z, \Gamma_2\hat{f}_\zeta)_{\mathcal{H}} - \\ -(\Gamma_2\hat{f}_z, M(\zeta)\Gamma_2\hat{f}_\zeta)_{\mathcal{H}} = ((M(z)-M^*(\zeta))\Gamma_2\hat{f}_z, \Gamma_2\hat{f}_\zeta)_{\mathcal{H}} \tag{41}$$

According to the definition of the functions $\hat{\gamma}(z)$ and $\gamma(z)$, we have the equalities

$$\hat{\gamma}(z)\Gamma_2\hat{f}_z = \hat{f}_z, \ \gamma(z)\Gamma_2\hat{f}_z = f_z, \ \gamma(\zeta)\Gamma_2\hat{f}_\zeta = f_\zeta. \tag{42}$$

Therefore, setting $\Gamma_2\hat{f}_z = h_1$, $\Gamma_2\hat{f}_\zeta = h_2$, from (41), (42) we obtain

$$((M(z)-M^*(\zeta))h_1, h_2) = (z-\bar{\zeta})(\gamma(z)h_1, \gamma(\zeta)h_2) = (z-\bar{\zeta})(\gamma^*(\zeta)\gamma(z)h_1, h_2). \tag{43}$$

Since $\Gamma_2\hat{\mathfrak{N}}_z = \Gamma_2\hat{\mathfrak{N}}_\zeta = \mathcal{H}$, equality (39) is proved.

From (39) for $\zeta = z$ we conclude that $[M(z) - M^*(z)]/(z - \bar{z}) \geq 0$. Therefore, $-i \in \rho(M(z))$ and, by virtue of (37), (38), we have

$$C(\lambda) = (M(\lambda) - i)(M(\lambda) + i)^{-1}, \ \lambda \in \mathbb{C}_+. \tag{44}$$

From (44) it is obvious that $C(\lambda)$ is well defined, it is holomorphic in $\mathbb{C}_+$, and it is a contraction in $\mathbb{C}_+$. The proposition is proved.

COROLLARY 5. The Weyl function $M(\lambda) = M^*(\bar{\lambda})$ and the characteristic function $C(\lambda)$, corresponding to the SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$, have the following properties:

1) $M(\lambda) \in (R)_{\mathcal{H}} (\Leftrightarrow \operatorname{Im}\lambda \operatorname{Im} M(\lambda) > 0 \ \forall\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-)$;

2) $0 \in \rho(\operatorname{Im} M(\lambda)) \ \forall\lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \ (\operatorname{Im} M(\lambda) \doteq \dfrac{M(\lambda) - M^*(\lambda)}{2i})$;

3) $\|C(\lambda)\| < 1 \ \forall\lambda \in \mathbb{C}_+$.

COROLLARY 6. Two simple Hermitian operators A′ and A″ with, in general, nondense domains of definition are isometrically equivalent if and only if, for some choice of SBV $\{\mathcal{H}', \Gamma_1', \Gamma_2'\}$ and $\{\mathcal{H}'', \Gamma'', \Gamma_2''\}$ of the relations $(A')^*$ and $(A'')^*$, the corresponding Weyl functions $M'(\lambda)$ and $M''(\lambda)$ or the characteristic functions $C'(\lambda)$ and $C''(\lambda)$ coincide. In this case the extensions $\tilde{A}_i' = \ker \Gamma_i'$ and $\tilde{A}_i'' = \ker \Gamma_i''$ are also unitarily equivalent.

LEMMA 4. Let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a SBV of the relation A^*, and let $\hat{\mathfrak{N}}_\lambda''(\tilde{A}) = \{\hat{f}_\lambda; f_\lambda \in \mathfrak{N}_\lambda''(\tilde{A})\}$. Then the following relations hold:

$$\gamma^*(\bar{\lambda}) = \Gamma_1\{(\tilde{A}_2 - \lambda)^{-1}, I + \lambda(\tilde{A}_2 - \lambda)^{-1}\}; \tag{45}$$

$$\ker(\Gamma_i \upharpoonright \hat{\mathfrak{N}}) = \{\{0, f\}, f \in A_i(0)\}, i = 1, 2; \tag{46}$$

$$\Gamma_i\hat{\mathfrak{N}}_\lambda''(\tilde{A}_i) = \Gamma_i\hat{\mathfrak{N}}(\tilde{A}_i) = \mathcal{H}_i (\doteq \Gamma_i\hat{\mathfrak{N}}); \tag{47}$$

$$\hat{\gamma}(\lambda)\mathcal{H}_i = \hat{\mathfrak{N}}_\lambda''(\tilde{A}_i), \ \gamma(\lambda)\mathcal{H}_i = \mathfrak{N}_\lambda''(\tilde{A}_i). \tag{48}$$

Proof. From (33) there follows the equality

$$\hat{\gamma}(z) - \hat{\gamma}(\zeta) = (z-\zeta)\{(\tilde{A}_2 - z)^{-1}, I + z(\tilde{A}_2 - z)^{-1}\}\gamma(\zeta). \tag{49}$$

Applying the operator Γ_1 to (49) and taking into account (40), we obtain

$$M(z)-M(\zeta)=\Gamma_1(\hat{\gamma}(z)-\hat{\gamma}(\zeta))=(z-\zeta)\Gamma_1\{(\tilde{A}_2-z)^{-1},\ I+z(\tilde{A}_2-z)^{-1}\}\gamma(\zeta). \tag{50}$$

The Weyl function M(z), being a Q-function, satisfies the identity (39), which, combined with (50), yields (45). We mention also that equality (45) can be obtained directly from the Green formula.

Further, equality (46) is obvious, while (47) follows from relation (29) and the identity

$$\{(\tilde{A}_i-\lambda)^{-1}h,\lambda(\tilde{A}_i-\lambda)^{-1}h\}=\{0,-h\}+\{(\tilde{A}_i-\lambda)^{-1}h,h+\lambda(\tilde{A}_i-\lambda)^{-1}h\}, \tag{51}$$

in which $\{(\tilde{A}_i-\lambda)^{-1}h,h+\lambda(\tilde{A}_i-\lambda)^{-1}h\}\in\tilde{A}_i=\ker\Gamma_i$, $i=1,2$.

Relations (48) follow from the equalities (47) and (32). The lemma is proved.

2. We characterize the exclusion relation V_Γ in terms of the limiting values of the Weyl function. For this we introduce the operator

$$V''_\Gamma \dotplus \Gamma\hat{\mathfrak{N}}''(\tilde{A}_2)=\{\Gamma_2,\Gamma_1\}\hat{\mathfrak{N}}''(\tilde{A}_2),\ \hat{\mathfrak{N}}''(\tilde{A}_2)=\{0,\mathfrak{N}''(\tilde{A}_2)\}. \tag{52}$$

The fact that the relation V''_Γ from (52) is indeed an operator follows from the relations

$$\begin{gathered}\mathfrak{N}=\mathfrak{N}'(\tilde{A}_2)\oplus\mathfrak{N}''(\tilde{A}_2),\\ V_\Gamma=\Gamma\hat{\mathfrak{N}}=\Gamma\{0,\tilde{A}_2(0)\}\dotplus\Gamma\hat{\mathfrak{N}}''(\tilde{A}_2)=\{0,V_\Gamma(0)\}\dotplus V''_\Gamma.\end{gathered} \tag{52'}$$

In general, the operator V''_Γ does not coincide with the operator part V'_Γ of the relation V_Γ, but $V''_\Gamma=V'_\Gamma=V_\Gamma$ for $V_\Gamma(0)=\{0\}$.

THEOREM 1. Let $\{\mathcal{H},\Gamma_1{}^*,\Gamma_2\}$ be a SBV of the relation A^*, and let $\mathcal{H}_i\dotplus\Gamma_i\hat{\mathfrak{N}}$, $i=1,2$. Then

1) $h\in\mathcal{H}_2=\mathcal{D}(V_\Gamma)\Leftrightarrow\lim_{y\uparrow\infty}y\,\mathrm{Im}\,(M(iy)h,h)<\infty\Leftrightarrow\lim_{y\uparrow\infty}y\,\mathrm{Im}\,(M(-iy)h,h)<\infty$; (53)

2) for each $h\in\mathcal{H}_2$ there exist the strong limits

$$M(i\infty)\dotplus s-\lim_{y\uparrow\infty}M(iy)h=s-\lim_{y\uparrow\infty}M(-iy)h\ \ u\ \ M(i\infty)h=V''_\Gamma h, \tag{54}$$

and, moreover, $M(i\infty)h=V_\Gamma h$, if V_Γ is an operator (i.e., $V_\Gamma(0)=\{0\}$);

3) for each $h\in\mathcal{H}\ominus V_\Gamma(0)$ we have the equality

$$s-\lim_{y\uparrow\infty}\frac{M(iy)}{iy}h=0; \tag{55}$$

4) for each $h\in\bar{V}_\Gamma(0)\setminus\{0\}$ there exists the strong limit

$$B_Mh\dotplus s-\lim_{y\uparrow\infty}\frac{M(iy)}{iy}h=\gamma^*(\lambda)P_{A_2(0)}\gamma(\lambda)h\neq 0\ \ (\forall\lambda\in\mathbb{C}_+\cup\mathbb{C}_-), \tag{56}$$

and, moreover, $\mathfrak{R}(B_M)\subset V_\Gamma(0)$, $\overline{\mathfrak{R}(B_M)}=\overline{V_\Gamma(0)}$.

Proof. 1) Let E_t be the resolution of the identity for the operator part $\tilde{A}'_2$ of the relation $\tilde{A}_2=\ker\Gamma_2$. From (39) we have

$$y\left(\frac{M(iy)-M(iy)^*}{2i}h,h\right)=y^2(\gamma^*(iy)\,\gamma(iy)\,h,h)=y^2\|\gamma(iy)h\|^2. \tag{57}$$

According to (33),

$$\gamma(iy)h=[I+i(y-1)(\tilde{A}_2-iy)^{-1}]\gamma(i)h=P_{A_2(0)}\gamma(i)h+(\tilde{A}'_2-i)(\tilde{A}'_2-iy)^{-1}\gamma(i)h. \tag{58}$$

From the relations (57), (58) we derive

$$y\,\mathrm{Im}\,(M(iy)h,h)=y^2\|P_{A_2(0)}\gamma(i)h\|^2+y^2\int_{-\infty}^{\infty}\frac{t^2+1}{t^2+y^2}\,d(E_t\gamma(i)h,\gamma(i)h). \tag{59}$$

The condition $P_{A_2(0)}\gamma(i)h=0$ is necessary for the uniform boundedness for each $y>0$ of the right-hand side in (59).

Assuming that this is satisfied, from (59) and the Lebesgue monotone convergence theorem we conclude that

$$\lim_{y\uparrow\infty} y^2 \|\gamma(iy)h\|^2 = \int_{-\infty}^{\infty}(t^2+1)\, d(E_t\gamma(i)h, \gamma(i)h) = \left\|\tilde{A}_2'\gamma(i)h\right\|^2 + \|\gamma(i)h\|^2. \tag{60}$$

Thus, the left-hand side in (60) is finite exactly when $\gamma(i)h \in \mathcal{D}(\tilde{A}_2) = \mathcal{D}(\tilde{A}_2')$, i.e., $\gamma(i)h \in \mathfrak{N}_i''(\tilde{A}_2)$ $(= \mathcal{D}(\tilde{A}_2) \cap \mathfrak{N}_i)$. The equivalence (53) follows now from the relation $\mathfrak{N}_i''(\tilde{A}_2) = \gamma(i)\mathcal{H}_2$ (see (48)).

2) Let $h_2 \in \mathcal{H}_2$. Then there exists a unique vector $n \in \mathfrak{N}''(\tilde{A}_2)$ such that $h_2 = \Gamma_2\{0, -n\}$. By virtue of (52), $h_1 \mathrel{\dot{=}} \Gamma_1\{0, -n\} = V_\Gamma'' h_2$. Therefore, from the definition (37) of the Weyl function and from (47), (51) we obtain

$$M(\lambda)h_2 = \Gamma_1\hat{\gamma}(\lambda)\Gamma_2\{0,-n\} = \Gamma_1\hat{\gamma}(\lambda)\Gamma_2\{(\tilde{A}_2-\lambda)^{-1}n, \lambda(\tilde{A}_2-\lambda)^{-1}n\} = \Gamma_1\{(\tilde{A}_2-\lambda)^{-1}n, \lambda(\tilde{A}_2-\lambda)^{-1}n\}.$$

Consequently, for each $h_2 \in \mathcal{H}_2$ there exists the strong limit

$$\begin{aligned} s-\lim_{y\uparrow\infty} M(iy)h_2 &= \lim_{y\uparrow\infty}\Gamma_1\{(\tilde{A}_2-iy)^{-1}n, iy(\tilde{A}_2-iy)^{-1}n\} = \\ &= \Gamma_1\{0,-n\} = h_1 = V_\Gamma'' h_2. \end{aligned} \tag{61}$$

Equality (61) proves (54) and statement 2.

3, 4) From Lemma 4 we obtain the following equivalence:

$$h \in \mathcal{H}_1 \ominus V_\Gamma(0) \Leftrightarrow \gamma(\lambda)h \perp A_2(0) \quad \forall\lambda \in \rho(\tilde{A}_2). \tag{62}$$

Indeed, by virtue of (45), we have $\Gamma_1\{0, f\} = \gamma^*(\lambda)f$ $\forall f \in A_2(0)$, $\lambda \in \rho(\tilde{A}_2)$. Since $V_\Gamma(0) = \{\Gamma_1\{0, f\};\ f \in A_2(0)\}$, we have $h \perp V_\Gamma(0) \Leftrightarrow h \perp \gamma^*(\lambda)A_2(0) \Leftrightarrow \gamma(\lambda)h \perp A_2(0)$. Besides, the equivalence (62) follows easily from the Green formula (16) for $\hat{f} = \hat{f}_\lambda = \{\gamma(\lambda)h, \lambda\gamma(\lambda)h\}$ and $\hat{g} = \hat{n} = \{0, n\} \in \tilde{A}_2$.

Further, from the identities (39) and (33) we find

$$\frac{M(\lambda)}{\lambda}h = \frac{M^*(i)}{\lambda}h + \left(1+\frac{i}{\lambda}\right)\gamma^*(i)\gamma(\lambda)h = \left(1+\frac{i}{\lambda}\right)\gamma^*(i)\left\{P_{A_2(0)}\gamma(i)h + \left[I+(\lambda-i)(\tilde{A}_2'-\lambda)^{-1}\right]\gamma(i)h\right\}. \tag{63}$$

Taking in (63) the limit as $\lambda = iy \to i\infty$ and taking into account (45), we obtain

$$s-\lim_{y\uparrow\infty}\frac{M(iy)}{iy}h = \gamma^*(i)P_{A_2(0)}\gamma(i)h = \Gamma_1\left\{0, P_{A_2(0)}\gamma(i)h\right\}. \tag{64}$$

Relations (55), (56) as well as the equality $B_M = \gamma^*(i)P_{A_2(0)}\gamma(i)$ and the inclusion $\mathfrak{R}(B_M) \subset V_\Gamma(0)$ follow from (64) and (62). The theorem is proved.

Remark 2. The existence of the strong limit in (54) (but without the connection with V_Γ) for vectors h, satisfying condition (53), is valid for any R-function and follows from its integral representation

$$M(z) = A + Bz + \int_{-\infty}^{\infty}\frac{1+tz}{t-z}dF(t),\ A = A^*,\ B \geq 0,\ \int_{-\infty}^{\infty} dF(t) \in [\mathcal{H}]. \tag{65}$$

Indeed,

$$y\,\mathrm{Im}\,(M(iy)h, h) = y^2\int_{-\infty}^{\infty}\frac{1+t^2}{t^2+y^2}d(F(t)h,h) + y^2(Bh,h) \tag{66}$$

and, consequently, the limit in (66) is finite exactly when $h \in \ker B \cap \Sigma_2(F)$, where $\Sigma_2(F) = \left\{h \in \mathcal{H};\ \int_{-\infty}^{\infty} t^2\, d(F(t)h, h) < \infty\right\}$. In this case

$$M(i\infty)h \mathrel{\dot{=}} s-\lim_{y\uparrow\infty} M(iy)h = s-\lim_{y\uparrow\infty}\int_{-\infty}^{\infty}\frac{ity+1}{t-iy}dF(t)h = -\int_{-\infty}^{\infty}dF(t)h. \tag{67}$$

Remark 3. Statements 3 and 4 of Theorem 1 can be derived also from (53), (54) by passing to another SBV $\{\mathcal{H}, \Gamma_1', \Gamma_2'\}$, where $\Gamma_1' = -\Gamma_2$, $\Gamma_2' = \Gamma_1$, or from Lemma 4.

We note that $\dim\mathcal{H}_2 + \dim V_\Gamma(0) = \dim\mathfrak{N}$, and the equality (55) is satisfied for each $h \in \mathcal{H}$ exactly when $\tilde{A}_{22}$ is an operator. In this case we have the equivalence

$$\overline{\mathcal{D}(A)} = \mathfrak{H} \Leftrightarrow \lim_{y \uparrow \infty} y \operatorname{Im}(M(iy)h, h) = \infty \quad \forall h \in \mathcal{H} \setminus \{0\}. \tag{68}$$

Further, the relation (56), equivalent to the presence of the linear term $B_M z \dotplus Bz$ in (65), characterizes extensions $\tilde{A}_2$ that are not operators (i.e., $A_2(0) \neq \{0\}$). Due to condition (56), the Weyl functions of the extensions $\tilde{A}_2 \in \tilde{\mathfrak{C}}(\mathfrak{H}) \setminus \mathfrak{C}(\mathfrak{H})$ play an essential role in B. S. Pavlov's and his collaborators' investigation [13].

Remark 4. The limits along the imaginary axis in (54)-(56) can be replaced by the angular limits for $\lambda \to \infty$, $\lambda \in \Pi_\varepsilon = \{\lambda \in \mathbb{C}_+;\ 0 < \varepsilon < \arg\lambda < \pi - \varepsilon\}$.

3. We characterize the angular boundary values of the Weyl function for $\lambda \to a = \bar{a}$. For this we introduce the notations

$$\mathfrak{N}_a = \mathfrak{M}_a^{\perp}, \hat{\mathfrak{N}}_a = \{\mathfrak{N}_a, a\mathfrak{N}_a\},$$
$$\hat{\mathfrak{N}}'_a(\tilde{A}_2) = \tilde{A}_2 \cap \hat{\mathfrak{N}}_a, \hat{\mathfrak{N}}''_a(\tilde{A}_2) = \hat{\mathfrak{N}}_a \ominus \hat{\mathfrak{N}}''_a(\tilde{A}_2) \tag{69}$$

and we define the Hermitian relation V_a and the operator V''_a (compare with (52)):

$$V_a \dotplus \Gamma\hat{\mathfrak{N}}_a = \{\{\Gamma_2\hat{f}_a, \Gamma_1\hat{f}_a\};\ \hat{f}_a \in \hat{\mathfrak{N}}_a\},$$
$$V''_a = \Gamma\hat{\mathfrak{N}}''_a = \{\{\Gamma_2\hat{f}_a, \Gamma_1\hat{f}_a\};\ \hat{f}_a \in \hat{\mathfrak{N}}''_a\}. \tag{70}$$

From (70) we obtain relations that are similar to the relations (52):

$$V_a = \Gamma\hat{\mathfrak{N}}_a = \Gamma(\hat{\mathfrak{N}}'_a(\tilde{A}_2) \oplus \hat{\mathfrak{N}}''_a(\tilde{A}_2)) = \Gamma\hat{\mathfrak{N}}'_a(\tilde{A}_2) + \Gamma\hat{\mathfrak{N}}''_a(\tilde{A}_2) = \Gamma\hat{\mathfrak{N}}'_a(\tilde{A}_2) + V''_a. \tag{70'}$$

Clearly, $\Gamma\hat{\mathfrak{N}}'_a(\tilde{A})$ is the indeterminate part of the relation V_a, while V''_a is an operator which, in general, does not coincide with the operator part V'_a of the relation V_a ($V''_a = V'_a = V_a$ if $V_a(0) = \{0\}$). In addition,

$$\tilde{A}_a \dotplus A + \hat{\mathfrak{N}}_a = \tilde{A}_{V_a} \quad (\hat{\mathfrak{N}}_a = \{\hat{f} \dotplus \{f_a, af_a\}; f_a \in \mathfrak{N}_a\}). \tag{71}$$

The following statement is obtained from the Hermitian property of the operator $\tilde{A}_a$ of the form (71) and it is well known (see [3]) in the case $\overline{\mathcal{D}(A)} = \mathfrak{H}$.

Statement 4. Let A be a Hermitian operator in $\mathfrak{H}$, $\overline{\mathcal{D}(A)} = \mathfrak{H}_0 \subset \mathfrak{H}$, $P_0 \dotplus P_{\mathfrak{H}_0}$, $n_+(A) = n_-(A) = m < \infty$ and $a = \bar{a} \notin \sigma_p(A)$. Then

$$\dim\ker(A^*_{0p} - aP_0) = \operatorname{codim}[(A - a)\,\mathcal{D}(A)] \le m.$$

Proposition 5. Let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a SBV of the relation A^*. Then

1) $h \in \mathcal{D}(V_a) \Leftrightarrow \lim_{y \downarrow 0} \frac{1}{y} \operatorname{Im}(M(a+iy)h, h) < \infty \Leftrightarrow \lim_{y \downarrow 0} \frac{1}{y} \operatorname{Im}(M(a-iy)h, h) < \infty$; (72)

2) for each $h \in \mathcal{D}(V_a)$ there exist the strong limits

$$M(a)h \dotplus s - \lim_{y \downarrow 0} M(a + iy)h = s - \lim_{y \downarrow 0} M(a - iy)h = V''_a h, \tag{72'}$$

and, moreover, $M(a)h = V''_a h = V_a h$ if V_a is an operator (i.e., $V_a(0) = \{0\}$).

The **proof** is analogous to the proof of statements 1 and 2 of Theorem 1. However, it can be derived directly from Theorem 1. Let $(A - a)^{-1}$ and $(A^* - a)^{-1}$ be the relations that are the inverses of $A - a$ and $A^* - a$, respectively. We define a SBV $\Pi' = \{\mathcal{H}, \Gamma_1', \Gamma_2'\}$ of the relation $(A^* - a)^{-1}$ by setting

$$\Gamma'_i\{f' - af, f\} = (-1)^i \Gamma_i \hat{f}, \ ((A^* - a)^{-1} = = \{\{f' - af, f\};\ \hat{f} = \{f, f'\} \in A\}),\ i = 1, 2. \tag{73}$$

From the obvious equality

$$0 = (f' - \bar{\lambda} f, f_\lambda) = (f' - af - (\bar{\lambda} - a)f, f_\lambda) = -(\bar{\lambda} - a)\left(f - \frac{1}{\bar{\lambda} - a}(f' - af), f_\lambda\right),$$

in which $\{f, f'\} \in A$, $f_\lambda \in \mathfrak{N}_\lambda(A)$, there follows that $\mathfrak{N}_\lambda((A - a)^{-1}) = \mathfrak{N}_{1/(\lambda - a)}(A)$, where $\mathfrak{N}_\lambda((A - a)^{-1})$ and $\mathfrak{N}_\lambda(A)$ are the defect subspaces of the relations $(A - a)^{-1}$ and A. Therefore, by virtue of (73), the Weyl functions $M'(\lambda)$ and $M(\lambda)$, corresponding to the SBV $\{\mathcal{H}, \Gamma_1', \Gamma_2''\}$ and $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$, are connected by the equality

$$M'(\lambda) = \Gamma_1'\gamma(\lambda) = \Gamma_1'(\Gamma_2' \upharpoonright \hat{\mathfrak{N}}_\lambda((A-a)^{-1}))^{-1} = -\Gamma_1(\Gamma_2 \upharpoonright \mathfrak{N}_{1/(\lambda-a)}(A))^{-1} = -M\left(\frac{1}{\lambda-a}\right). \quad (74)$$

Further, the indeterminate part $\hat{\mathfrak{N}}' = \{0, \mathfrak{N}'\}$ of the relation $(A^* - a)^{-1}$ coincides with $\hat{\mathfrak{N}}_a(A)$. Therefore, the exclusion relation $V_{\Gamma'}$ and the operator $V_{\Gamma'}''$ are connected with the relation V_a and the operator V_a'' from (70) in the following manner:

$$V_{\Gamma'} = \{\Gamma_2', \Gamma_1'\}\hat{\mathfrak{N}}' = \{\Gamma_2, -\Gamma_1\}\hat{\mathfrak{N}}_a(A) = \{\{h_2, -h_1\}, h_i = \Gamma_i \hat{f}_a, \hat{f}_a \in \hat{\mathfrak{N}}_a\} = -V_a,$$
$$V_{\Gamma'}'' = \{\Gamma_2', \Gamma_1'\}\hat{\mathfrak{N}}''(\tilde{A}_2) = \{\Gamma_2, -\Gamma_1\}\hat{\mathfrak{N}}_a''(\tilde{A}_2) = -V_a''. \quad (75)$$

Now the relations (72) and (73) follow from the relations (53), (54) and (74), (75).

Remark 5. Relations (72) and (72′) in terms of the integral representation (65) of the function $M(\lambda)$ obtain the form

$$\int_{-\infty}^{\infty} \frac{1+t^2}{(t-a)^2} d(F(t)h, h) < \infty, \quad s - \lim_{y \downarrow 0} M(a+iy)h = \int_{-\infty}^{\infty} \frac{1+ta}{t-a} dF(t)h.$$

From Theorem 1 and Proposition 5 we can obtain the following characterization of the relations V_Γ and V_a (see (70)) in terms of the characteristic function $C(\lambda)$.

THEOREM 1′. Let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a SBV of the relation A^*, and let $V_\Gamma = \Gamma\hat{\mathfrak{N}}$. Then:

1) $h = h_1 + ih_2$, where $\{h_2, h_1\} \in V_\Gamma \Leftrightarrow \lim_{y \uparrow \infty}(\|h\| - \|C(iy)h\|) < \infty;$

2) for each $h = h_1 + ih_2$ ($\{h_2, h\} \in V_\Gamma + i$) there exists the strong limit

$$C(i\infty)h \doteq s - \lim_{y \uparrow \infty} C(iy)h = h - 2i(V_\Gamma + i)^{-1}h;$$

3) $h = h_1' + ih_2'$, where $\{h_2', h_1'\} \in V_a \Leftrightarrow \lim_{y \downarrow 0} \frac{\|h\| - \|C(a+iy)h\|}{y} < \infty;$

4) for each $h = h_1' + ih_2'$ ($\{h_2', h\} \in V_a + i$) there exists the strong limit

$$C(a)h \doteq s - \lim_{y \downarrow 0} C(a+iy)h = h - 2i(V_a + i)^{-1}h.$$

Proposition 6. Let $M_1(\lambda)$ and $M_2(\lambda)$ be the Weyl functions, corresponding to the SBV $\{\mathcal{H}_1, \Gamma_1', \Gamma_2'\}$ and $\{\mathcal{H}_2, \Gamma_1^2, \Gamma_2^2\}$ of the relation A^*, and let U be an isometry from $\mathcal{H}_2$ onto $\mathcal{H}_1$. Then

$$M_1(\lambda) = (X_{11}\, U\, M_2(\lambda)\, U^{-1} + X_{12})\,(X_{21}\, U\, M_2(\lambda)\, U^{-1} + X_{22})^{-1},$$

where $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ — is a J-unitary operator in $\mathcal{H}_1 \oplus \mathcal{H}_1, J = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}$.

COROLLARY 7. Suppose that, under the assumptions of Proposition 6, we have $\ker \Gamma_2' = \ker \Gamma_2^2$. Then $M_1(\lambda) = CM_2(\lambda)C^* + K$, where $C = X_{11}U$, $K = X_{12}X_{11}^* = K^*$.

The proof of Proposition 6 is analogous to the proof in [14, 15] for the case $\overline{\mathcal{D}(A)} = \mathfrak{H}$.

Remark 6. For the case $\overline{\mathcal{D}(A)} = \mathfrak{H}$ $C(\lambda)$ has been introduced in [14], and the Weyl function $M(\lambda)$ in [16, 17]. In [16] (see also [18, 19]) Lemma 3 and Proposition 4 have been proved for this case.

4. The Formula for the Generalized Resolvents.

1. **LEMMA 5.** Let $\lambda \in \hat{\rho}(A)$, $A \subset \tilde{A} \subset A^*$, and let $\tilde{A}_\lambda = A \dotplus \hat{\mathfrak{N}}_\lambda$. Then the following equivalences hold:

$$\lambda \notin \sigma_p(\tilde{A}) \Leftrightarrow \tilde{A} \text{ and } \tilde{A}_\lambda \text{ — are disjoint;} \quad (76)$$

$$\lambda \in \rho(\tilde{A}) \Leftrightarrow \tilde{A} \text{ and } \tilde{A}_\lambda \text{ — are transversal.} \quad (77)$$

In addition,

$$\ker(\tilde{A}-\lambda)=\pi_1(\tilde{A}\cap\hat{\mathfrak{N}}_\lambda),\ \dim\ker(\tilde{A}-\lambda)=\dim(\tilde{A}\cap\hat{\mathfrak{N}}_\lambda). \tag{78}$$

Proof. 1) Since $\lambda\in\hat{\rho}(A)$, there exists an extension $\tilde{A}_2=\tilde{A}_2^*$ such that $\lambda\in\rho(\tilde{A}_2)$. Let $\lambda\in\sigma_p(\tilde{A})$, i.e., $\exists\{f,\lambda f\}\in\tilde{A}$. According to (34), we have $\{f,\lambda f\}=\{g,g'\}+\{f_\lambda,\lambda f_\lambda\}$, where $\{g,g'\}\in\tilde{A}_2$, $\hat{f}_\lambda\in\hat{\mathfrak{N}}_\lambda$. From here $g'=\lambda g$ and, therefore, $g=0$ since $\lambda\notin\sigma_p(\tilde{A}_2)$. Consequently, $\{f,\lambda f\}=\hat{f}_\lambda\in\tilde{A}$. Conversely, if $\exists\{f,f'\}=\{f_A+f_\lambda, Af_A+\lambda f_\lambda\}\in\tilde{A}$, then $\hat{f}_\lambda=\{f,f'\}-\{f_A,Af_A\}\in\tilde{A}$. The equivalence (76) has been proved.

2) Let $\lambda\in\rho(\tilde{A})$ and $\{f,f'\}\in\tilde{A}$. Setting

$$g=(\tilde{A}-\lambda)^{-1}(f'-\lambda f),\ g'=[I+\lambda(\tilde{A}-\lambda)^{-1}](f'-\lambda f), \tag{79}$$

we can see that $\{g,g'\}\in\tilde{A}$ and

$$((A-\bar{\lambda})f_A,g-f)=((A-\bar{\lambda})f_A,(\tilde{A}-\lambda)^{-1}(f'-\lambda f)-f)=(f_A,f'-\lambda f)-((A-\bar{\lambda})f_A,f)=(f_A,f')-(Af_A,f)=0\ \ \forall f_A\in\mathcal{D}(A),$$

i.e., $f_\lambda\doteq g-f\in\mathfrak{N}_\lambda$. But then from (79) we obtain $g'-f'=\lambda(g-f)=\lambda f_\lambda$. Therefore, $\{f,f'\}=\{g,g'\}+\{f_\lambda,\lambda f_\lambda\}$, which, taking into account (34) and (76), proves the transversality of $\tilde{A}$ and $\tilde{A}_\lambda$.

Conversely, if $\tilde{A}$ and $\tilde{A}_\lambda$ are transversal, then for each $\{g,g'\}\in\tilde{A}_2$ we have the equality $\{f_2,f_2'\}=\{f,f'\}+\hat{f}_\lambda$, where $\{f,f'\}\in\tilde{A}$, $\hat{f}_\lambda\in\hat{\mathfrak{N}}_\lambda$. From here, $f'-\lambda f=f_2'-\lambda f_2$ and, consequently, $\mathfrak{R}(\tilde{A}-\lambda)=\mathfrak{R}(\tilde{A}_2-\lambda)=\mathfrak{H}$, since $\lambda\in\rho(\tilde{A}_2)$.

Proposition 7. Let $\{\mathcal{H},\Gamma_1,\Gamma_2\}$ be a SBV of the relation A^*, let $M(\lambda)$ be the corresponding Weyl function, let $\theta\in\tilde{\mathfrak{C}}(\mathcal{H})$ and $\lambda\in\rho(\tilde{A}_2)$. Then:

1) $\lambda\in\rho(\tilde{A}_\theta)\Leftrightarrow 0\in\rho(\theta-M(\lambda))$;

2) $\lambda\in\sigma_i(\tilde{A}_\theta)\Leftrightarrow 0\in\sigma_i(\theta-M(\lambda))$, $i=p,c,r$; (80)

3) $\dim\ker(\tilde{A}_\theta-\lambda)=\dim\ker(\theta-M(\lambda))$, $\operatorname{codim}[(\tilde{A}_\theta-\lambda)\mathcal{D}(A)]=\operatorname{codim}[(\theta-M(\lambda))\mathcal{D}(\theta)]$, and, moreover, $\hat{f}=\{f,\lambda f\}\in\tilde{A}_\theta\Leftrightarrow\Gamma_2\hat{f}\in\ker(\theta-M(\lambda))$.

Proof. Since $\tilde{A}_\lambda=\tilde{A}_{M(\lambda)}$ (i.e., $\hat{f}\in\tilde{A}_\lambda\Leftrightarrow\Gamma\hat{f}\in\operatorname{gr}M(\lambda)$), the chain of equivalences

$$\lambda\notin\sigma_p(\tilde{A}_\theta)\Leftrightarrow\tilde{A}_\theta\cap\tilde{A}_\lambda=\mathcal{D}(A)\Leftrightarrow\theta\cap\operatorname{gr}M(\lambda)=\{0\}\Leftrightarrow 0\notin\sigma_p(\theta-M(\lambda)) \tag{81}$$

following from Lemmas 1, 5, and Proposition 3, proves (80) for $i=p$. Now, statement 2 for $i=r$ is a consequence of the obvious relations

$$\lambda\in\sigma_r(\tilde{A}_\theta)\Leftrightarrow\bar{\lambda}\in\sigma_p(\tilde{A}_\theta),\ (\tilde{A}_\theta)^*=\tilde{A}_{\theta^*},\ M(\bar{\lambda})=M(\lambda)^*.$$

Now, from the same lemmas and Proposition 3 we obtain that $\lambda\in\rho(\tilde{A}_\theta)\Leftrightarrow\tilde{A}_\theta$ and $\tilde{A}_{M(\lambda)}$ are transversal $\Leftrightarrow\theta\cap\operatorname{gr}M(\lambda)=\mathcal{H}\oplus\mathcal{H}\Leftrightarrow 0\in\rho(\theta-M(\lambda))$. Since $\sigma_c(T)=\sigma(T)\setminus(\sigma_p(T)\cup\sigma_r(T))\ \ \forall T\in\tilde{\mathfrak{C}}(\mathfrak{H})$, the equivalence (80) is proved for $i=c$. Statement 3 is a consequence of the equivalences

$$\hat{f}=\{f,\lambda f\}\in\tilde{A}_\theta\Leftrightarrow\hat{f}=\hat{f}_\lambda\in\tilde{A}_\theta\cap\hat{\mathfrak{N}}_\lambda\Leftrightarrow\Gamma\hat{f}\in\theta\cap\operatorname{gr}M(\lambda)\Leftrightarrow\Gamma_2\hat{f}\in\ker(\theta-M(\lambda)).$$

2. We recall that an operator-valued function $\mathbb{R}_\lambda=P(\tilde{A}-\lambda)^{-1}\upharpoonright\mathfrak{H}$ holomorphic in $\mathbb{C}_+\cup\mathbb{C}_-$, is said to be a generalized pseudoresolvent of the operator A, written $\mathbb{R}_\lambda\in P\Omega_A$, if $\tilde{A}\in\tilde{\mathfrak{C}}(\tilde{\mathfrak{H}})$ is a self-adjoint extension of the operator A, acting in a Hilbert space $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, P being the orthoprojection in $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}$. The set of generalized resolvents $\mathbb{R}_\lambda\in P\Omega_A$, for which $\tilde{A}\in\mathfrak{C}(\tilde{\mathfrak{H}})$ (i.e., $\tilde{A}\in\tilde{A}^*$ is an operator), will be denoted by Ω_A.

In the collection of the extensions $\tilde{A}\in\tilde{A}^*\in\mathfrak{C}(\tilde{\mathfrak{H}})$, generating the resolvent $\mathbb{R}_\lambda\in P\Omega_A$, there exist minimal ones, i.e., such that $\tilde{\mathfrak{H}}$ is generated by the lineals $\mathfrak{H}$ and $(\tilde{A}-\lambda)^{-1}\mathfrak{H}$ $(\lambda\in\mathbb{C}_+\cup\mathbb{C}_-)$. Any two minimal extensions are unitarily isomorphic.

Following [20], the family of relations $T(\lambda)\in\tilde{\mathfrak{C}}(\mathfrak{H})$ is said to be holomorphic at the point λ_0 if there exist $\zeta\in\rho(T(\lambda))$ and $\varepsilon>0$ such that the resolvent $(T(\lambda)-\zeta)^{-1}$ is bounded-holomorphic [20] for $|\lambda-\lambda_0|<\varepsilon$.

THEOREM 2. Let $\{\mathcal{H},\Gamma_1,\Gamma_2\}$ be a SBV of the relation A^*, let $M(\lambda)$ be the corresponding Weyl function, and let $\lambda\in\rho(\tilde{A}_2)$, $\mathfrak{H}_0=\mathcal{D}(A)$, $\mathfrak{N}=\mathfrak{H}_0^{\perp}$. Then:

1) the equality

$$(\tilde{A}_\theta-\lambda)^{-1}=(A_2-\lambda)^{-1}+\gamma(\lambda)(\theta-M(\lambda))^{-1}\dot{\gamma}^*(\bar{\lambda}) \tag{82}$$

establishes a bijective correspondence between the resolvents of the proper extensions $\tilde{A}_\theta$ of the operator A and the closed linear relations θ in $\mathcal{H}$;

2) $\mathbb{R}_\lambda g$ is a solution of the following problem with a spectral parameter $\tau(\lambda) \in (\tilde{R})_{\mathcal{H}}$ in the boundary condition:

$$(A^*_{0p} - \lambda)f = g - n\,(\Leftrightarrow f' - \lambda f = g),\ \{\Gamma_2 \hat{f}, -\Gamma_1 \hat{f}\} \in \tau(\lambda),\ \hat{f} = \{f, f'\},\ n = P_{\mathfrak{N}} f', \tag{83}$$

i.e.,

$$\mathbb{R}_\lambda = (\tilde{A}_{-\tau(\lambda)} - \lambda)^{-1},\ \tau(\lambda) = \tau_*(\bar{\lambda}) \in (\mathbb{R})_{\mathcal{H}}\ (\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-), \tag{84}$$

where $\tilde{A}_{-\tau(\lambda)}$ is a holomorphic family of (proper) maximal accumulative extensions of the form (20);

3) the formula (for generalized resolvents)

$$\begin{aligned}\mathbb{R}_\lambda &\doteq P(\tilde{A} - \lambda)^{-1} \restriction \mathfrak{H} = (\tilde{A}_{-\tau(\lambda)} - \lambda)^{-1} = \\ &= (\tilde{A}_2 - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda) + M(\lambda))^{-1}\gamma^*(\bar{\lambda})\end{aligned} \tag{85}$$

establishes a bijective correspondence between the generalized resolvents $\mathbb{R}_\lambda \in P\Omega_A$ and the functions $\tau(\lambda) \in (\tilde{R})_{\mathcal{H}}$. Moreover, when $\tilde{A}_2$ is an operator, then $\mathbb{R}_\lambda \in \Omega_A$ if and only if $\tau(\lambda)$ is M-admissible, i.e.,

$$s - \lim_{y \uparrow \infty} y^{-1} (\tau(iy) + M(iy))^{-1} = 0. \tag{86}$$

Proof. 1) Let $f \in \mathfrak{H}$. Then

$$\begin{aligned}\hat{f}_\theta(\lambda) &\doteq \{(\tilde{A}_\theta - \lambda)^{-1}f, f + \lambda(\tilde{A}_\theta - \lambda)^{-1}f\} \in \tilde{A}_\theta, \\ \hat{f}_2(\lambda) &\doteq \{(\tilde{A}_2 - \lambda)^{-1}f, \tilde{A}_2(\tilde{A}_2 - \lambda)^{-1}f\} \in \tilde{A}_2.\end{aligned} \tag{87}$$

Since the extension $\tilde{A}_\theta$ is proper, it follows that $f_\lambda \doteq (\tilde{A}_\theta - \lambda)^{-1}f - (\tilde{A}_2 - \lambda)^{-1}f \in \mathfrak{N}_\lambda$ and, consequently, $\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} = \hat{f}_\theta(\lambda) - \hat{f}_2(\lambda) \in \hat{\mathfrak{N}}_\lambda$.

Assume first that $\theta = B \in \mathcal{C}(\mathcal{H})$. Since $\lambda \in \rho(\tilde{A}_B)$, it follows that $\exists\ (B - M(\lambda))^{-1} \in [\mathcal{H}]$. Therefore, from Lemma 4 we obtain

$$(\Gamma_1 - B\Gamma_2)(\hat{f}_\theta(\lambda) - \hat{f}_2(\lambda)) = -\Gamma_1 \hat{f}_2(\lambda) = -\gamma^*(\bar{\lambda})f, \tag{88}$$

$$(\Gamma_1 - B\Gamma_2)\hat{\gamma}(\lambda)(B - M(\lambda))^{-1}\gamma^*(\bar{\lambda})f = (M(\lambda) - B)(B - M(\lambda))^{-1}\gamma^*(\bar{\lambda})f = -\gamma^*(\bar{\lambda})f. \tag{89}$$

Formula (82) follows from combining the formulas (88) and the fact that $\Gamma_1 - B\Gamma_2$ is an isomorphism from $\hat{\mathfrak{N}}_\lambda$ onto $\mathcal{H}$, since $\lambda \in \rho(\tilde{A}_B)$.

If $\theta \in \tilde{\mathcal{C}}(\mathcal{H}) \setminus \mathcal{C}(\mathcal{H})$, then in this case from (87) and Lemma 4 we obtain

$$\{\Gamma_2, \Gamma_1\}\hat{f}_\theta(\lambda) = \{\Gamma_2, \Gamma_1\}(\hat{f}_2 + \hat{f}_\lambda) = \{\Gamma_2 \hat{f}_\lambda, \gamma^*(\bar{\lambda})f + M(\lambda)\Gamma_2 f_\lambda\}. \tag{90}$$

Since $\{\Gamma_2, \Gamma_1\}\hat{f}_\theta(\lambda) \in \theta$, from (90) $\Rightarrow \{\Gamma_2 \hat{f}_\lambda, \gamma^*(\bar{\lambda})f\} \in \theta - M(\lambda)$. According to Proposition 7, we have $0 \in \rho(\theta - M(\lambda))\ (\Leftrightarrow \lambda \in \rho(\tilde{A}_\theta))$. Therefore,

$$\Gamma_2 \hat{f}_\lambda = (\theta - M(\lambda))^{-1}\gamma^*(\bar{\lambda})f \Rightarrow \hat{f}_\lambda = \gamma(\lambda)(\theta - M(\lambda))^{-1}\gamma^*(\bar{\lambda})f. \tag{91}$$

Combining (91) and (87), we obtain formula (82).

2) Let $\mathbb{R}_\lambda = P(\tilde{A} - \lambda)^{-1} \restriction \mathfrak{H},\ \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, and let $\tilde{A}'$ be the operator part of the relation $\tilde{A}$. Since $\mathbb{R}_\lambda(A - \lambda)f = f$ ($\forall f \in \mathcal{D}(A)$), it follows that $\forall\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ the relation $T(\lambda) \doteq \mathbb{R}_\lambda^{-1} + \lambda = T(\bar{\lambda})^* = (\mathbb{R}_{\bar{\lambda}}^{-1})^* + \bar{\lambda}$ is an extension of the operator A. We show that it is maximal accumulative (dissipative) for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$). If $\{f, f'\} \in T(\lambda)$, then $\{f' - \lambda f, f\} \in (T(\lambda) - \lambda)^{-1} = \mathbb{R}_\lambda$. Therefore, if we set $f' - \lambda f = g$, $\varphi = (\tilde{A} - \lambda)^{-1}g$, then we obtain ($\beta \doteq \operatorname{Im}\lambda < 0$)

$$\begin{aligned}\operatorname{Im}(f', f) &= \operatorname{Im}(f' - \lambda f, f) + \operatorname{Im}(\lambda f, f) = \operatorname{Im}(g, \mathbb{R}_\lambda g) + \beta\|\mathbb{R}_\lambda g\|^2 = \\ &= \operatorname{Im}((\tilde{A}' - \lambda)\varphi, \varphi) + \beta\|P\varphi\|^2 = -\beta[\|\varphi\|^2 - \|P\varphi\|^2] = -\beta\|(I - P)\varphi\|^2 \le 0.\end{aligned}$$

According to statement 4 of Proposition 3, the extensions $T(\lambda)$ are proper: $T(\lambda) = \tilde{A}_{-\tau(\lambda)}$, where $\tau(\lambda) = \tau(\bar{\lambda})^*$ is a function with values in the set of maximal dissipative (for $\lambda \in \mathbb{C}_+$) relations in $\mathcal{H}$. For the proof of its holomorphy we make use of the identity, following from (82),

$$(\tau(\lambda)+M(\lambda))^{-1}=\gamma(\lambda)^{-1}[(\tilde{A}_2-\lambda)^{-1}-\mathbb{R}_\lambda]\gamma^*(\bar{\lambda})^{-1}. \tag{92}$$

From (92) we conclude that $\tau(\lambda)+\mathrm{M}(\lambda)$ is holomorphic in $\mathbb{C}_+\cup\mathbb{C}_-$ ($0\in\rho(\tau(\lambda)+\mathrm{M}(\lambda))\ \forall\lambda\in\mathbb{C}_+\cup\mathbb{C}_-$). Since $\mathrm{M}(\lambda)$ is bounded-holomorphic in $\mathbb{C}_+\cup\mathbb{C}_-$, it follows (see [20]) that $\tau(\lambda)$ is also holomorphic in $\mathbb{C}_+\cup\mathbb{C}_-$. Thus, $\tau(\lambda)=\tau(\bar{\lambda})^*\in(\tilde{\mathrm{R}})_{\mathcal{H}}$ and the relation (84) as well as its equivalent (83) are proved.

Conversely, if $\tau(\lambda)=\tau(\bar{\lambda})^*\in(\tilde{\mathrm{R}})_{\mathcal{H}}$, then $\tilde{\mathrm{A}}_{-\tau(\lambda)}$ is a holomorphic (by virtue of (82)) family of maximal accumulative (for $\lambda\in\mathbb{C}_+$) relations from $\tilde{c}(\mathfrak{H})$. Then $\mathfrak{H}_2\doteq\tilde{\mathrm{A}}_{-\tau(\lambda)}(0)$ does not depend on $\lambda\in\mathbb{C}_+\cup\mathbb{C}_-$ (see [21, 22]) and we have the relations

$$\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2,\ \tilde{A}_{-\tau(\lambda)}=\tilde{A}'_{-\tau(\lambda)}\oplus\hat{\mathfrak{H}}_2\ (\hat{\mathfrak{H}}_2\doteq\{0,\mathfrak{H}_2\}),$$

$$(\tilde{A}_{-\tau(\lambda)}-\lambda)^{-1}=(\tilde{A}'_{-\tau(\lambda)}-\lambda)^{-1}\oplus 0_{\mathcal{H}_2}, \tag{93}$$

in which $\tilde{\mathrm{A}}_{-\tau(\lambda)}'$ and $\hat{\mathfrak{H}}_2$ are the operator and the indeterminate parts of the relation $\tilde{\mathrm{A}}_{-\tau(\lambda)}$, while $(\tilde{A}'_{-\tau(\lambda)}-\lambda)^{-1}\in(R)_{\mathfrak{H}_1}$. From (93) we conclude that

$$s-\lim_{y\uparrow\infty} iy\mathbb{R}_{iy}f=s-\lim_{y\uparrow\infty} iy(\tilde{A}'_{-\tau(\lambda)}-iy)^{-1}f=-P_1f\quad\forall f\in\mathfrak{H}, \tag{94}$$

where P_1 is the orthoprojection in $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ onto $\mathfrak{H}_1$. According to Naimark's known lemma [3] (more precisely, its generalization to a holomorphic family of relations [21]), there exist a separable Hilbert space $\tilde{\mathfrak{H}}$ and a relation $\tilde{\mathrm{A}}$, self-adjoint in $\tilde{\mathfrak{H}}$, such that $(\tilde{\mathrm{A}}_{-\tau(\lambda)}-\lambda)^{-1}=P(\tilde{A}-\lambda)^{-1}\upharpoonright\mathfrak{H}$ ($P=P_{\mathfrak{H}}$ is the orthoprojection in $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}$). It is easy to show that $\tilde{\mathrm{A}}\supset\mathrm{A}$. Relation (84) is proved.

3) Since $\tilde{\mathrm{A}}_2$ is an operator, from the equality (45) we obtain

$$\lim_{y\uparrow\infty}(\pm iy)\gamma^*(\mp iy)f=$$
$$=\lim_{y\uparrow\infty}\Gamma_1\{(\pm iy)(\tilde{A}_2-iy)^{-1}f,(\pm iy)(\tilde{A}_2-iy)^{-1}\tilde{A}_2f\}=-\Gamma_1\{f,\tilde{A}_2f\}. \tag{95}$$

If $\tau(\lambda)\in(\tilde{\mathrm{R}})_{\mathcal{H}}$, then $G(\lambda)\doteq-(\tau(\lambda)+M(\lambda))^{-1}\in(R)_{\mathcal{H}}$ and assumes values in $[\mathcal{H}]$. From the integral representation (65) of the operator-valued function $\mathrm{G}(\lambda)$ there follows the existence of the strong limit

$$s-\lim_{\lambda\to\infty}\frac{G(\lambda)}{\lambda}=s-\lim_{\lambda\to\infty}\left[-\frac{(\tau(\lambda)+M(\lambda))^{-1}}{\lambda}\right]\doteq B_G=B_G^*,\ B_G\in[\mathcal{H}]. \tag{96}$$

From (95) and (96) we have

$$\exists\lim_{y\uparrow\infty}(iy)(\gamma(iy)(\tau(iy)+M(iy))^{-1}\gamma^*(-iy)f,f)=\lim_{y\uparrow\infty}\left(\frac{(\tau(iy)+M(iy))^{-1}}{iy}iy\gamma^*(-iy)f,(-iy)\gamma^*(iy)f\right)=-\left\|B_G\Gamma_1\{f,\tilde{A}_2,f\}\right\|^2. \tag{97}$$

Here we have used an elementary statement: if $\mathrm{f}_n\in\mathcal{H}$ and $\exists\lim_{n\to\infty}f_n=f$, while $\mathrm{T}_n\in[\mathcal{H}]$ and $\exists\ s-\lim_{n\to\infty}\mathrm{T}_n=\mathrm{T}$, then $\exists\lim_{n\to\infty}\mathrm{T}_n\mathrm{f}_n=\mathrm{Tf}$. Since $s-\lim_{y\uparrow\infty}iy(\tilde{A}_2-iy)^{-1}=-I$, from formulas (82), (97), and (94) we obtain

$$-\|P_1f\|^2=\lim_{y\uparrow\infty}iy(\mathbb{R}_{iy}f,f)=\lim_{y\uparrow\infty}iy((\tilde{A}_{-\tau(iy)}-iy)^{-1}f,f)=-\|f\|^2+\|B_G^{1/2}\Gamma_1\{f,\tilde{A}_2f\}\|^2. \tag{98}$$

According to (94), $\mathbb{R}_\lambda\in\Omega_\mathrm{A}$ (i.e., in (82) $\tilde{\mathrm{A}}\in\mathfrak{C}(\tilde{\mathfrak{H}})$) exactly when $\mathrm{P}_1=\mathrm{I}_{\mathfrak{H}}$. Therefore, by virtue of (98) we have the equivalence

$$\mathbb{R}_\lambda\in\Omega_A\Leftrightarrow s-\lim_{y\uparrow\infty}iy\mathbb{R}_{iy}=-I_{\mathfrak{H}}\Leftrightarrow B_G^{1/2}\Gamma_1\{f,\tilde{A}_2f\}=0\ \forall f\in\mathcal{D}(\tilde{A}_2). \tag{99}$$

Since the extensions $\tilde{\mathrm{A}}_1$ and $\tilde{\mathrm{A}}_2$ are transversal, it follows that $\Gamma_1(\mathrm{gr}\,\tilde{\mathrm{A}}_2)=\mathcal{H}$ and the equivalence (99), assuming the form $\mathbb{R}_\lambda\in\Omega_A\Leftrightarrow B_G=$, means, taking into account (96), the M-admissibility of $\tau(\lambda)$, i.e., the equality (86).

COROLLARY 8. Let $\{\mathcal{H},\Gamma_1,\Gamma_2\}$ be a SBV of the relation A^*, let $\theta_i\in\tilde{\mathfrak{C}}(\mathcal{H})$, $i=1,2$, $\lambda\in\rho(\tilde{A}_2)\cap\rho(\tilde{A}_{\theta_1})\cap\rho(\tilde{A}_{\theta_2})$, and let $\mathfrak{S}_p(\mathfrak{H})$ be the Neumann–Schatten ideals in $[\mathfrak{H}]$. Then:

1) $\forall\zeta\in\rho(\theta_1)\cap\rho(\theta_2)$, we have

$$(\tilde{A}_{\theta_1}-\lambda)^{-1}-(\tilde{A}_{\theta_2}-\lambda)^{-1}\in\mathfrak{S}_p(\mathfrak{H})\Leftrightarrow(\theta_1-\zeta)^{-1}-(\theta_2-\zeta)^{-1}\in\mathfrak{S}_p(\mathcal{H}); \tag{100}$$

2) $\tilde{A}_{\theta_1}$ and $\tilde{A}_{\theta_2}$ are transversal $\Leftrightarrow 0\in\rho(\theta_1-\zeta)^{-1}-(\theta_2-\zeta)^{-1})$.

Proof. From the resolvent formula (82) we obtain

$$(\tilde{A}_{\theta_1}-\lambda)^{-1}-(\tilde{A}_{\theta_2}-\lambda)^{-1}\in\mathfrak{S}_p(\mathfrak{H})\Leftrightarrow(\theta_1-M(\lambda))^{-1}-(\theta_2-M(\lambda))^{-1}\in\mathfrak{S}_p(\mathcal{H}). \tag{101}$$

Since $\lambda\in\rho(\tilde{A}_{\theta_i})\cap\rho(\tilde{A}_2)$, it follows that $0\in\rho(\theta_i-M(\lambda))$, i = 1, 2. Therefore, $\forall\lambda\in\rho(\tilde{A}_\theta)$ and $\forall\zeta\in\rho(\theta)$ we have the identities

$$\begin{aligned}&[I+(\zeta-M(\lambda))(\theta-\zeta)^{-1}]^{-1}=I+(M(\lambda)-\zeta)(\theta-M(\lambda))^{-1},\\&[I+(\theta-\zeta)^{-1}(\zeta-M(\lambda))]^{-1}=I+(\theta-M(\lambda))^{-1}(M(\lambda)-\zeta),\end{aligned} \tag{102}$$

and, taking these into account. we find

$$\begin{aligned}&(\theta_1-M(\lambda))^{-1}-(\theta_2-M(\lambda))^{-1}=(\theta_2-\zeta)^{-1}[I+(\zeta-M(\lambda))(\theta_2-\zeta)^{-1}]^{-1}-\\&-[I+(\theta_1-\zeta)^{-1}(\zeta-M(\lambda))^{-1}(\theta_2-\zeta)^{-1}]=[I+(\theta_1-\zeta)^{-1}(\zeta-M(\lambda))^{-1}]^{-1}\times\\&\times[(\theta_2-\zeta)^{-1}-(\theta_1-\zeta)^{-1}]\,[I+(\zeta-M(\lambda))(\theta_2-\zeta)^{-1}]^{-1}.\end{aligned} \tag{103}$$

From here we obtain the equivalence (100).

Statement 2 is a consequence of the relations (23), (103) and of the equivalence

$$0\in\rho((\tilde{A}_{\theta_1}-\lambda)^{-1}-(\tilde{A}_{\theta_2}-\lambda)^{-1}))\Leftrightarrow 0\in\rho((\theta_1-M(\lambda))^{-1}-(\theta_2-M(\lambda))^{-1}),$$

which, just as (101), is derived from the resolvent formula (82).

COROLLARY 9. If $\theta_i = B_i \in [\mathcal{H}]$, then we have the equivalence

$$(\tilde{A}_{B_1}-\lambda)^{-1}-(\tilde{A}_{B_2}-\lambda)^{-1})\in\mathfrak{S}_p(\mathfrak{H})\Leftrightarrow B_1-B_2\in\mathfrak{S}_p(\mathcal{H}).$$

Remark 7. In the case $\overline{\mathcal{D}(A)}=\mathfrak{H}$ and only in this case, all $\tau(\lambda)\in(\tilde{R})_{\mathcal{H}}$ are M-admissible. In this case ($\overline{\mathcal{D}(A)}=\mathfrak{H}$) formula (85), being a generalization of M. G. Krein's formula for the resolvent [22], and Corollaries 8, 9 have been obtained in [16-19]. For the connection with A. V. Shtraus' formula [23], see Sec. 5.

Remark 8. In Theorem 2 the formula (86) has been obtained from (84). We show how, starting from (86), one can obtain formula (84). Let G(λ) be the right-hand side of the equality (86). Then $T(\lambda)\doteq G(\lambda)^{-1}+\lambda I$ is a proper extension of the operator A for each $\lambda\in\mathbb{C}_+\cup\mathbb{C}_-$. Therefore, $T(\lambda)=\tilde{A}_{\theta(\lambda)}$, where $\theta(\lambda)\doteq\Gamma T(\lambda)\in\tilde{\mathfrak{C}}(\mathcal{H})$. From Lemma 4, taking into account the notations (87), we obtain, assuming for the sake of simplicity that $\tau(\lambda)\in\mathfrak{C}(\mathcal{H})$ (i.e., $\tau(\lambda)(0)=\{0\}$),

$$\Gamma_2\hat{f}_\theta=\Gamma_2(\hat{f}_\theta-\hat{f}_2)=-\Gamma_2\gamma(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\bar{\lambda})f=-(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\bar{\lambda})f,$$

$$\Gamma_1\hat{f}_\theta=\Gamma_1\hat{f}_2-\Gamma_1\gamma(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\bar{\lambda})f=[I-M(\lambda)(\tau(\lambda)+M(\lambda))^{-1}]\gamma^*(\bar{\lambda})f\;=\tau(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\bar{\lambda})f.$$

From here $\Gamma\hat{f}_\theta=\{\Gamma_2\hat{f}_\theta,\Gamma_1\hat{f}_\theta\}\in-\tau(\lambda)$, i.e., $\theta(\lambda)=-\tau(\lambda)$ and $T(\lambda)=\tilde{A}_{-\tau(\lambda)}$.

3. Let $\mathfrak{L}$ be a subspace of $\mathfrak{H}$. We recall that the operator-valued function $P_{\mathfrak{L}}\mathbb{R}_\lambda\upharpoonright\mathfrak{L}$, where $\mathbb{R}_\lambda\in P\Omega_A$ ($\mathbb{R}_\lambda\in\Omega_A$), is called a $\mathfrak{L}$-pseudoresolvent ($\mathfrak{L}$-resolvent) of the operator A and their collection is denoted by $P\Omega_A^{\mathfrak{L}}$ ($\Omega_A^{\mathfrak{L}}$). The monotone operator-valued function $\Sigma(t)\doteq P_{\mathfrak{L}}E(t)\upharpoonright\mathfrak{L}=\Sigma(t-0)$ is called an $\mathfrak{L}$-spectral function of the operator A if E(t) is a generalized (with output in $\tilde{\mathfrak{H}}\supset\mathfrak{H}$) spectral function of the operator A. The function Σ(t) is said to be orthogonal if E(t) is orthogonal. The relationship between $\mathfrak{L}$-pseudoresolvents and $\mathfrak{L}$-spectral functions is given by the formula

$$P_{\mathfrak{L}}\mathbb{R}_\lambda\upharpoonright\mathfrak{L}=\int_{-\infty}^{\infty}\frac{d\Sigma(t)}{t-\lambda}\;\Big(\mathbb{R}_\lambda\doteq P_{\mathfrak{H}}(\tilde{A}-\lambda)^{-1}\upharpoonright\mathfrak{H}=\int_{-\infty}^{\infty}\frac{dE(t)}{t-\lambda}\Big).$$

If P_1 is the orthoprojection in $\mathfrak{H}$ onto the operator part $\mathfrak{H}_1$ of the relation $\mathbb{R}_\lambda$, then

$$E(\infty)\doteq s-\lim_{t\uparrow\infty}E(t)=P_1,\;\Sigma(\infty)\doteq s-\lim_{t\uparrow\infty}\Sigma(t)=P_{\mathfrak{L}}P_1\upharpoonright\mathfrak{L}$$

(see (94)) and, moreover, if $\mathfrak{L}\not\subset\mathfrak{N}=\mathfrak{H}_0^\perp$, then $\Sigma(\infty)=I_{\mathfrak{L}}\Leftrightarrow\mathbb{R}_\lambda\in\Omega_A$.

We recall that the matrix-valued function

$$U_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix} \doteq \begin{pmatrix} M(\lambda) & \gamma^*(\bar{\lambda})\upharpoonright\mathfrak{L} \\ P_{\mathfrak{L}}\gamma(\lambda) & P_{\mathfrak{L}}(\tilde{A}_2 - \lambda)^{-1}\upharpoonright\mathfrak{L} \end{pmatrix}, \ \lambda \in \rho(\tilde{A}_2),$$

is called the $\Pi\mathfrak{L}$-pseudoresolvent matrix of the operator A, corresponding to the SBV $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$. To describe the collections $P\Omega_A^{\mathfrak{L}}$ and $\Omega_A^{\mathfrak{L}}$ ($\dim\mathfrak{L} = n_\pm(A)$) we introduce the $\Pi\mathfrak{L}$-resolvent matrix $W_{\Pi\mathfrak{L}}(\lambda)$, corresponding to the SBV $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$,

$$W_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} \omega_{11}(\lambda) & \omega_{12}(\lambda) \\ \omega_{21}(\lambda) & \omega_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} a_{22}a_{12}^{-1} & a_{22}a_{12}^{-1}a_{11} - a_{21} \\ a_{12}^{-1} & a_{12}^{-1}a_{11} \end{pmatrix}. \tag{104}$$

It is holomorphic on the set of $\mathfrak{L}$-regular points $\rho(A;\mathfrak{L})$ of the operator A ($\lambda_0 \in \rho(A;\mathfrak{L}) \Leftrightarrow \mathfrak{H} = (A-\lambda_0)\mathfrak{D}(A) \dotplus \mathfrak{L}$ and assumes values in $[\mathfrak{L} \oplus \mathfrak{L}]$ by virtue of the equivalence $0 \in \rho\,(a_{12}(\lambda_0)) \Leftrightarrow \lambda_0 \in \rho(A;\mathfrak{L})$.

THEOREM 3. Let $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ be a SBV of the relation A^*, let $W_{\Pi\mathfrak{L}}(\lambda)$ be the corresponding $\Pi\mathfrak{L}$-resolvent matrix (104). Then the formula

$$P_{\mathfrak{L}}(\tilde{A} - \lambda)^{-1}\upharpoonright\mathfrak{L} = [\omega_{11}(\lambda)\tau(\lambda) + \omega_{12}(\lambda)][\omega_{21}(\lambda)\tau(\lambda) + \omega_{22}(\lambda)]^{-1}$$

establishes a bijective correspondence between the $\mathfrak{L}$-pseudoresolvents $P_{\mathfrak{L}}\mathbb{R}_\lambda\upharpoonright\mathfrak{L} \in P\Omega_A^{\mathfrak{L}}$ and $\tau(\lambda) \in (\tilde{R})_{\mathfrak{L}}$. Moreover, $P_{\mathfrak{L}}\mathbb{R}_\lambda\upharpoonright\mathfrak{L} \in \Omega_A^{\mathfrak{L}} \Leftrightarrow \tau(\lambda)$ is M-admissible.

Proof. Since $P_{\mathfrak{L}}$ maps isomorphically $\mathfrak{N}_\lambda$ onto $\mathfrak{L}$ $\forall\lambda \in \rho(A;\mathfrak{L})$, it follows that $a_{12}(\lambda_0)^{-1} \in [\mathfrak{L}]$. Therefore, from (82) we obtain

$$P_{\mathfrak{L}}\mathbb{R}_\lambda\upharpoonright\mathfrak{L} = P_{\mathfrak{L}}(\tilde{A} - \lambda)^{-1}\upharpoonright\mathfrak{L} = P_{\mathfrak{L}}(\tilde{A}_2 - \lambda)^{-1}\upharpoonright\mathfrak{L} - P_{\mathfrak{L}}\gamma(\lambda)(\tau(\lambda) +$$

$$+ M(\lambda))^{-1}\gamma^*(\bar{\lambda})\upharpoonright\mathfrak{L} = a_{22}(\lambda) - a_{21}(\lambda)(a_{11}(\lambda) + \tau(\lambda))^{-1}a_{12}(\lambda) = a_{22}(\lambda) - a_{21}(\lambda)[a_{12}^{-1}(\lambda)(\tau(\lambda) + a_{11}(\lambda))]^{-1} =$$

$$= \left\{a_{22}\left[a_{12}^{-1}\tau(\lambda) + a_{12}^{-1}a_{11}\right] - a_{21}\right\}\left[a_{12}^{-1}\tau(\lambda) + a_{12}^{-1}a_{11}\right]^{-1} = [\omega_{11}(\lambda)\tau(\lambda) + \omega_{12}(\lambda)][\omega_{21}(\lambda)\tau(\lambda) + \omega_{22}(\lambda)]^{-1}.$$

A reference to Theorem 2 concludes the proof.

5. The Relationship with the Classical Approach. 1. From the results of Sec. 2 there follows easily the description of the various classes of extensions of the operator A in terms of the von Neumann formulas. We consider for this the canonical SBV $\{\mathcal{H}, \Gamma_1^0, \Gamma_2^0\}$ of the form (17):

$$\mathcal{H} = \mathfrak{N}_i,\ \Gamma_1^0 = \pi_1(\mathcal{P}_i + \tilde{U}_0\mathcal{P}_{-i}),\ \Gamma_2^0 = -i\pi_1(\mathcal{P}_i + \tilde{U}_0\mathcal{P}_{-i}). \tag{105}$$

According to Proposition 2, $\forall\hat{n} = \{0, n\} \in \hat{\mathfrak{N}}$ we have

$$\frac{1}{2i}\begin{pmatrix} 0 \\ n \end{pmatrix} = \begin{pmatrix} f_A \\ Af_A \end{pmatrix} + \begin{pmatrix} P_{\mathfrak{N}_i}n \\ iP_{\mathfrak{N}_i}n \end{pmatrix} + \begin{pmatrix} -P_{\mathfrak{N}_{-i}}n \\ iP_{\mathfrak{N}_{-i}}n \end{pmatrix}. \tag{106}$$

From (105) and (106) we obtain

$$\Gamma_1\hat{n} = 2i(P_{\mathfrak{N}_i} - U_0P_{\mathfrak{N}_{-i}})n,\ \Gamma_2\hat{n} = 2(P_{\mathfrak{N}_i} + U_0P_{\mathfrak{N}_{-i}})n,$$

$$V_\Gamma = \{\{(P_{\mathfrak{N}_i} + U_0P_{\mathfrak{N}_{-i}})n,\ i(P_{\mathfrak{N}_i} - U_0P_{\mathfrak{N}_{-i}})n\};\ n \in \mathfrak{N} = \mathfrak{H}_0^\perp\}.$$

Therefore, the operator

$$V_e' = I - 2i(V_\Gamma + i)^{-1} = \{\{P_{\mathfrak{N}_i}n,\ U_0P_{\mathfrak{N}_{-i}}n\};\ n \in \mathfrak{N}\},\ (V_3' \in [\mathfrak{N}_i''])\tag{107}$$

coincides, to within the isometry $-U_0$, with the exclusion operator V_e ($\in [\mathfrak{N}_i'', \mathfrak{N}_{-i}'']$) of the form (12): $V_e' = -U_0V_e$. Further, if $A \subset \tilde{A} \subset A^*$, then

$$\hat{f} \in \tilde{A} = \tilde{A}_\theta \Leftrightarrow \Gamma\hat{f} \in \theta \doteq \Gamma\tilde{A} = \{\{-if_i + iU_0f_{-i}, f_i + U_0f_{-i}\}\};\ \hat{f} = \hat{f}_A + \hat{f}_i + \hat{f}_{-i} \in \tilde{A}\}.$$

From here we find the Cayley transform Φ' of the relation θ under the condition $-i \notin \sigma_p(\theta)$:

$$\Phi' \doteq I - 2i(\theta + i)^{-1} = \{\{f_i, U_0 f_{-i}\};\ \hat f_{\pm i} = \mathcal{P}_{\pm i}\hat f,\ \hat f = \hat f_A + \hat f_i + \hat f_{-i} \in \tilde A\}. \tag{108}$$

Setting $\Phi \doteq -U_0^*\Phi'$ and taking into account (107), (108), and the obvious equivalences

$$\theta \cap V_\Gamma = \{0\} \Leftrightarrow \ker(V_e' - \Phi') = \{0\} \Leftrightarrow \ker(V_e - \Phi) = \{0\}, \tag{109}$$

we conclude that the admissibility condition assumes the known [4, 7] form: $(V_e - \Phi)f = 0 \Leftrightarrow f = 0$. We sum up the presented facts.

Proposition 8. The formulas

$$\mathcal{D}(\tilde A) = \mathcal{D}(A) \dotplus (I - \Phi)\mathcal{D}(\Phi),\ \mathcal{D}(\Phi) \subset \mathfrak{N}_i, \tag{110}$$

$$\tilde A(f_A + f_i - \Phi f_i) = Af_A + i(f_i + \Phi f_i),\ f_{\tilde A} = f_A + f_i - \Phi f_i, \tag{111}$$

establish a bijective correspondence between the collection of proper extensions $\tilde A \in \tilde{\mathcal{C}}(\mathfrak{H})$, for which $-i \notin \sigma_p(\tilde A)$, and the collection of admissible operators $\Phi \in \mathcal{C}(\mathfrak{N}_i, \mathfrak{N}_{-i})$. Moreover:

1) $\Phi = (\tilde A - i)(\tilde A + i)^{-1} \upharpoonright \mathfrak{N}_i \cap \mathfrak{R}(\tilde A + i);$ (112)

2) $\tilde A$ is Hermitian (dissipative) $\Leftrightarrow$ Φ is an admissible isometry (contraction) from $\mathcal{D}(\Phi) \subset \mathfrak{N}_i$ into $\mathfrak{R}(\Phi) \subset \mathfrak{N}_{-i}$;

3) $\tilde A$ is self-adjoint (maximal dissipative) $\Leftrightarrow$ Φ is an admissible isometry (contraction), for which $\mathcal{D}(\Phi) = \mathfrak{N}_i$, $\mathfrak{R}(\Phi) = \mathfrak{N}_{-i}$ ($\mathcal{D}(\Phi) = \mathfrak{N}_i$).

Proof. The relations (110), (111) follow from the first Neumann formula (14) and the relations (108), (109). One has only to make use of the obvious equivalence $-i \notin \sigma_p(\theta) \Leftrightarrow -i \in \sigma_p(\tilde A_\theta)$, which, besides, is a consequence of the equality $M(i) = iI_{\mathfrak{N}_i}$ [see formula (118)] and Proposition 7.

Further, formula (112) follows from the relations (111), rewritten in the form $2if_i = (\tilde A + i)(f_{\tilde A} - f_A)$, $2i\Phi f_i = (\tilde A - i)(f_{\tilde A} - f_A)$. Statements 2 and 3 are consequences of formulas (110), (111) and of Proposition 3.

Remark 9. The description of the Hermitian extensions $\tilde A \in \tilde{\mathcal{C}}(\mathfrak{H})$ in the form (110), (111) has been obtained in [4], the dissipative ones in [7, 24], and all proper ones in [24] (see also [25]). In these investigations the relations (110)-(112) have been established with the aid of the Cayley transform. We mention also that the relations (110), (111) in the case $-i \in \sigma_p(\tilde A)$ can be replaced (see [24]) by the following ones:

$$\mathcal{D}(\tilde A) = \mathcal{D}(A) \dotplus \{f_i - g_{-i}, \{f_i, g_{-i}\} \in \Phi\},\ \tilde A(f_A + f_i - g_{-i}) = Af_A + i(f_i + g_{-i}).$$

2. Let $C_0(\lambda)$ be the characteristic function corresponding to the "canonical" SBV of the form (105). Then $\Gamma_1^0 + i\Gamma_2^0 = 2\pi_1\mathcal{P}_i$, $\Gamma_1^0 - i\Gamma_2^0 = 2\pi_1\tilde U_0\mathcal{P}_{-i}$. If $\hat f_\lambda = \{f_i, \lambda f_\lambda\} \in \hat{\mathfrak{N}}_\lambda$, then, according to (14), we have

$$\hat f_\lambda = \hat f_A(\lambda) + \hat f_i(\lambda) - \hat f_{-i}(\lambda)\ \ (\hat f_A (= \{f_A, Af_A\}, \hat f_{\pm i} = \{f_i, \pm if_i\} \in \hat{\mathfrak{N}}_{\pm i}). \tag{113}$$

From here $\pi_1\mathcal{P}_i\hat f_\lambda = f_i$, $\pi_1\tilde U_0\mathcal{P}_{-i}\hat f_\lambda = U_0 f_{-i}$ and $C_0(\lambda)f_i(\lambda) = -U_0 f_{-i}(\lambda)$. Writing relation (113) in the form of the system of equalities

$$f_\lambda = f_A + f_i - f_{-i},\ \lambda f_\lambda = Af_A + if_i - if_{-i} \tag{114}$$

and taking into account that $\tilde A_\lambda \doteq A \dotplus \hat{\mathfrak{N}}_\lambda$ (see Lemma 5), from (114) we obtain

$$-2if_\pm = Af_A - \lambda f_\lambda \pm i(f_A - f_\lambda) = (\tilde A_\lambda \pm i)(f_A - f_\lambda).$$

From here, $f_{-i} = (\tilde A_\lambda - i)(\tilde A_\lambda + i)^{-1}f_i$ and, consequently,

$$C_0(\lambda) = -U_0(\tilde A_\lambda - i)(\tilde A_\lambda + i)^{-1} \upharpoonright \mathfrak{N}_i. \tag{115}$$

Equality (115) means that $C_0(\lambda)$ differs only by the isometric factor $-U_0$ from the characteristic function in [7]. Thus, in the case of a SBV of the form (105), by virtue of (115) and the equality $V_e' = -U_0V_e$, Theorem 1′ and Proposition 5 coincide basically with Shtraus' results, proved in a different way in [7].

3. We find the expression for the Weyl function $M_0(\lambda)$, corresponding to a SBV of the form (105), assuming that $\tilde A_2 = \ker \Gamma_2$ is an operator. According to (34) and (28), we have

$$\hat{f} \doteq \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} = \begin{pmatrix} U_{i\lambda} f_i \\ \lambda U_{i\lambda} f_i \end{pmatrix} = \begin{pmatrix} f_i \\ i f_i \end{pmatrix} + (\lambda - i) \begin{pmatrix} (\tilde{A}_2 - \lambda)^{-1} f_i \\ f_i + \lambda(\tilde{A}_2 - \lambda)^{-1} f_i \end{pmatrix}, f_i \in \mathfrak{N}_i. \tag{116}$$

From here we obtain

$$\Gamma_2^0 \hat{f}_\lambda = -i f_i \Rightarrow \gamma(\lambda) f_i = i f_\lambda,\ \gamma(\lambda) = U_{i\lambda}\gamma(i) = i U_{i\lambda} I_{\mathfrak{N}_i}. \tag{117}$$

Now from (116), (117), and Lemma 4 we obtain

$$\Gamma_1^0 \hat{f} = f_i + (\lambda - i)\Gamma_1^0 \begin{pmatrix} (\tilde{A}_2 - \lambda)^{-1} f_i \\ f_i + \lambda(\tilde{A}_2 - \lambda)^{-1} f_i \end{pmatrix} = f_i + (\lambda - i)\gamma^*(\bar{\lambda}) f_i =$$
$$= [I - i(\lambda - i) P_{\mathfrak{N}_i} U^*_{i\bar{\lambda}}] f_i.$$

Consequently,

$$M_0(\lambda) = \Gamma_1^0 \gamma(\lambda) = P_{\mathfrak{N}_i}[i + (\lambda - i)U^*_{i\bar{\lambda}}] \upharpoonright \mathfrak{N}_i = P_{\mathfrak{N}_i}(\lambda \tilde{A}_2 + I)(\tilde{A}_2 - \lambda)^{-1} \upharpoonright \mathfrak{N}_i. \tag{118}$$

Thus, in the SBV (105) (under the condition that $\tilde{A}_2$ is an operator), formula (85) for the generalized resolvents assumes the form

$$\mathbb{R}_\lambda = (\tilde{A}_2 - \lambda)^{-1} - (\tilde{A}_2 - i)(\tilde{A}_2 - \lambda)^{-1}[\tau(\lambda) + M_0(\lambda)]^{-1} P_{\mathfrak{N}_i}(\tilde{A}_2 + i)(\tilde{A}_2 - \lambda)^{-1}, \tag{119}$$

where $M_0(\lambda)$ is defined by the equality (118). In the form (119) it has been derived by Shmul'yan [26] from Shtraus' formula [23] (see also [27], where a formula, close to (119), has been derived also from Shtraus' formula).

We note also that, although the characteristic function and the Q-function of a Hermitian operator A have been defined (for the case $\overline{\mathcal{D}(A)} = \mathfrak{H}$ and $n_\pm(A) = 1$) already in 1944 by Livshits and Krein with the aid of the equalities (115) and (118), respectively, the connection between them has been detected by Krein and Langer only in 1973. On the other hand, Definition 7 makes this connection obvious.

Let $\tau(\lambda) \in (\tilde{R})_{\mathcal{H}}$. The indeterminate part $\mathcal{H}^2 \doteq \tau(\lambda)(0)$ of the function $\tau(\lambda)$ does not depend on $\lambda \in \mathbb{C}_+$ [22]. Consequently [21, 22],

$$\tau(\lambda) = \tau_1(\lambda) \oplus \hat{\mathcal{H}}^2,\ \hat{\mathcal{H}}^2 = \mathbf{0} \oplus \mathcal{H}^2,\ \mathcal{H}^2 \doteq \tau(\lambda)(0),\ \mathcal{H}^1 = \mathcal{H} \ominus \mathcal{H}^2,$$

where $\tau_1(\lambda)$ is the operator part of the relation $\tau(\lambda)$ $(\tau_1(\lambda) \in \mathfrak{C}(\mathcal{H}^1)\ \ \forall\lambda \in \mathbb{C}_+)$. If $\tau_1(\lambda)$ takes values in $[\hat{\mathcal{H}}^1]$, then we define the Hermitian relation $\tau(i\infty)$ by setting $\tau(i\infty) = \tau_1(i\infty) \oplus \hat{\mathcal{H}}^2$, where

$$\tau_1(i\infty) f = s - \lim_{y \uparrow \infty} \tau_1(iy) f \quad \forall f \in \mathcal{H}^1_{\tau_1} \doteq \{f \in \mathcal{H}^1;\ \lim_{y\uparrow\infty} y \ \mathrm{Im}\,(\tau_1(iy)f, f) < \infty\}.$$

We give without proof the following M-admissibility criterion.

Proposition 9. Let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a SBV, let $M(\lambda)$ and V_Γ be the corresponding Weyl function and exclusion relation, and let $\tau(\lambda) \in (\tilde{R})_{\mathcal{H}}$. If the operator-valued function $\tau_1(\lambda)$ takes values in $[\mathcal{H}_1]$, then the M-admissibility condition of $\tau(\lambda)$ is equivalent to the condition of admissibility of the relation $-\tau(i\infty)$, i.e.,

$$s - \lim_{y\uparrow\infty} y^{-1}(\tau(iy) + M(iy))^{-1} = 0 \Leftrightarrow -\tau(i\infty) \cap V_\Gamma = \{0\}.$$

6. Examples and Applications. 1. Let e_0 be a unit vector of the Hilbert space $\mathfrak{H}$, $\mathfrak{N} = \{e_0\}$, $\mathfrak{H}_0 = \mathfrak{N}^\perp = \mathfrak{H} \ominus \mathfrak{N}$, $\tilde{A} = \tilde{A}^* \in \mathfrak{C}(\mathfrak{H})$. The operator $A = \tilde{A} \upharpoonright \mathfrak{H}_0$ is Hermitian, $n_\pm(A) = 1$, $A^*_{0p} = P_{\mathfrak{H}_0}\tilde{A}$. It is easy to see that $f_\lambda \doteq (\tilde{A} - \lambda)^{-1} e_0 \in \mathfrak{N}_\lambda$, i.e., $\mathfrak{N}_\lambda(A) = \mathfrak{N}''_\lambda(\tilde{A})$. We define a SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ of the relation $A^* = \{\{f, A^*_{0p} f + c e_0\},\ f \in D(A^*_{0p}), c \in \mathbb{C}\}$, setting

$$\mathcal{H} = \mathbb{C}\ \ \Gamma_1 \hat{f} = (f, e_0),\ \Gamma_2 \hat{f} = (f, A e_0) - c\left(\hat{f} = \{f, A^*_{0p} f + c e_0\}\right). \tag{120}$$

From (120) we obtain

$$A^*_{0p} f_\lambda = P_{\mathfrak{H}_0} \tilde{A}(\tilde{A} - \lambda)^{-1} e_0 = \lambda P_{\mathfrak{H}_0}(\tilde{A} - \lambda)^{-1} e_0 = \lambda f_\lambda - \lambda(f_\lambda, e_0) e_0,$$
$$\Gamma_1 \hat{f}_\lambda = \Gamma_1\{f_\lambda, \lambda f_\lambda\} = (f_\lambda, e_0),\ \Gamma_2 f_\lambda = (f_\lambda, \tilde{A} e_0) - \lambda(f_\lambda, e_0) = 1.$$

Consequently, the Weyl function has the form

$$M(\lambda)=(f_\lambda,e_0)=\left((\tilde{A}-\lambda)^{-1}e_0,e_0\right)=\int_{-\infty}^{\infty}\frac{d(E(t)e_0,e_0)}{t-\lambda}. \tag{121}$$

The case dim $\mathfrak{N}$ = n < ∞ is examined in a similar manner. Setting $A=\tilde{A}\restriction\mathfrak{H}_0$ ($\mathfrak{H}_0=\mathfrak{H}\ominus\mathfrak{N}$), we obtain a Hermitian operator, for which $n_\pm(A)=n$, $\mathfrak{N}_\lambda(A)=\mathfrak{N}''_\lambda(\tilde{A})=(\tilde{A}-\lambda)^{-1}\mathfrak{N}$. The SBV $\{\mathcal{H},\Gamma_1,\Gamma_2\}$ of the relation $A^*=\{\{f,\tilde{A}f+n\};\ f\in\mathcal{D}(\tilde{A})=\mathcal{D}(A^*_{0p}),\ n\in\mathfrak{N}\}$ is defined more conveniently than in (120): $\mathcal{H}=\mathfrak{N}$, $\Gamma_1\hat{f}=P_\mathfrak{N}f$, $\Gamma_2\hat{f}=-n$ ($\hat{f}=\{f,\tilde{A}f+n\}$). Since $\mathfrak{N}_\lambda=\{(\tilde{A}-\lambda)^{-1}n;\ n\in\mathfrak{N}\}$, we have $\Gamma_1\hat{f}_\lambda=\Gamma_1\{(\tilde{A}-\lambda)^{-1}n,\ \lambda(\tilde{A}-\lambda)^{-1}n\}=P_\mathfrak{N}(\tilde{A}-\lambda)^{-1}n$, $\Gamma_2\hat{f}_\lambda=n$. Thus, the corresponding Weyl function M(λ) coincides with one of the $\mathfrak{N}$-resolvents of the operator A:

$$M(\lambda)=P_\mathfrak{N}(\tilde{A}-\lambda)^{-1}\restriction\mathfrak{N}. \tag{121'}$$

2. Let $\mathfrak{H}=l_2[0,\infty)$, let $\{e_k\}_0^\infty$ be the standard basis in $l_2[0,\infty)$, and let $\tilde{A}=\tilde{A}^*$ be the operator generated in l_2 by the Jacobi matrix associated with a certain moment problem:

$$\tilde{A}e_k=b_{k-1}e_k+a_ke_k+b_ke_{k+1}\ (b_{-1}=0,\ b_k>0, a_k=\bar{a}_k, k\in\mathbb{Z}_+). \tag{122}$$

Assume, as before, that $A=\tilde{A}\restriction\mathfrak{H}_0$ ($\mathfrak{H}_0=\mathfrak{H}\ominus e_0$). Then the equation $A^*_{0p}y=\lambda P_{\mathfrak{H}_0}y$ ($A^*_{0p}=P_{\mathfrak{H}_0}\tilde{A}$) is equivalent to the finite-difference equation

$$b_{k-1}y_{k-1}+a_ky_k+b_ky_{k+1}=\lambda y_k,\ k=1,2,3,\dots . \tag{123}$$

Therefore, from the condition $n_\pm(A)=1$ there follows the existence (and uniqueness) of the solution $\{y_k(\lambda)\}_0^\infty$, belonging to $l_2[0,\infty)$, of the equation (123). If $P_k(\lambda)$ and $Q_k(\lambda)$ are orthogonal polynomials of the first and second kind [28, 29], then the solution $\{y_k(\lambda)\}_0^\infty$ can be represented in the form

$$(f_\lambda=)\sum_{k=0}^{\infty}y_k(\lambda)e_k=\sum_{k=0}^{\infty}[P_k(\lambda)\omega(\lambda)+Q_k(\lambda)]e_k\ \ (\in\mathfrak{N}_\lambda),$$

where the function ω(λ) is such that

$$\sum_{k=0}^{\infty}|P_k(\lambda)\omega(\lambda)+Q_k(\lambda)|^2<\infty. \tag{124}$$

From (120) we obtain $\Gamma_1\hat{f}_\lambda=\Gamma_1\{f_\lambda,\lambda f_\lambda\}=\omega(\lambda)$. Consequently, by virtue of (121) we have $\omega(\lambda)=M(\lambda)=((\tilde{A}-\lambda)^{-1}e_0,e_0)$ and, therefore, $\omega(\lambda)\in(R)$.

Remark 10. The fact of the existence of a function ω(λ) (∈ (R)), satisfying condition (124) (and also its properties), constitutes the content of the finite-difference analogue of H. Weyl's theorem, involving the Sturm–Liouville equation on the semiaxis. Thus, in the above given operator proof of this theorem, the analogue of the Sturm–Liouville minimal operator on the semiaxis is a nondensely defined operator A. According to the equality $\omega(\lambda)=((\tilde{A}-\lambda)^{-1}e_0,e_0)$, the measure $\sigma(t)=d(E(t)e_0,e_0)$ from the integral representation (121) of the function ω(λ) = M(λ) is a solution of the Hamburger moment problem

$$s_k=\int_{-\infty}^{\infty}t^k\,d\sigma(t)\ (=(\tilde{A}^ke_0,e_0)),\ k\in\mathbb{Z}_+, \tag{125}$$

generated by the Eq. (122) (e_0 is the generating vector).

3. Let A be the Hermitian operator in $\mathfrak{H}=l_2[0,\infty]$, generated by the Jacobi matrix associated with the moment problem (125). From statement 4 one obtains the following proposition, due to Hamburger (see [28]).

Proposition 10. In order that the moment problem (125) be determinate it is necessary and sufficient that at least one of the series

$$\sum_{k=0}^{\infty}P_k(a)^2,\ \ \sum_{k=1}^{\infty}Q_k(a)^2,\ \ a=\bar{a}. \tag{126}$$

be divergent.

Proof. Necessity. The indeterminacy of the problem (125) is equivalent (see [28, 29]) to the relations $n_{\pm}(A) = 1 \Leftrightarrow \sum_{k=0}^{\infty}|P_k(\lambda)|^2 < \infty$, from which, by virtue of (124), there follows $\sum_{k=1}^{\infty}|Q_k(\lambda)|^2 < \infty$.

Sufficiency. Assume that the series (126) converge. Introducing the operator $A_0 \doteq A \upharpoonright \mathfrak{H}_0$ ($\mathfrak{H}_0 \doteq \mathfrak{H} \ominus e_0$), we show that $n_{\pm}(A_0) = 2$. Since $f_a \doteq \{P_k(a)\}_0^{\infty} \in \ker(A - a)$ and $f_a \notin \mathcal{D}(A_0)$, it follows that $\ker(A_0 - a) = \{0\}$. Indeed, otherwise we would have $\dim \ker(A - a) \geq 2$, contradicting the fact that the spectrum $\sigma(A)$ of the operator A is simple ($\sigma(A)$ is simple since e_0 is a generating vector for A). Further, $f_a = \{P_k(a)\}_0^{\infty}$ and $g_a = \{Q_k(a)\}_1^{\infty}$ are solutions of (123) for $\lambda = a$ and, consequently, $f_a, g_a \in \mathfrak{N}_a(A_0)$, i.e., $\dim \mathfrak{N}_a(A_0) = 2$. According to statement 4, we have $n_{\infty}(A_0) = 2$. But then $n_{\pm}(A) = 1$ and the problem (125) is indeterminate.

Remark 11. Analogous arguments can be applied to higher-order finite-difference equations. We mention also that the consideration of the operator A_0 gives the possibility to give a natural (operator) interpretation to the polynomials of the second kind: $g_\lambda \doteq \{Q_k(\lambda)\}_1^{\infty}$ ($\in \mathfrak{N}_\lambda(A_0) \cap \mathfrak{H}_0$) is a semidefect vector of the operator A_0.

4. Let $\{s_k\}_0^{2n}$ be a strictly positive sequence, let $\mathfrak{H}$ be the Euclidean space of the polynomials $\mathbb{C}_n[t]$ of degree at most n with the inner product

$$(f, g) = \sum_{j,k=0}^{n} s_{j+k}\alpha_j\bar{\beta}_k \quad \left(f(t) = \sum_{k=0}^{n}\alpha_k t^k,\ g(t) = \sum_{k=0}^{n}\beta_k t^k \in \mathbb{C}_n[t]\right). \tag{127}$$

Assume, further, that $\{P_k(t)\}_0^n$ are polynomials orthogonal with respect to the sequence $\{s_k\}_0^{2n}$ of polynomials (of the first kind). We consider in $\mathfrak{H}_0 \doteq \mathfrak{H} \ominus \{P_n(t)\}$ the Hermitian operator A ($\in [\mathfrak{H}_0, \mathfrak{H}]$) of multiplication by t. Then $A^* = \{\{f, A_{\mathrm{op}}^* f + cP_n\}; f \in \mathfrak{H}, c \in \mathbb{C}\}$, where $A_{\mathrm{op}}^* = [\mathfrak{H}, \mathfrak{H}_0]$ and

$$A\, e_k \doteq tP_k(t) = b_{k-1}P_{k-1}(t) + a_kP_k(t) + b_kP_{k+1}(t),\ 0 \leq k \leq n-1,$$

$$A_{\mathrm{op}}^* e_k = Ae_k\ (0 \leq k \leq n-2),\ A_{\mathrm{op}}^* e_{n-1} = b_{n-2}e_{n-2} + a_{n-1}e_{n-1},\ A_{\mathrm{op}}^* e_n = b_{n-1}e_{n-1}, \tag{128}$$

where $b_{-1} = 0$, $b_k > 0$, $a_k = \bar{a}_k$, $0 \leq k \leq n$. We shall identify the operator $A_{\mathrm{op}}^* \in [\mathfrak{H}, \mathfrak{H}_0]$ with the operator $\tilde{A}_0^* \in [\mathfrak{H}]$, where $\tilde{A}_0$ is the zero extension of the operator A ($\tilde{A}_0 e_n = 0$). Setting

$$\mathcal{H} = \mathbb{C},\ \Gamma_1\hat{f} = (f, P_n),\ \Gamma_2\hat{f} = (f, b_{n-1}P_{n-1} + a_nP_n) - c,\ \hat{f} = \{f, A_{\mathrm{op}}^* f + cP_n\}, \tag{129}$$

we obtain a SBV of the relation A^*. It is easy to see that

$$f_\lambda(t) = h(\lambda, t) \doteq \sum_{k=0}^{n} P_k(\lambda)P_k(t) \in \mathfrak{N}_\lambda,\ A_{\mathrm{op}}^* f_\lambda(t) = \lambda f_\lambda(t) - \lambda P_n(\lambda)P_n(t), \tag{130}$$

$$\Gamma_1\hat{f}_\lambda = P_n(\lambda),\ \Gamma_2\hat{f}_\lambda = b_{n-1}P_{n-1}(\lambda) + a_nP_n(\lambda) - \lambda P_n(\lambda) = -b_nP_{n+1}(\lambda). \tag{131}$$

Here $\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\}$, $h(\lambda, \mu) = (f_\lambda, f_\mu)$ is the polynomial kernel, corresponding to the system $\{P_k(t)\}_0^n$ (it is also the reproducing kernel of the space $\mathfrak{H} = \mathbb{C}_n[t]$). The selection of the moment S_{2n+1} for the determination of the polynomial $P_{n+1}(t)$ is immaterial; its replacement by S'_{2n+1} changes b_n and $P_{n+1}(t)$ into b'_n and $P'_{n+1}(t)$ without changing their product: $b_nP_{n+1}(t) = b'_nP'_{n+1}(t)$.

Now from (129) and (131) we find the Weyl function $M(\lambda)$ and the operator V_Γ:

$$M(\lambda) = -\frac{P_n(\lambda)}{b_nP_{n+1}(\lambda)},\ \mathcal{H}_2 \doteq \Gamma_2\{0, \mathfrak{N}\} = \mathbb{C},\ \Gamma_1\{0, \mathfrak{N}\} = \{0\},\ V_\Gamma = \{0\}, \tag{132}$$

where $\mathfrak{N} = \{P_n(t)\}$. The obtained relations illustrate Theorem 1: a) $\lim_{y\uparrow\infty} M(iy) = V_\Gamma = 0$; b) the Weyl function $M_1(\lambda)$, corresponding to the SBV $\{\mathbb{C}, \Gamma_2, -\Gamma_1\}$, contains a linear term:

$$M_1(\lambda) = -M(\lambda)^{-1} = \frac{b_nP_{n+1}(\lambda)}{P_n(\lambda)} = b_n\sqrt{\frac{D_n}{D_{n+1}}}\lambda + a + \frac{G_{n-1}(\lambda)}{P_n(\lambda)}\quad (\deg G_{n-1} = n-1).$$

We mention that the alternation of the zeros of the polynomials $P_n(\lambda)$ and $P_{n+1}(\lambda)$ and their simplicity are consequences of the equality (132) and of the inclusion $M(\lambda) \in (R)$. The alternation and the simplicity of the zeros of the quasi-orthogonal polynomials $P_{n+1}(\lambda, \tau_i) \doteq P_{n+1}(\lambda) - \tau_iP_n(\lambda)$ ($i = 1, 2$) are also consequences of the Nevanlinna property of the Weyl function $M_2(\lambda) = P_{n+1}(\lambda, \tau_1)/P_{n+1}(\lambda, \tau_2)$, $\tau_1 > \tau_2$, corresponding to the SBV $\{\mathcal{H}, \Gamma_1^1, \Gamma_2^1\}$, where $\Gamma_i^1 = (\tau_1 -$

$\tau_2)(\tau_i\Gamma_1 + \Gamma_2)$, $i = 1, 2$. We note also that if $-\infty \le \alpha \le \tilde{A}_2 \le \beta \le \infty$, then the zeros of the polynomial $P_k(\lambda)$ are situated in (α, β).

The application of Theorem 3 to the truncated moment problem

$$s_k = \int_{-\infty}^{\infty} t^k d\sigma(t),\ 0 \le k \le 2n-1,\ \int_{-\infty}^{\infty} t^{2n} d\sigma(t) \le s_{2n} \tag{133}$$

gives us the possibility to describe the collection $\tilde{V}(S; \mathbb{R})$ of all its measure-solutions $\sigma(t)$ and also the subclass $V(S, \mathbb{R})$ of those $\sigma(t) \in \tilde{V}(S; \mathbb{R})$ for which in (133) we have

$$s_{2n} = \int_{-\infty}^{\infty} t^{2n} d\sigma(t). \tag{134}$$

Proposition 11 [28]. Let $s = \{s_k\}_0^{2n}$ be a strictly positive sequence in $\mathbb{R}$, let $P_k(\lambda)$ and $Q_k(\lambda)$ be orthogonal polynomials of the first and second kind, respectively. Then the Nevanlinna formula

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-\lambda} = -\frac{Q_{n+1}(\lambda)\tau(\lambda) - Q_n(\lambda)}{P_{n+1}(\lambda)\tau(\lambda) - P_n(\lambda)} = \frac{Q_n(\lambda)\omega(\lambda) + Q_{n+1}(\lambda)}{P_n(\lambda)\omega(\lambda) + P_{n+1}(\lambda)} \tag{135}$$

establishes a bijective correspondence between $\sigma(t) \in \tilde{V}(S; \mathbb{R})$ and $\tau(\lambda)$ $(= -\omega(\lambda)^{-1}) \in (\tilde{R})$. Moreover, we have the equivalence

$$\sigma(t) \in V(s;\mathrm{R}) \Leftrightarrow \tau(\lambda) \in (\tilde{R}),\ \lim_{y\uparrow\infty} y\tau(iy) = \infty \left(\Leftrightarrow \lim_{y\uparrow\infty} \frac{\omega(iy)}{y} = 0 \right). \tag{136}$$

The proof follows from Theorem 3 and formula (132) for $M(\lambda)$ if we take $\mathfrak{L} = \{\mathbb{1}\}$ and we note that the $\mathfrak{L}$-spectral functions $\sigma(t) = (E(t)\mathbb{1}, \mathbb{1})$ of the extensions $\tilde{A} = \tilde{A}^*$ are the only solutions of the problem (133) and the equality (134) is satisfied precisely by those $\sigma(t)$ which are generated by the spectral measures of the operators $\tilde{A} = \tilde{A}^* \supset A$ $(\Leftrightarrow R_\lambda \in \Omega_A)$. It remains to note that the condition of the M-admissibility of the function $\tau(\lambda)$ assumes the form (136) since, by virtue of (132), $\lim_{y\uparrow\infty} M(iy) \ne \infty$. Setting $\omega(\lambda) = -\tau(\lambda)^{-1}$, we obtain the second equality in (135), in which $\omega(\lambda)$ satisfies already Nevanlinna's condition: $\lim_{y\uparrow\infty} \omega(iy)/y = 0$.

5. Let A be a Hermitian matrix in $\mathbb{C}^n$ $(A \in [\mathbb{C}^n])$, let g be a vector $(\in \mathbb{C}^n)$. We consider the bordered matrix

$$\tilde{A}_1 = \tilde{A}_1^* = \begin{pmatrix} A & g \\ g^* & a \end{pmatrix} \in [\mathbb{C}^{n+1}]$$

as the extension of the operator $A \in [\mathbb{C}^n, \mathbb{C}^{n+1}]$ $(a = \bar{a})$, and we define the SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ of the relation (see [30]) $A^* = \{\{f, \tilde{A}_1 f + c\}; f \in \mathbb{C}^{n+1}, c \in \mathbb{C}\}$:

$$\mathcal{H} = \mathbb{C},\ \Gamma_1\{f, \tilde{A}_1 f + c\} = c,\ \Gamma_2\{f, \tilde{A}_1 f + c\} = (f, e_{n+1}). \tag{137}$$

Here, $\{e_k\}_1^{n+1}$ is a basis in $\mathbb{C}^{n+1}$, $e_{n+1} \in \mathbb{C}^{n+1} \ominus \mathbb{C}^n$. The Weyl function, corresponding to the SBV (137), has the form

$$M(\lambda) = \lambda - a + g^*(A-\lambda)^{-1} g,\ \left(\gamma(\lambda) = 1 \oplus (A-\lambda)^{-1} g\right). \tag{138}$$

From (138) we can easily see that

$$M(\lambda) = -\frac{\det(\tilde{A}_1 - \lambda)}{\det(A-\lambda)} \in (R). \tag{139}$$

From (13) there follows the alternation of the eigenvalues $\{\lambda_k\}_1^n$ and $\{\tilde{\lambda}_k\}_1^{n+1}$ of the matrices A and $\tilde{A}_1$:

$$\tilde{\lambda}_1 \le \lambda_1 \le \tilde{\lambda}_2 \le \lambda_2 \le \ldots \le \tilde{\lambda}_n \le \lambda_n \le \tilde{\lambda}_{n+1}.$$

This fact is usually derived from the Courant–Fischer variational principle.

From here we obtain, in particular, the alternation of the zeros of the orthogonal polynomials $P_k(\lambda)$ if we represent them in the form

$$P_k(\lambda)=\frac{\det(\lambda-J_k)}{b_0 b_1\cdots b_{k-1}},\quad J_k=\begin{pmatrix} a_0 & b_0 & 0 & \dots & 0 & 0\\ b_0 & a_1 & b_1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & b_{k-2} & a_{k-1}\end{pmatrix}.$$

From formula (139) one can derive the converse statement: for two alternating collections of numbers $\{\lambda_k\}_1^n$ and $\{\tilde\lambda_k\}_1^{n+1}$, there exist matrices A and $\tilde A_1=\tilde A_1^*=\begin{pmatrix} A & g\\ g^* & a\end{pmatrix}$ such that $\sigma(A)=\{\lambda_k\}_1^n$, $\sigma(\tilde A_1)=\{\tilde\lambda_k\}_1^{n+1}$.

6. Let $\tilde A_2 \doteq \tilde A=\tilde A^*\ge 0$, $\tilde A_1=\tilde A_2-K^*K$, where $\tilde A_i\in\mathfrak{C}(\mathfrak{H})$, $K^*K\in\mathfrak{C}[\mathfrak{H}]$, and $0\in\rho(K^*K)$. We define the bounded operators $\tilde A_2(n)\doteq\tilde A_2 E_{\tilde A_2}(-\infty,n)$, $\tilde A_1(n)=\tilde A_2(n)-K^*K$. Considering $\tilde A_i(n)$, i = 1, 2, as transversal ($0\in\rho(\tilde A(n)-\tilde A_1(n))$) extensions of the zero operator A = 0, we introduce the SBV $\{\mathcal{H},\Gamma_1^{(n)},\Gamma_2^{(n)}\}$ of the relation $A^*=\mathfrak{H}\oplus\mathfrak{H}$, by setting

$$\mathcal{H}=\mathfrak{H},\ \Gamma_i^{(n)}\hat f=(-1)^{i-1}(K^*)^{-1}(f'-\tilde A_i(n)f),\ (i=1,2)\ \hat f=\{f,f'\}. \tag{140}$$

Then $\mathfrak{N}_\lambda=(A_2(n)-\lambda)^{-1}\mathfrak{H}$ and $\forall\hat f_\lambda=\{f_\lambda,\lambda f_\lambda\}$ $(f_\lambda\doteq(A_2(n)-\lambda)^{-1}f)$ we obtain

$$\Gamma_1\hat f_\lambda=(K^*)^{-1}\left[-f+K^*K(A_2(n)-\lambda)^{-1}f\right],\ \Gamma_2\hat f_\lambda=(K^*)^{-1}f,\ f\in\mathfrak{H}.$$

From here we find the Weyl function $M_n(\lambda)$, corresponding to the SBV (140):

$$M_n(\lambda)=-I+K(A_2(n)-\lambda)^{-1}K^*,\ \gamma_n(\lambda)=(A_2(n)-\lambda)^{-1}K^*. \tag{141}$$

According to [19, 31], for each SBV $\{\mathcal{H},\Gamma_1,\Gamma_2\}$ such that $\tilde A_2\ge 0$ we have the equality

$$\dim E_{\tilde A_\theta}(-\infty,-\varepsilon)=\dim E_{\theta-M(-\varepsilon)}(-\infty,0)-\dim E_{\theta-M(-\infty)}(-\infty,0)\ \forall\varepsilon\ge 0, \tag{142}$$

where $M(0)=s-R-\lim_{\lambda\uparrow 0}M(\lambda)$, $M(-\infty)=s-R-\lim_{\lambda\downarrow-\infty}M(\lambda)$. In connection with the operator $\tilde A_1$ ($\Leftrightarrow\theta=0$), from the equality (142) we obtain

$$\dim E_{A_1(n)}(-\infty,-\varepsilon)=\dim E_{-M_n(-\varepsilon)}(-\infty,0)=\dim E_{M_n(-\varepsilon)}(0,+\infty)\ \forall\varepsilon\ge 0. \tag{143}$$

Under the additional condition $K(\tilde A_2+\lambda)^{-1/2}\in\mathfrak{S}_\infty$ the operators $K(\tilde A_2(n)+\lambda)^{-1/2}\in\mathfrak{S}_\infty$ $(\forall n\in\mathbb{Z}_+)$, and the spectrum $\sigma(\tilde A_1(n))$ in the interval $(-\infty,-\varepsilon)$ is discrete for each $\varepsilon>0$. Therefore, in the equality (143), both sides of which are finite for each $\varepsilon>0$, it is possible to take the limit from below for $n\to\infty$ and $\varepsilon\ge 0$ [32]:

$$\dim E_{\tilde A_1}(-\infty,-\varepsilon)=\dim E_{-M(-\varepsilon)}(-\infty,0)=\dim E_{M(-\varepsilon)}(0,+\infty), \tag{144}$$

where $M(\lambda)$, which, naturally, can be called the generalized Weyl function (it is not the usual one: $0\notin\rho(\Im M(i))$ if $\tilde A_2\notin[\mathfrak{H}]$), has the form

$$M(\lambda)=-I+K\left(\hat A_2-\lambda\right)^{-1}K^*. \tag{145}$$

It is easy to see that the relation (144) remains valid also for unbounded perturbations K^*K such that $\mathcal{D}(\mathfrak{t}_{K^*K})=\mathcal{D}[K^*K]=\mathcal{D}(K)\supseteq\mathcal{D}[\tilde A]=\mathcal{D}(\mathfrak{t}_{\tilde A})=\mathcal{D}(\tilde A^{1/2})$ and the form $\mathfrak{t}\doteq\mathfrak{t}_{\tilde A}-\mathfrak{t}_{K^*K}$ is closed and semibounded from below (here $\mathfrak{t}_T$ is the closed quadratic form, associated with the semibounded operator $T=T^*\ge m$: $\mathcal{D}(\mathfrak{t}_T)=\mathcal{D}((T-m)^{1/2})$, $\mathfrak{t}_T[u]=\|(T-m)^{1/2}u\|^2$). In this case the operator $\tilde A_1$, understood as the form-sum, is associated with the form $\mathfrak{t}:\mathfrak{t}_{\tilde A_1}=\mathfrak{t}=\mathfrak{t}_{\tilde A}-\mathfrak{t}_{K^*K}$, while the function $M(\lambda)$ of the form (145) has to be understood as:

$$M(\lambda) = -I + K(\tilde{A} - \lambda)^{-1/2}(K(\tilde{A} - \lambda)^{-1/2})^*. \tag{146}$$

We note that relation (144) can be proved directly, omitting the limiting process from (143), and without the condition $K(\tilde{A} + \varepsilon)^{-1/2} \in \mathfrak{S}_\infty$.

If $G(0) \doteq I + M(0) \in \mathfrak{S}_p$, then from (144) for $\varepsilon = 0$ there follows the estimate

$$N_-(\tilde{A}_1) \doteq \dim E_{\tilde{A}_1}(-\infty, 0) = \sum_{\lambda_j(G(0))>1} \lambda_j^p \leq \|G(0)\|^p_{\mathfrak{S}_p}. \tag{147}$$

If the negative part of the spectrum $\sigma(\tilde{A}_1)$ is discrete, but infinite, then for the eigenvalues $\lambda_n^-(\tilde{A}_1) < 0$ we have (see [31]) the equivalence

$$\lambda_n^-(\tilde{A}_1) = O(n^{-p}) \Leftrightarrow \lambda_n^-(I - G(0)) = O(n^{-p}). \tag{148}$$

Now from (144)-(147) there follows the known Birman–Schwinger principle. For its formulation we introduce the Hilbert spaces $D_\varepsilon[\tilde{A}]$ with the metric $\|u\|_\varepsilon^2 = \tilde{A}[u, u] + \varepsilon\|u\|^2$ ($\varepsilon > 0$) and also (assuming $\ker \tilde{A} = \{0\}$) the space $\mathfrak{H}_{\tilde{A}}$ which is the completion of $D_0[\tilde{A}]$ with respect to the $\tilde{A}$-metric.

Proposition 12 [33]. Let $A = A^* \geq 0$ ($A \in \mathfrak{C}(\mathfrak{H})$), $B = B^* \geq 0$, $\mathcal{D}[B] \supseteq \mathcal{D}[A]$ and assume that the form $\mathfrak{k} \doteq \mathfrak{k}_A - \mathfrak{k}_B$ is closed and semibounded from below. Then:

1) the total multiplicity of the spectrum of the operator $C = C^*$, associated with the form $\mathfrak{k}$ ($\mathfrak{k} = \mathfrak{k}_C$), in the interval $(-\infty, -\varepsilon)$, $\varepsilon > 0$, is equal to the total multiplicity of the spectrum in $(1, \infty)$ of the form $\mathfrak{k}_B$ in the space $D_\varepsilon[A]$;

2) if the form $\mathfrak{k}_B$ is compact in $D_1[A]$ ($\mathfrak{H}_A$), then the negative spectrum of the operator C is discrete (finite);

3) in order that the negative part of the spectrum of the operator C_h associated with the form $h\mathfrak{k}_A - \mathfrak{k}_B$ be discrete (finite) for each $h > 0$, it is necessary and sufficient that the form $\mathfrak{k}_B (= B[u, u])$ be compact in $D_1[A]$ ($\mathfrak{H}_A$).

Proof. Statement 1 is a consequence of the equalities (144)-(146). Further, from the compactness of the form $\mathfrak{k}_B$ in $D_1[A]$ there follow (see [33]) the closedness and the semiboundedness from below of the form $\mathfrak{k} \doteq \mathfrak{k}_A - \mathfrak{k}_B \Rightarrow \exists C = C^* \geq m$: $\mathfrak{k} = \mathfrak{k}_C$. The remaining statements follow now from the chain of equivalences: the form $\mathfrak{k}_B$ is compact in $D_\varepsilon[A]$ $\Leftrightarrow$ an arbitrary $(A + \varepsilon)$-bounded set is B-compact $\Leftrightarrow$ the operator $T = B^{-1/2}(A + \varepsilon)^{-1/2} \in \mathfrak{S}_\infty \Leftrightarrow TT^* = I + M(-\varepsilon) \in \mathfrak{S}_\infty$. In particular, the compactness of the form $\mathfrak{k}_B$ in $\mathfrak{H}_A$ is equivalent to the compactness of the operator $I + M(0)$ ($M(0) \doteq s - R - \lim_{\lambda \uparrow 0} M(\lambda)$).

We illustrate relation (147) by two known examples.

a) Let $\mathfrak{H} = L_2[0, \infty)$, $Ay = -y''$, $\mathcal{D}(A) = \overset{\circ}{W}{}_2^2$ ($y \in \overset{\circ}{W}{}_2^2 \Leftrightarrow y \in W_2^2$, $y(0) = 0$), $By = q(x)y$, $C \doteq A - B$ ($q(x) \geq 0$, $q \in C_{[0,\infty)}$). In this case

$$M(\lambda)f = -f + \int_0^\infty \sqrt{q(x)q(t)}\, G(x,t,\lambda) f(t)\,dt,$$

where $G(x,t,\lambda) = \frac{1}{\sqrt{-\lambda}} \operatorname{sh} t\sqrt{-\lambda} \exp\left(-x\sqrt{-\lambda}\right)$, $t \leq x$. We find M(0):

$$M(0)f = -f + \int_0^\infty \sqrt{q(x)q(t)}\, G(x,t,0) f(t)\,dt, \quad G(x,t,0) = \begin{cases} t, & t \leq x; \\ x, & t \geq x. \end{cases}$$

From here and from (147) for $p = 1$ there follows the Bargmann bound estimate [33-35]:

$$N_-(C) = \dim E_C(-\infty, 0) \leq \operatorname{sp}(I + M(0)) = \int_0^\infty x q(x)\,dx.$$

b) Let $\mathfrak{H} = L_2(\mathbb{R}^3)$, $A = -\Delta$, $By = q(x)y$, $C = A - B$, $By = \sqrt{q(x)}\,y$ and assume that $q(x)$ (≥ 0) belongs to the Rollnik class [34]. Then

$$M(\lambda) = -f + \int_{\mathbb{R}^3} \frac{\sqrt{q(x)q(t)} \exp\left(-\sqrt{-\lambda}\,|x - t|\right)}{4\pi|x - t|} f(t)\,dt.$$

Therefore, for M(0) we obtain the expression

$$M(0)f = -f + \int_{\mathbb{R}^3} \frac{\sqrt{q(x)q(t)}}{4\pi|x-t|} f(t)dt. \tag{149}$$

From (149) and (147) for p = 2 we obtain the Birman–Schwinger bound [33, 34]:

$$N_{-}(C) = \dim E_C(-\infty, 0) \leq \| M(0) + I \|^2_{\mathfrak{S}_2} = \iint \frac{q(x)q(t)}{16\pi^2 |x-t|^2} dxdt.$$

From relation (148) we can obtain information on the asymptotic behavior of the negative spectrum of the Schrödinger operator.

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